CHARACTER VARIETIES, A-POLYNOMIALS, AND THE AJ CONJECTURE

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Abstract. We establish some facts about the behavior of the rational-geometric subvariety of the $SL_2(\mathbb{C})$ or $PSL_2(\mathbb{C})$ character variety of a hyperbolic knot manifold under the restriction map to the $SL_2(\mathbb{C})$ or $PSL_2(\mathbb{C})$ character variety of the boundary torus, and use the results to get some properties about the A-polynomials and to prove the AJ conjecture for certain class of knots in S^3 including in particular any 2-bridge knot over which the double branched cover of S^3 is a lens space of prime order.

1. Introduction

For a finitely generated group Γ, let $R(\Gamma)$ denote the $SL_2(\mathbb{C})$-representation variety of Γ, $X(\Gamma)$ the $SL_2(\mathbb{C})$-character variety of Γ, and $\mathrm{tr} : R(\Gamma) \to X(\Gamma)$ the map which sends a representation $\rho \in R(\Gamma)$ to its character $\chi_\rho \in X(\Gamma)$. When Γ is the fundamental group of a connected manifold W, we also write $R(W)$, $X(W)$ for $R(\pi_1(W))$, $X(\pi_1(W))$ respectively and call them the $SL_2(\mathbb{C})$-representation variety of W and the $SL_2(\mathbb{C})$-character variety of W. The counterparts of these notions when the target group $SL_2(\mathbb{C})$ is replaced by $PSL_2(\mathbb{C})$ are similarly defined and are denoted by $\overline{R}(\Gamma)$, $\overline{X}(\Gamma)$, $\overline{\mathrm{tr}}$, $\overline{\rho}$, $\chi_{\overline{\rho}}$, $\overline{R}(W)$, $\overline{X}(W)$ respectively. We refer to [CS] for basics about $SL_2(\mathbb{C})$-representation and character varieties and to [BZ1] in $PSL_2(\mathbb{C})$ case.

In this paper, a *variety* V is a closed complex affine algebraic set, i.e. a subset of $\mathbb{C}^n$ which is the zero locus of a set of polynomials in $\mathbb{C}[x_1, \ldots, x_n]$. If among the sets of polynomials which define the same variety V there is one whose elements all have rational coefficients, we say that V is *defined over* $\mathbb{Q}$. Similarly a regular map between two varieties is said to be *defined over* $\mathbb{Q}$ if the map is given by a tuple of polynomials with coefficients in $\mathbb{Q}$. Note that $R(\Gamma)$, $X(\Gamma)$, tr, $\overline{R}(\Gamma)$, $\overline{X}(\Gamma)$, $\overline{\mathrm{tr}}$ are all defined over $\mathbb{Q}$.

In this paper irreducible varieties will be called $\mathbb{C}$-irreducible varieties. Recall that a variety is $\mathbb{C}$-*irreducible* if it is not a union of two proper subvarieties. Any variety V can be presented as an irredundant union of $\mathbb{C}$-irreducible subvarieties, each is called a $\mathbb{C}$-*component* of V. Similarly, a variety defined over $\mathbb{Q}$ is $\mathbb{Q}$-*irreducible* if it is not a union of two proper subvarieties defined over $\mathbb{Q}$. Any variety V defined over $\mathbb{Q}$ can be presented as an irredundant union of $\mathbb{Q}$-irreducible subvarieties, each is called a $\mathbb{Q}$-*component* of V. In general a $\mathbb{Q}$-component can be further decomposed into $\mathbb{C}$-components.

T. L. is supported in part by National Science Foundation grant DMS-1406419.

2010 *Mathematics Classification:* Primary 57M25.

Key words and phrases: character variety, A-polynomial, AJ conjecture.

