THE SKEIN MODULE OF TWO-BRIDGE LINKS

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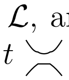
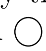
Abstract. We calculate the Kauffman bracket skein module (KBSM) of the complement of all two-bridge links. For a two-bridge link, we show that the KBSM of its complement is free over the ring $\mathbb{C}[t^{\pm 1}]$ and when reducing $t = -1$, it is isomorphic to the ring of regular functions on the character variety of the link group.

0. Introduction

The theory of Kauffman bracket skein module (KBSM) was introduced by Przytycki [Pr] and Turaev [Tu] as a generalization of the Kauffman bracket [Ka] in S^3 to an arbitrary 3-manifold. The KBSM of a knot complement contains a lot, if not all, of information about the colored Jones polynomial. It also contains a lot of information about classical geometric invariants such as the character variety, and has been instrumental in the study of the AJ conjecture which relates the colored Jones polynomial and the A-polynomial of a knot, see [FGL, Ge, Ga, Le, LT]. The calculation of the KBSM of a knot complement is a difficult task. At the moment, the KBSM has been calculated only for two-bridge knots [Le] (with earlier work for twist knots [BL]) and torus knots [Ma] (with earlier work for $(2, 2m + 1)$-torus knots [Bu1]). In this paper, we calculate the KBSM of the complement of all two-bridge links. Applications to the theory of AJ conjecture for links will be discussed in a subsequent work.

0.1. **Skein modules.** A *framed link* in an oriented 3-manifold Y is a disjoint union of embedded circles, equipped with a non-zero normal vector field. Framed links are considered up to isotopy. In all figures we will draw framed links, or part of them, by lines as usual, with the convention that the framing is blackboard. Let $\mathcal{L}$ be the set of isotopy classes of framed links in the manifold Y, including the empty link. Consider the free $\mathbb{C}[t^{\pm 1}]$-module with basis $\mathcal{L}$, and factor it by the smallest submodule containing all expressions of the form  $- t$  $- t^{-1}$  and  $+ (t^2 + t^{-2})\emptyset$, where the links in each expression are identical except in a ball in which they look like depicted. This quotient is denoted by $\mathcal{S}(Y)$ and is called the Kauffman bracket skein module, or just skein module, of Y.

If $Y_1 \subset Y_2$, then the embedding $Y_1 \hookrightarrow Y_2$ induces a linear map $\mathcal{S}(Y_1) \to \mathcal{S}(Y_2)$.

For an oriented surface Σ we define $\mathcal{S}(\Sigma) = \mathcal{S}(Y)$, where $Y = \Sigma \times [0, 1]$, the cylinder over Σ. The skein module $\mathcal{S}(\Sigma)$ has an algebra structure induced by the operation of gluing one cylinder on top of the other.

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0.2. **Main Results.** A two-bridge link is a two-component link $L \subset S^3$ such that there is a 2-sphere $S^2 \subset S^3$ separating S^3 into 2 balls B_1 and B_2, and the intersection of L and each ball is isotopic to 2 trivial arcs in the ball. The branched double covering of S^3 along a two-bridge link is a lens space $L(2p, q)$, which is obtained by doing a $2p/q$ surgery on the unknot. Such a two-bridge link is denoted by $\mathfrak{b}(2p, q)$. Here $\gcd(q, 2p) = 1$, and one can always assume that $2p > q \geq 1$. It is known that $\mathfrak{b}(2p', q')$ is isotopic to $\mathfrak{b}(2p, q)$ if and only if $p' = p$ and $q' \equiv q^{\pm 1} \pmod{2p}$, see [BZ].

Assume the 3-ball B_1 is presented as a vertical cylinder $B_1 = D \times [0, 1]$, where D is a 2-dimensional disk, and the two arcs of L inside B_1 are two vertical line segments $U \times [0, 1]$ and $U' \times [0, 1]$, where U and U' are 2 interior points of D. Let $D_{**} = D \setminus \{U, U'\}$, then $B_1 \setminus L = D_{**} \times [0, 1]$. Hence $\mathcal{S}(B_1 \setminus L) = \mathcal{S}(D_{**})$ is an algebra. Let $x, x' \subset D_{**}$ are respectively small loops around U, U', and $y = \partial D \subset D_{**}$ is the boundary of D. We consider x, x', and y as elements of the algebra $\mathcal{S}(B_1 \setminus L)$. Using the embedding $\mathcal{S}(B_1 \setminus L) \subset \mathcal{S}(S^3 \setminus L)$ we will consider $x^a (x')^b y^c$ as an element of $\mathcal{S}(S^3 \setminus L)$.

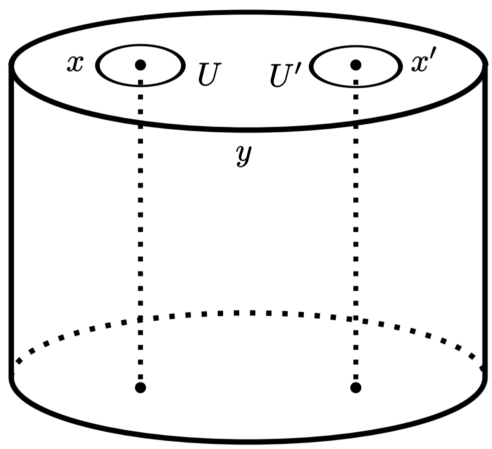

FIGURE 1. The loops x, x' and y

Theorem 1. *For the two-bridge link $L = \mathfrak{b}(2p, q)$, the skein module $\mathcal{S}(S^3 \setminus L)$ is free over $\mathbb{C}[t^{\pm 1}]$ with basis $\{x^a (x')^b y^c \mid 0 \leq a, b, 0 \leq c \leq p\}$.*

0.3. **The universal character ring.** Let $\varepsilon(\mathcal{S}(Y))$ be the quotient of $\mathcal{S}(Y)$ by the relation $t = -1$. An important result [Bu2, PS] in the theory of skein modules is that $\varepsilon(\mathcal{S}(Y))$ has a natural $\mathbb{C}$-algebra structure and is isomorphic to the universal SL_2-character algebra of the fundamental group of Y. For a defintion of the universal character algebra, see [BH, LM]. The product of 2 links in $\varepsilon(\mathcal{S}(Y))$ is their union. Using the skein relation with $t = -1$, it is easy to see that the product is well-defined, and that the value of a knot in the skein module depends only on the homotopy class of the knot in Y. The isomorphism between $\varepsilon(\mathcal{S}(Y))$ and the universal SL_2-character algebra of $\pi_1(Y)$ is given by $K(\rho) = -\operatorname{tr} \rho(K)$, where K is a homotopy class of a knot in Y, represented by an element, also denoted by K, of $\pi_1(Y)$, and $\rho : \pi_1(Y) \to SL_2(\mathbb{C})$ is a representation of $\pi_1(Y)$. The quotient of $\varepsilon(\mathcal{S}(Y))$ by its nilradical is canonically isomorphic to $\mathbb{C}[\chi(\pi_1(Y))]$, the ring of regular functions on the SL_2-character variety of $\pi_1(Y)$.

The above fact has been exploited in the work of Frohman, Gelca, and Lofaro [FGL] where they defined the non-commutative A-ideal of a knot, and in our proof of the AJ conjecture [Ga] for some classes of two-bridge knots and pretzel knots in [Le, LT]. In our work on the AJ conjecture, it is important to know whether the universal character algebra $\varepsilon(\mathcal{S}(Y))$ is reduced, i.e. whether its nilradical is 0. Although it is difficult to construct a group whose universal character algebra is not reduced (see [LM]), so far there are a few groups for which the universal character algebra is known to be reduced: free groups [Si], surface groups [CM], two-bridge knot groups [PS], torus knot groups [Ma], some pretzel knot groups [LT].

As a consequence of Theorem 1, we will show the following.

Proposition 1. *For a two-bridge link L, the universal SL_2-character algebra $\varepsilon(\mathcal{S}(S^3 \setminus L))$ is reduced, and hence $\varepsilon(\mathcal{S}(S^3 \setminus L))$ is canonically isomorphic to the ring of regular functions on the SL_2-character variety of $\pi_1(S^3 \setminus L)$.*

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1. Proof of Theorem 1 and Proposition 1

We change the picture and will present the ball $B_1 \subset \mathbb{R}^3$ as the closed ball of radius $\sqrt{2}$ centered at the origin, i.e. $B_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 \le 2\}$. We suppose that the two-bridge link $L = \mathfrak{b}(2p, q)$ intersects the interior of B_1 in two straight intervals UV and $U'V'$ in the x_1x_2-plane, where $U = (-1, 1, 0)$, $U' = (1, 1, 0)$, $V = (-1, -1, 0)$ and $V' = (1, -1, 0)$, see Figure 2. After an isotopy, we assume that the part of L outside the interior of B_1 are 2 non-intersecting arcs $\mathfrak{u}$ and $\mathfrak{u}'$ on the sphere $S = \partial B_1$, where $\mathfrak{u}$ connects U and V, and $\mathfrak{u}'$ connects U' and V'. If one cuts S along the arc $\mathfrak{u}$, then one obtains a disk, hence the other arc $\mathfrak{u}'$, is uniquely determined by $\mathfrak{u}$, up to isotopy.

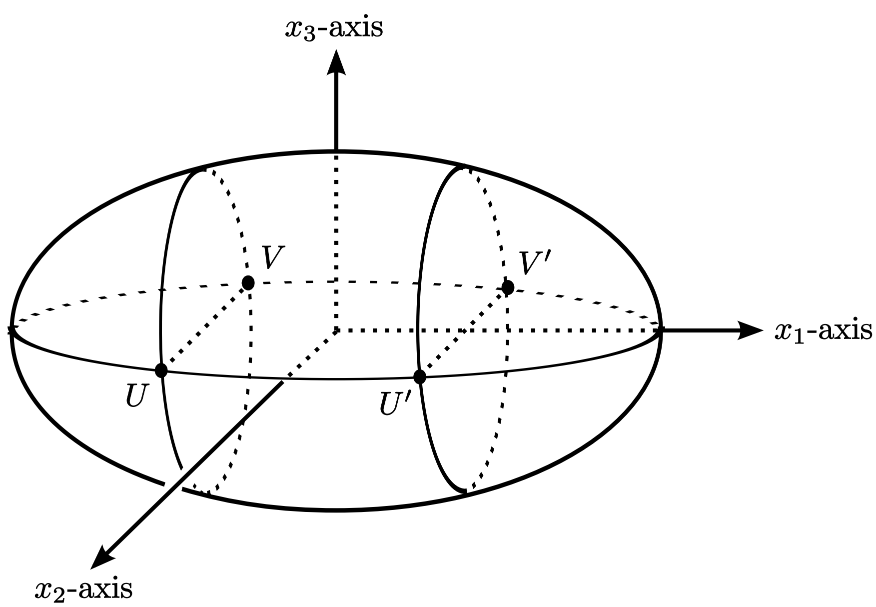

Figure 2. The ball B_1

For a set $Z \subset \mathbb{R}^3$ let $Z[\alpha, \beta]$ be the part of Z in the strip $\{\alpha \le x_1 \le \beta\}$, i.e. $Z[\alpha, \beta] := Z \cap \{(x_1, x_2, x_3) \mid \alpha \le x_1 \le \beta\}$.

Let $\tilde{S}$ be the 2-fold covering of S branched along the 4 points U, U', V, V'. Note that $\tilde{S}$ is a torus, with the following preferred meridian and longitude. The plane passing through U, U', V, V' (i.e. the x_1x_2 plane) intersects $S[-\sqrt{2}, -1]$ in an arc $\mathfrak{m}$ that connects U and V. In other words, $\mathfrak{m}$ is the shortest arc on the sphere S connecting U and V, see Figure 3. The total lift $\tilde{\mathfrak{m}}$ of $\mathfrak{m}$ is a closed curve on the the torus $\tilde{S}$ which will serve as the meridian, see Figure 4. Let $\mathfrak{l}$ be the shortest arc on S connecting U and U'. The total lift $\tilde{\mathfrak{l}}$ of $\mathfrak{l}$ is a closed curve serving as the longitude. It is easy to see that $\tilde{\mathfrak{m}}$ and $\tilde{\mathfrak{l}}$ form a basis of $H_1(\tilde{S}, \mathbb{Z})$.

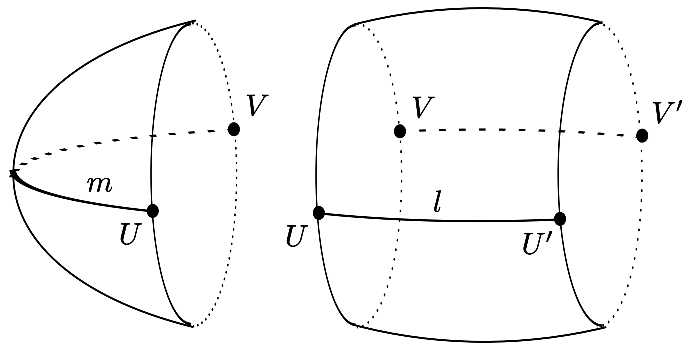

FIGURE 3. The curves $\mathfrak{m}$ and $\mathfrak{l}$.

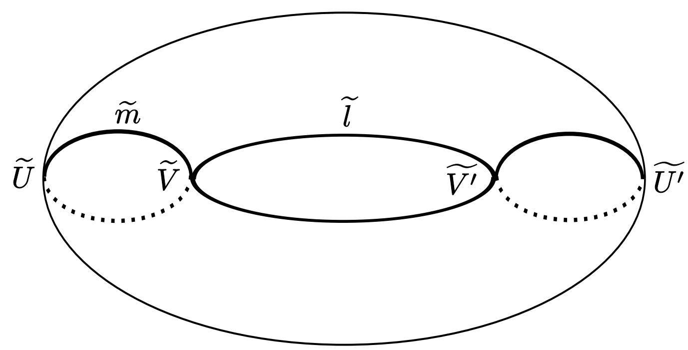

FIGURE 4. The total lifts $\tilde{\mathfrak{m}}, \tilde{\mathfrak{l}}$ of $\mathfrak{m}$ and $\mathfrak{l}$ on the torus $\tilde{S}$ respectively.

According to [BZ, Chapter 12], the isotopy class of the pair of arcs $(\mathfrak{u}, \mathfrak{u}')$ in the ball B_2 is uniquely determined by the homology class of the total lift $\tilde{\mathfrak{u}}$ of the curve $\mathfrak{u}$ in $H_1(\tilde{S}, \mathbb{Z})$. Moreover, the homology class of $\tilde{\mathfrak{u}}$ is equal to $2p\,\tilde{\mathfrak{m}} + q'\tilde{\mathfrak{l}}$ for some $q' \in \mathbb{Z}$ satisfying the condition $q' \equiv q^{\pm 1} \pmod{2p}$. We will describe explicitly the arc $\mathfrak{u}$ in the next subsection.

1.1. **Description of $\mathfrak{u}$.** We will present $\mathfrak{u}$ by describing 3 parts of it: the left part $\mathfrak{u}_l$, the middle part $\mathfrak{u}_m$, and the right part $\mathfrak{u}_r$, which are respectively the intersection of $\mathfrak{u}$ with $S_l := S[-\sqrt{2}, -1]$, $S_m := S[-1, 1]$, and $S_r := S[1, \sqrt{2}]$. For two non-antipodal points A, B on the sphere S let $\gamma(AB)$ be the shortest geodesic on S connecting A and B.

The boundary $C_l := \partial S_l$ is a circle containing U and V. On the circle C mark $2p$ points $A_0 = V, A_1, \dots, A_{2p-1}$ which are: (i) counter-clockwise in that order if viewing from the origin of the coordinate system, and (ii) uniformly distributed on the circle C.

Then $A_p = U$, and for $1 \le j \le p-1$, the segment $A_{p-j}A_{p+j}$ is parallel to the x_3-axis. The shortest geodesic $\gamma(A_{p-j}A_{p+j})$ lies in S_l. Let $\mathfrak{u}_{0,l}$ be the union of all the disjoint $\gamma(A_{p-j}A_{p+j})$, $1 \le j \le p-1$. See Figure 5.

Let E_j be the midpoint of the arc A_jA_{j+1} on the circle C (indices are taken modulo $2p$). In other words, E_j is the image of A_j under the rotation by $2\pi/4p$ about the x_1-axis, counter-clockwise if viewing from the origin.

Let E'_j be the reflection of E_j through the x_2x_3-plane. Note that all the points E'_j are on the circle $C' = \partial S_r$. The p geodesics $\gamma(E'_{p-j}E'_{p+j-1})$, $j = 1, \dots, p$, are disjoint and are in S_r. Let $\mathfrak{u}_{0,r}$ be the union of the p geodesics $\gamma(E'_{p-j}E'_{p+j-1})$, $j = 1, \dots, p$.

On S_m let $\mathfrak{u}_{0,m}$ be the union of $2p$ geodesics $\gamma(A_jE'_{j+(q-1)/2})$, $j = 0, 1, \dots, 2p-1$ (indices taken modulo $2p$). Note that the $2p$ components of $\mathfrak{u}_{0,m}$ are obtained from each other by rotations by $2j\pi/2p$ about the x_1-axis.

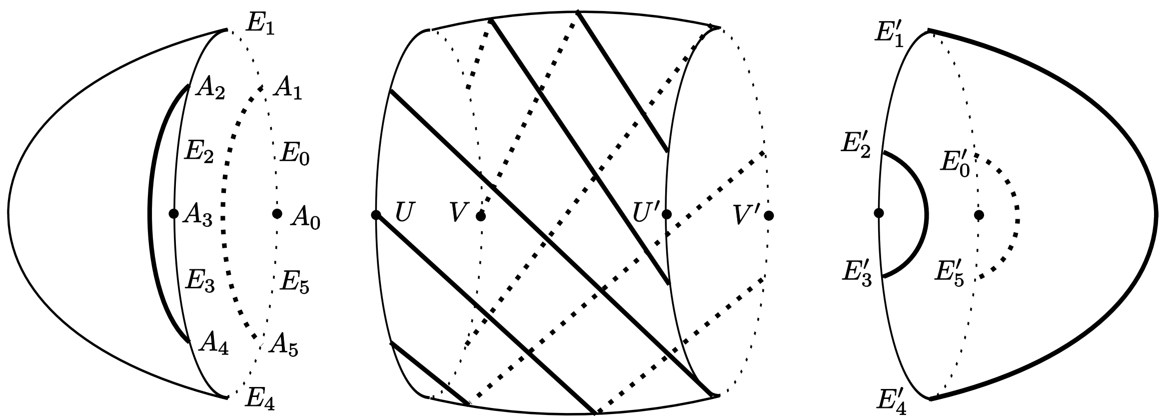

FIGURE 5. $\mathfrak{u}_{0,l}, \mathfrak{u}_{0,m}, \mathfrak{u}_{0,r}$ for $2p = 6$ and $q' = 5$

Let $\mathfrak{u}_0$ be the arc on S obtained by combining $\mathfrak{u}_{0,l}, \mathfrak{u}_{0,m}$ and $\mathfrak{u}_{0,r}$; it connects U and V. Up to isotopy there is a unique arc $\mathfrak{u}'_0$ on S connecting V and V', and disjoint from $\mathfrak{u}_0$.

Lemma 1.1. *The pair* $(\mathfrak{u}_0, \mathfrak{u}'_0)$ *is isotopic, relative endpoints, to* $(\mathfrak{u}, \mathfrak{u}')$ *in the ball* B_2.

Proof. It is easy that the homology class of the total lift of the arc $\mathfrak{u}_0$ in $H_1(\tilde{S}^2, \mathbb{Z})$ is equal to $2p\,\tilde{\mathfrak{m}} + q'\tilde{\mathfrak{l}}$, which is exactly equal to the homology class of the total lift of the arc $\mathfrak{u}$ in $H_1(\tilde{S}^2, \mathbb{Z})$. According to [BZ, Chapter 12], $(\mathfrak{u}_0, \mathfrak{u}'_0)$ is isotopic, relative endpoints, to $(\mathfrak{u}, \mathfrak{u}')$ in the ball B_2. □

From now on we identify $(\mathfrak{u}, \mathfrak{u}')$ and $(\mathfrak{u}_0, \mathfrak{u}'_0)$. Without loss of generality we also assume $q' = q$.

1.2. **The link complement.** Let ω be the boundary curve of a small normal neighborhood of the arc $\mathfrak{u}$ in $S = \partial B_1$. Let $X_1 := B_1 \setminus (UV \cup U'V')$, which is homeomorphic to the cylinder over a two-punctured disk D_{**} in Subsection 0.2. Then the complement X of the link L is obtained from X_1 by gluing a 2-handle to X_1 along ω. Up to isotopy, ω can be described as follows.

On the circle $C = \partial S_l$ mark $4p$ points $F_0, F_1, \dots, F_{4p-1}$ which are: (i) counter-clockwise in that order if viewing from the origin of the coordinate system, and (ii) uniformly distributed on the circle C, and such that (iii) V is the midpoint of the arc $F_{4p-1}F_0$ on

C. We can also say that F_{2j} is the midpoint of the arc A_jE_j, and F_{2j+1} is the midpoint of the arc E_jA_{j+1} on C. See Figures 6 and 7.

For $1 \le j \le 2p$, the segment $F_{2p-j}F_{2p+j-1}$ is parallel to the x_3-axis. The shortest geodesic $\gamma(F_{2p-j}F_{2p+j-1})$ lies in S_l. Let ω_l be the union of all the disjoint $\gamma(F_{2p-j}F_{2p+j-1})$, $1 \le j \le 2p$.

Let F'_j be the reflection of F_j through the x_2x_3-plane. We can also say that F'_{2j} is the midpoint of the arc $A'_jE'_j$, and F'_{2j+1} is the midpoint of the arc $E'_jA'_{j+1}$ on C, where A'_j is the reflection of A_j through the x_2x_3-plane. Note that all the points F'_j are on the circle $C' = \partial S_r$. For $1 \le j \le 2p$, the segment $F'_{2p-j}F'_{2p+j-1}$ is parallel to the x_3-axis. The shortest geodesic $\gamma(F'_{2p-j}F'_{2p+j-1})$ lies in S_r. Let ω_r be the union of all the disjoint $\gamma(F'_{2p-j}F'_{2p+j-1})$, $1 \le j \le 2p$.

Note that we can also say that ω_r is the reflection of ω_l through the x_2x_3-plane.

On S_m, let ω_m is the union of $4p$ geodesics $\gamma(E_jE'_{q+j})$, $j = 0, 1, \dots, 4p-1$ (indices taken modulo $4p$). Note that the $4p$ components of ω_m are obtained from each other by rotations by $2j\pi/4p$ about the x_1-axis.

Then, up to isotopy, ω is obtained by combining ω_l, ω_m and ω_r.

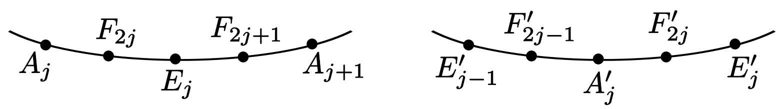

FIGURE 6. The distribution of the points A_j, F_j, E_j and A'_j, F'_j, E'_j on the circles C and C' respectively

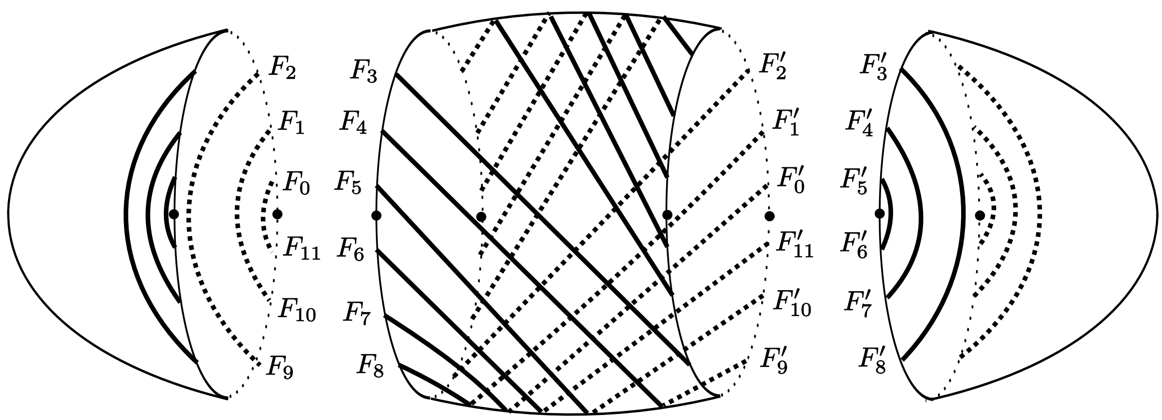

FIGURE 7. $\omega_l, \omega_m, \omega_r$ for $2p = 6$ and $q = 5$

Let ψ be the rotation by 180^o about the x_1-axis. One has $\psi(B_1) = B_1$. Up to isotopy, we can assume that $\psi(\omega) = \omega$.

Let $P = F_{p+(q+1)/2}$, $P' = F_{3p+(q+1)/2}$ and $Q = F'_{p+1-(q+1)/2}$, $Q' = F'_{3p+1-(q+1)/2}$. See Figure 8. Note that $\psi(P) = P'$ and $\psi(Q) = Q'$.

1.3. **Relative skein modules.** Let us recall the definition of the relative skein module $\mathcal{S}(X_1; P, Q')$ (see [BL, Le]). A *type 1 tangle* is the disjoint union of a framed link and a framed arc in X_1 such that the parts of the arc near the two end points are on the boundary ∂X_1, and the framing on these parts are given by vectors normal to ∂X_1. Type 1 tangles are considered up to isotopy relative the endpoints. Then $\mathcal{S}(X_1; P, Q')$ is the $\mathbb{C}[t^{\pm 1}]$-module generated by type 1 tangles with endpoints at P, Q' modulo the usual skein relations, like in the definition of $\mathcal{S}(X)$. One defines in a similar way the relative Kauffman bracket skein module $\mathcal{S}(\partial X_1; P, Q') := \mathcal{S}(\partial X_1 \times [0,1]; P, Q')$, where we identify $\partial X_1 \times [0,1]$ with a collar of ∂X_1 in X_1.

There is a natural bilinear map $\mathcal{S}(\partial X_1; P, Q') \otimes \mathcal{S}(X_1) \to \mathcal{S}(X_1; P, Q')$, where $\ell \otimes \ell' \to \ell \star \ell'$, which is the disjoint union of ℓ and ℓ'.

The pair P, Q divide ω into two arcs, the one that is fully drawn in Figure 8 (and that goes around the points U and V' exactly once) is denoted by ω_s. Similarly, the pair P', Q' divide ω into two arcs, the one that is fully drawn in Figure 8 (and that goes around the points U' and V exactly once) is denoted by ω'_s.

Let $P_c = F_{3p+1-(q+1)/2}, P'_c = F_{p+1-(q+1)/2}$ and $Q_c = F'_{3p+(q+1)/2}, Q'_c = F'_{p+(q+1)/2}$. Then ω_s consists of 3 parts: the left part is an arc on S_l connecting P and P_c, the middle part is an arc on S_m connecting P_c and Q_c, and the right part is an arc on S_r connecting Q_c and Q. Similarly, ω'_s also consists of 3 parts: the left part is an arc on S_l connecting P' and P'_c, the middle part is an arc on S_m connecting P'_c and Q'_c, and the right part is an arc on S_r connecting Q'_c and Q'.

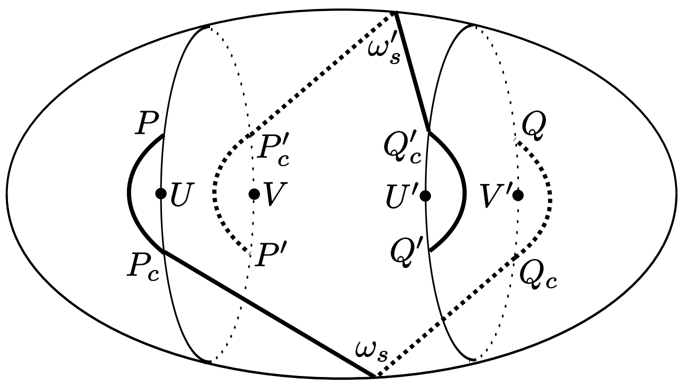

FIGURE 8. ω_s connects P, Q, and ω'_s connects P', Q'

Let $\gamma_{\text{in}}(PQ'), \gamma_{\text{in}}(P'Q)$ be respectively the shortest arcs on the surface $S = \partial B_1$ connecting P and Q', P' and Q, whose interiors are slightly pushed inside the interior of B_1 (to avoid intersections with other arcs on S) and whose framings are given by vectors normal to S.

Let $\mathfrak{d}_{\text{in}}(PP'), \mathfrak{d}_{\text{in}}(QQ')$ be respectively the straight intervals connecting P and P', Q' and Q, whose interiors are slightly pushed into the interior of $B_1[-\sqrt{2}, -1]$ and the interior of $B_1[1, \sqrt{2}]$ respectively (to avoid intersections with the straight lines UV and $U'V'$ respectively).

Let $\mathfrak{a}_1$ be $\gamma_{\text{in}}(PQ')$; $\mathfrak{a}_2$ be ω_s followed by $\mathfrak{d}_{\text{in}}(QQ')$; $\mathfrak{a}_3$ be $\mathfrak{d}_{\text{in}}(PP')$ followed by ω'_s; and $\mathfrak{a}_4$ be ω_s followed by $\gamma_{\text{in}}(QP')$ then followed by ω'_s.

Lemma 1.2. *The relative skein $\mathcal{S}(X_1; P, Q')$ is equal to the union $\cup_{i=1}^{4}(\mathfrak{a}_i \star \mathcal{S}(X_1))$.*

Proof. Let $\mathfrak{a}'_1 = \mathfrak{a}_1$. Let $\mathfrak{a}'_2$ be $\gamma(PP_c)$ folowed by $\gamma_{\text{in}}(P_cQ')$; $\mathfrak{a}'_3$ be $\gamma_{\text{in}}(PQ'_c)$ followed by $\gamma(Q'_cQ')$; and $\mathfrak{a}'_4$ be $\gamma(PP_c)$ followed by $\gamma_{\text{in}}(P_cQ'_c)$ then followed by $\gamma(Q'_cQ')$. Here

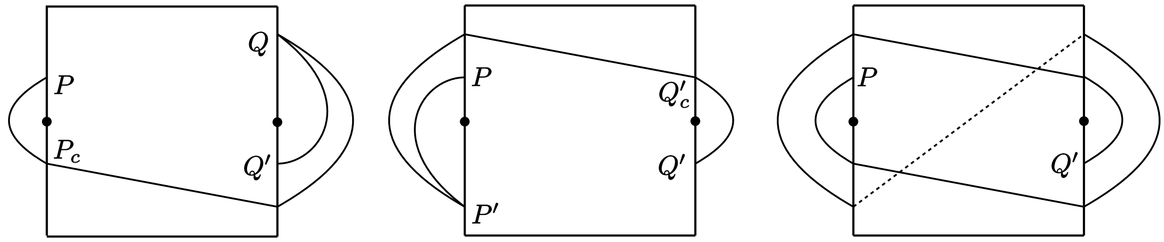

FIGURE 9. The projections of $\mathfrak{a}_2, \mathfrak{a}_3$ and $\mathfrak{a}_4$ onto the x_1x_3-plane

$\gamma_{\rm in}(P_cQ')$, $\gamma_{\rm in}(PQ'_c)$, $\gamma_{\rm in}(P_cQ'_c)$ are respectively the shortest arcs on S connecting P_c and Q', P and Q'_c, P_c and Q'_c, whose interiors are slightly pushed inside the interior of B_1 (to avoid intersections with other arcs on S) and whose framings are given by vectors normal to S.

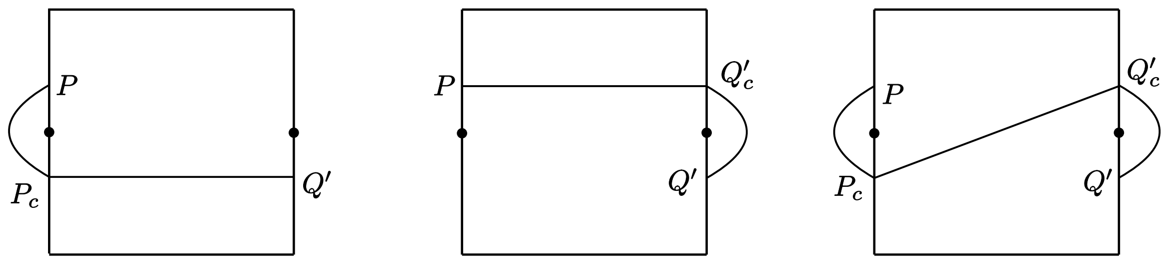

FIGURE 10. The projection of $\mathfrak{a}'_2, \mathfrak{a}'_3$ and $\mathfrak{a}'_4$ onto the x_1x_3-plane

Using the skein relations one can simplify the arc part of elements in $\mathcal{S}(X_1; P, Q')$, showing that the arc part is one of the four $\mathfrak{a}'_i, i = 1, 2, 3, 4$. More precisely, by similar arguments as in the proof of [BL, Lemma 3.1] one can show that the relative skein $\mathcal{S}(X_1; P, Q')$ is equal to the union $\cup_{i=1}^4(\mathfrak{a}'_i \star \mathcal{S}(X_1))$.

It is easy to see that $\mathfrak{a}_i$ is isotopic to $\mathfrak{a}'_i$ for all $1 \leq i \leq 3$. Use the skein relations to resolve all the crossings of $\mathfrak{a}_4$ one can easily show that the set $\mathfrak{a}_4 \star \mathcal{S}(X_1)$ is equal to $\mathfrak{a}'_4 \star \mathcal{S}(X_1)$, modulo the union $\cup_{i=1}^3(\mathfrak{a}'_i \star \mathcal{S}(X_1))$. The lemma follows. □

1.4. **From $\mathcal{S}(X_1)$ to $\mathcal{S}(X)$ through sliding.** Recall that X is obtained from X_1 by attaching a 2-handle along the curve ω. Note that $\mathcal{S}(X_1) = \mathcal{S}(D_{**})$ is isomorphic to the commutative algebra $\mathcal{R}[x, x', y]$, see [Pr]. The embedding of X_1 into X gives rise to a linear map from $\mathcal{S}(X_1) \equiv \mathcal{R}[x, x', y]$ to $\mathcal{S}(X)$. It is known that the map is surjective, and its kernel $\mathcal{K}$, see [Pr, BL], can be described through slides as follows.

Suppose $\mathfrak{a}$ is a type 1 tangle whose 2 endpoints are on ω such that outside a small neighborhood of the 2 endpoints $\mathfrak{a}$ is in the interior of X_1 and in a small neighborhood of the endpoints $\mathfrak{a}$ is on the boundary $S = \partial B_1$. The two end points of $\mathfrak{a}$ divide ω into 2 arcs ω_1 and ω_2. The loop ω partitions S, which is a sphere, into 2 parts; the one not containing U, U' is called the *outside one*. Let us isotope $\mathfrak{a}$ (relatively to the endpoints) to $\mathfrak{a}'$ so that in a small neighborhood of the endpoints, $\mathfrak{a}'$ is in the outside part of ω.

Let $sl(\mathfrak{a})$ be $\mathfrak{a}' \cdot \omega_1 - \mathfrak{a}' \cdot \omega_2$, considered as an element of the skein module $\mathcal{S}(X_1)$. Here $\mathfrak{a}' \cdot \omega_1$ is the framed link obtained by combining $\mathfrak{a}'$ and ω_1. Note that $sl(\mathfrak{a})$ is defined up to a factor $\pm t^{3n}, n \in \mathbb{Z}$. The exchange $\omega_1 \leftrightarrow \omega_2$ changes the sign, and isotopies in neighborhoods of the endpoints change the framing, which results in a factor equal to a power of $(-t^3)$.

It is clear that as framed links in X, $\mathfrak{a}' \cdot \omega_1$ is isotopic to $\mathfrak{a}' \cdot \omega_2$, since one is obtained from the other by sliding over the 2-handle attached to the curve ω. Hence we always have $sl(\mathfrak{a}) \in \mathcal{K}$. It was known that $\mathcal{K}$ is spanned by all possible $sl(\mathfrak{a})$, where $\mathfrak{a}$ can be chosen among all type 1 tangles with pre-given two endpoints on ω.

From the description of $\mathcal{S}(X_1; P, Q')$ in Lemma 1.2 we have

Lemma 1.3. *The kernel $\mathcal{K}$ is equal to the $\mathbb{C}[t^{\pm 1}]$-span of $\{sl(\mathfrak{a}_i) \star \mathcal{S}(X_1), i = 1, 2, 3, 4\}$.*

Lemma 1.4. *One has*

$$\begin{aligned} sl(\mathfrak{a}_1) &= sl(\gamma_{\rm in}(PQ')), \\ sl(\mathfrak{a}_2) &= sl(\mathfrak{d}_{\rm in}(PP')), \\ sl(\mathfrak{a}_3) &= sl(\mathfrak{d}_{\rm in}(QQ')), \\ sl(\mathfrak{a}_4) &= sl(\gamma_{\rm in}(P'Q)). \end{aligned}$$

Proof. The first identity is a tautology. The last three follows from trivially a simple isotopy of the links involved. □

Lemma 1.5. *For every $\ell \in \mathcal{S}(X_1)$, one has $\psi(\ell) = \ell$.*

Proof. This is because x, x' and y are invariant under the rotation ψ. □

Lemma 1.6. *One has*

$$\begin{aligned} sl(\gamma_{\rm in}(PQ')) \star \mathcal{S}(X_1) &= sl(\gamma_{\rm in}(P'Q)) \star \mathcal{S}(X_1), \\ sl(\mathfrak{d}_{\rm in}(PP')) \star \mathcal{S}(X_1) &= 0, \\ sl(\mathfrak{d}_{\rm in}(QQ')) \star \mathcal{S}(X_1) &= 0. \end{aligned}$$

Proof. Since $\psi(P) = P'$ and $\psi(Q) = Q'$, we have $\psi(sl(\gamma_{\rm in}(PQ'))) = sl(\gamma_{\rm in}(P'Q))$. Hence $sl(\gamma_{\rm in}(PQ')) \star \mathcal{S}(X_1) = sl(\gamma_{\rm in}(P'Q)) \star \mathcal{S}(X_1)$ by Lemma 1.5.

Since both $\mathfrak{d}_{\rm in}(PP')$ and ω is invariant under ψ, we have $\psi(\mathfrak{d}_{\rm in}(PP') \cdot \omega_1(P, P')) = \mathfrak{d}_{\rm in}(PP') \cdot \omega_2(P, P')$, where $\omega_1(P, P')$ and $\omega_2(P, P')$ are the two arcs of ω obtained by dividing ω using the two points P, P'. It implies that

$$sl(\mathfrak{d}_{\rm in}(PP')) \star \mathcal{S}(X_1) = (\mathfrak{d}_{\rm in}(PP') \cdot \omega_1(P, P') - \mathfrak{d}_{\rm in}(PP') \cdot \omega_2(P, P')) \star \mathcal{S}(X_1) = 0.$$

This completes the proof of the lemma. □

1.5. **Proof of Theorem 1.** Let $\mathcal{R} = \mathbb{C}[t^{\pm 1}]$. We have $\mathcal{S}(X) = \mathcal{R}[x, x', y]/\mathcal{K}$, where $\mathcal{K}$ is the $\mathcal{R}$-span of $sl(\mathfrak{a}_1) \star \mathcal{R}[x, x', y]$, by Lemmas 1.3, 1.4 and 1.6. Note that there is a natural $\mathcal{R}[x, x']$-module structure on $\mathcal{S}(X)$: Here x, x' are meridians, thus belong to the boundary of X. Over $\mathcal{R}[x, x']$, $\mathcal{R}[x, x', y]$ is spanned by $1, y, y^2, \dots$. Hence $\mathcal{K}$, as an $\mathcal{R}[x, x']$-module, is spanned by $sl(\mathfrak{a}_1) \star y^k = (\mathfrak{a}_1 \cdot \omega_1 - \mathfrak{a}_1 \cdot \omega_2) \star y^k, k = 0, 1, 2, \dots$.

Note that $\mathfrak{a}_1 \cdot \omega_1$ is the closure in the sense of [Le, Section 1.5] of a braid on $(2p + 2)$ strands, while $\mathfrak{a}_1 \cdot \omega_2$ is the closure of a braid on $(2p - 2)$ strands. Moreover, $(\mathfrak{a}_1 \cdot \omega_1) \star y^k$ is the closure of a braid on $(2p + 2) + 2k$ strands, while $(\mathfrak{a}_1 \cdot \omega_2) \star y^k$ is the closure of of a braid on $(2p - 2) + 2k$ strands. Lemma 1.1 in [Le] then shows that $(\mathfrak{a}_1 \cdot \omega_1 - \mathfrak{a}_1 \cdot \omega_2) \star y^k$, as an element of $\mathcal{R}[x,x',y]$, has y-degree $(p+1)+k$, with highest coefficient invertible and of the form a power of t. Hence when we factor out $\mathcal{R}[x,x',y]$ by $\mathcal{K}$, we get a free $\mathcal{R}[x,x']$-module with representatives $y^l, l=0,1,2,\dots,p$ as a basis.

1.6. **Proof of Proposition 1.** From Theorem 1 it follows that $\varepsilon(\mathcal{S}(X))$ is the quotient of the ring $\mathbb{C}[\bar{x},\bar{x'},\bar{y}]$ by the ideal I generated by $\varepsilon(sl(\mathfrak{a}_1)) \star \mathbb{C}[\bar{x},\bar{x'},\bar{y}]$, where $\bar{z}$ denotes the negative of the trace of the loop z.

Note that $\varepsilon(\mathcal{S}(X))$ has a natural $\mathbb{C}$-algebra structure and $\star$ is just the multiplication of this algebra. It implies that I can be generated by only one element which is $\varepsilon(sl(\mathfrak{a}_1))$. Hence $\varepsilon(\mathcal{S}(X)) = \mathbb{C}[\bar{x},\bar{x'},\bar{y}]/(\varphi)$, where $\varphi = \varepsilon(sl(\mathfrak{a}_1)) \in \mathbb{C}[\bar{x},\bar{x'},\bar{y}]$ is a polynomial with $\bar{y}$-degree $p+1$. Moreover, the coefficient of $\bar{y}^{p+1}$ in φ is ± 1.

We claim that φ has no repeated factors. Since φ is a polynomial of $\bar{y}$-degree $p+1$ with leading coefficient ±1, it suffices to show that $\varphi(0,0,\bar{y})$ has no repeated factors.

Lemma 1.7. *One has*

$$\varphi(0,0,\bar{y}) = \pm(\bar{y}^2-4)S_{p-1}(\bar{y}),$$

where $S_n(\bar{y})$ are the Chebyshev polynomials defined by $S_0(\bar{y})=1, S_1(\bar{y})=\bar{y}$ and $S_{n+1}(\bar{y}) = \bar{y}S_n(\bar{y}) - S_{n-1}(\bar{y})$ for all integer n.

Proof. By [BZ], the fundamental group of the two-bridge link $L = \mathfrak{b}(2p,q)$ is

$$\pi_1(L) = \langle \tilde{x}, \tilde{x'} \mid \tilde{x}w = w\tilde{x} \rangle,$$

where $w = (\tilde{x'})^{\varepsilon_1}(\tilde{x})^{\varepsilon_2}\cdots(\tilde{x})^{\varepsilon_{2p-2}}(\tilde{x'})^{\varepsilon_{2p-1}}$ and $\varepsilon_k = (-1)^{\lfloor \frac{kq}{2p} \rfloor}$. Here $\tilde{x}, \tilde{x'}$ are meridians of the link L, and are conjugate to x, x' respectively.

The character variety of the free group in 2 letters $\tilde{x}$ and $\tilde{x'}$ is isomorphic to $\mathbb{C}^3$, by the Fricke-Klein-Vogt theorem. For every word z, the trace of z is a polynomial in 3 variables $\operatorname{tr}\tilde{x} = -\bar{x}, \operatorname{tr}\tilde{x'} = -\bar{x'}$ and $\operatorname{tr}(\tilde{x}\tilde{x'}) = -\bar{y}$.

Note that the traces of the words $(\tilde{x})^{-1}w\tilde{x}(\tilde{x'})^{-1}$ and $w(\tilde{x'})^{-1}$ are equal. Hence

$$\eta = \operatorname{tr}\left((\tilde{x})^{-1}w\tilde{x}(\tilde{x'})^{-1} - w(\tilde{x'})^{-1}\right)$$

is divisible by φ in $\mathbb{C}[\bar{x},\bar{x'},\bar{y}]$.

Suppose from now on $\bar{x} = \bar{x'} = 0$. We have $(\tilde{x})^{-1} + \tilde{x} = \operatorname{tr}\tilde{x} = -\bar{x} = 0$, i.e. $(\tilde{x})^{-1} = -\tilde{x}$, by the Cayley-Hamilton theorem applying for matrices in $SL_2(\mathbb{C})$. Here we identify $\tilde{x}$ with its representation matrix in $SL_2(\mathbb{C})$. Similarly, $(\tilde{x'})^{-1} = -\tilde{x'}$.

Let k be the the number of times the power -1 appears in the word $w(\tilde{x'})^{-1}$. Then, since $\varepsilon_{2p-1} = 1$, the number of times the power -1 appears in the word $(\tilde{x})^{-1}w\tilde{x}(\tilde{x'})^{-\varepsilon_{2p-1}}$ is $k+2$. If we replace $(\tilde{x})^{-1}$ and $(\tilde{x'})^{-1}$ in $w(\tilde{x'})^{-1}$ by $\tilde{x}$ and $\tilde{x'}$ respectively then we pick up the sign $(-1)^k$, i.e. we have $w(\tilde{x'})^{-1} = (-1)^k(\tilde{x'}\tilde{x})^{p-1}$. Similarly, $(\tilde{x})^{-1}w\tilde{x}(\tilde{x'})^{-1} = (-1)^{k+2}(\tilde{x}\tilde{x'})^{p+1}$. It implies that $\eta(0,0,\bar{y}) = (-1)^k \operatorname{tr}\left((\tilde{x}\tilde{x'})^{p+1} - (\tilde{x'}\tilde{x})^{p-1}\right)$.

Let $\delta_p = \operatorname{tr}\left((\tilde{x}\tilde{x'})^{p+1} - (\tilde{x'}\tilde{x})^{p-1}\right)$. By the Cayley-Hamiton, $\tilde{x}\tilde{x'} + (\tilde{x}\tilde{x'})^{-1} = \operatorname{tr}(\tilde{x}\tilde{x'}) = -\bar{y}$. This implies that $\delta_{n+1} = -\bar{y}\delta_n - \delta_{n-1}$. It is easy to check that $\delta_1 = \bar{y}^2-4$, $\delta_2 = -(\bar{y}^2-4)\bar{y}$. Hence $\delta_p = (-1)^{p-1}(\bar{y}^2-4)S_{p-1}(\bar{y})$ where $S_n(\bar{y})$ are the Chebyshev polynomials defined by $S_0(\bar{y}) = 1, S_1(\bar{y}) = \bar{y}$ and $S_{n+1}(\bar{y}) = \bar{y}S_n(\bar{y}) - S_{n-1}(\bar{y})$ for all integer n.

We have $\eta(0,0,\bar{y}) = (-1)^k\delta_p = (-1)^{k+p-1}(\bar{y}^2-4)S_{p-1}(\bar{y})$, which is a polynomial of degree $p+1$ in $\bar{y}$ with leading coefficient $(-1)^{k+p-1}$. Since η is divisible by φ, and φ is also a polynomial of $\bar{y}$-degree $p+1$ with leading coefficient ±1, we must have $\varphi(0,0,\bar{y}) = \pm(\bar{y}^2-4)S_{p-1}(\bar{y})$ as desired. ∎

It is known that $S_{p-1}(\bar{y}) = \prod_{j=1}^{p-1}(\bar{y}-2\cos\frac{\pi j}{p})$ and hence $(\bar{y}^2-4)S_{p-1}(\bar{y})$ has no repeated factors. By Proposition 1.7, it follows that φ has no repeated factors either. Hence the nil-radical of $\varepsilon(\mathcal{S}(X))$ is zero, which means that $\varepsilon(\mathcal{S}(X))$ is exactly equal to $\mathbb{C}[\chi(\pi_1(X))]$. This completes the proof of Proposition 1.

Corollary 1.8. *The character ring of the two-bridge link* $\mathfrak{b}(2p,q)$ *is the quotient of the ring* $\mathbb{C}[\bar{x},\bar{x'},\bar{y}]$ *by the ideal generated by the polynomial* $\eta = \mathrm{tr}((\tilde{x})^{-1}w\tilde{x}(\tilde{x'})^{-1}) - \mathrm{tr}(w(\tilde{x'})^{-1})$, *where* $\bar{x}$, $\bar{x'}$, $\bar{y}$, $\tilde{x}$, $\tilde{x'}$ *and* w *are defined as in the proof of Lemma 1.7.*

Proof. We still use the notations in the proof of Lemma 1.7.

Since $w = (\tilde{x'})^{\varepsilon_1}(\tilde{x})^{\varepsilon_2}\cdots(\tilde{x})^{\varepsilon_{2p-2}}(\tilde{x'})^{\varepsilon_{2p-1}}$ and $\varepsilon_k = (-1)^{\lfloor\frac{kq}{2p}\rfloor} = \pm1$, it is easy to show that the traces of the words $(\tilde{x})^{-1}w\tilde{x}(\tilde{x'})^{-1}$ and $w(\tilde{x'})^{-1}$ have $\bar{y}$-degrees equal to $p+1$ and $p-1$ respectively, with leading coefficients ±1. It implies that the polynomial η has $\bar{y}$-degree $p+1$ with leading coefficient ±1. Since η is divisible by φ, we must have $\eta = \pm\varphi$. Hence, by Proposition 1, the character ring of $\mathfrak{b}(2p,q)$ is equal to the quotient of the ring $\mathbb{C}[\bar{x},\bar{x'},\bar{y}]$ by the ideal generated by the polynomial $\eta = \mathrm{tr}((\tilde{x})^{-1}w\tilde{x}(\tilde{x'})^{-1}) - \mathrm{tr}(w(\tilde{x'})^{-1})$. □

Remark 1.9. Corollary 1.8 was already obtained in [Ri] although it was not completely written in form of traces. The proof we present here essentially follows directly from Theorem 1.

One can easily show that the characters of abelian representations (into $SL_2(\mathbb{C})$) of the two-bridge link $\mathfrak{b}(2p,q)$ is determined by the polynomial

$$\eta_{ab} = \mathrm{tr}(\tilde{x}\tilde{x'}(\tilde{x})^{-1}(\tilde{x'})^{-1}) = \bar{y}^2 + \bar{x}^2 + \bar{x'}^2 - \bar{y}\bar{x}\bar{x'} - 4.$$

Hence, by Corollary 1.8, the characters of non-abelian representations of $\mathfrak{b}(2p,q)$ is determined by the polynomial $\eta_{nab} = \eta/\eta_{ab}$. The polynomial η_{nab} has $\bar{y}$-degree $p-1$ with leading coefficient ±1. After a suitable change of variables, it is exactly the polynomial $\Phi_{\pi L}$ in [Ri, Lemma 2], up to ±1.

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