If Γ_1 and Γ_2 are two finitely generated groups and $h : \Gamma_1 \to \Gamma_2$ is a group homomorphism, we use h^* to denote the induced regular map from $R(\Gamma_2)$, $X(\Gamma_2)$, $\overline{R}(\Gamma_2)$, or $\overline{X}(\Gamma_2)$ to $R(\Gamma_1)$, $X(\Gamma_1)$, $\overline{R}(\Gamma_1)$ or $\overline{X}(\Gamma_1)$ respectively. Note that h^* is defined over $\mathbb{Q}$.

Let M be a knot manifold, i.e. M is a connected compact orientable 3-manifold whose boundary ∂M is a torus. Let ι^* be the regular map from $R(M)$, $X(M)$, $\overline{R}(M)$ or $\overline{X}(M)$ to $R(\partial M)$, $X(\partial M)$, $\overline{R}(\partial M)$ or $\overline{X}(\partial M)$ respectively, induced from the inclusion induced homomorphism $\iota : \pi_1(\partial M) \to \pi_1(M)$.

We call a character χ_ρ (or $\overline{\chi}_{\overline{\rho}}$) reducible or irreducible or discrete faithful or dihedral if the corresponding representation ρ (or $\overline{\rho}$) has that property.

1.1. **Rational-geometric subvariety.** Suppose M is a hyperbolic knot manifold, i.e. a knot manifold whose interior has a complete hyperbolic metric of finite volume. There are precisely two discrete faithful characters in $\overline{X}(M)$ (which follows from the Mostow-Prasad rigidity) and there are precisely $2|H_1(M;\mathbb{Z}_2)|$ discrete faithful characters in $X(M)$ (which follows from a result of Thurston [CS, Proposition 3.1.1]). The *rational-geometric subvariety* $X^{\rm rg}(M)$ (respectively $\overline{X}^{\rm rg}(M)$) is the union of $\mathbb{Q}$-components of $X(M)$ (respectively $\overline{X}(M)$) each of which contains a discrete faithful character. The number of $\mathbb{Q}$-components of $X^{\rm rg}(M)$ is at most $|H_1(M;\mathbb{Z}_2)|$, and $\overline{X}^{\rm rg}(M)$ is $\mathbb{Q}$-irreducible (which will be explained in Section 2), but it is not known how many $\mathbb{C}$-components that $X^{\rm rg}(M)$ (respectively $\overline{X}^{\rm rg}(M)$) can possibly have.

In this paper we show

Theorem 1.1. *Let M be a hyperbolic knot manifold. Let $\overline{X}_1, ..., \overline{X}_l$ be the $\mathbb{C}$-components of $\overline{X}^{\rm rg}(M)$, and let $\overline{Y}_j$ be the Zariski closure of $\iota^*(\overline{X}_j)$ in $\overline{X}(\partial M)$, $j = 1, ..., l$.*
(1) For each j, $\overline{X}_j$ is a curve.
(2) The regular map $\iota^* : \overline{X}_j \to \overline{Y}_j$ is a birational map for each $j = 1, ..., l$.
(3) If the two discrete faithful characters of $\overline{X}(M)$ are contained in the same $\mathbb{C}$-component of $\overline{X}(M)$, then the curves $\overline{Y}_j$, $j = 1, ..., l$, are mutually distinct in $\overline{X}(\partial M)$.

In $SL_2(\mathbb{C})$-setting we have a similar result but we need some restriction on the knot manifold.

Theorem 1.2. *Suppose that M is a hyperbolic knot manifold which is the exterior of a knot in a homology 3-sphere. Let $X_1, ..., X_k$ be the $\mathbb{C}$-components of $X^{\rm rg}(M)$, and let Y_j be the Zariski closure of $\iota^*(X_j)$ in $X(\partial M)$, $j = 1, ..., k$.*
(1) For each j, X_j is a curve.
(2) The regular map $\iota^* : X_j \to Y_j$ is a birational map for each $j = 1, ..., k$.
(3) If the two discrete faithful characters of $\overline{X}(M)$ are contained in the same $\mathbb{C}$-component of $\overline{X}(M)$, then the curves Y_j, $j = 1, ..., k$, are mutually distinct in $X(\partial M)$.

Remark 1.3. Although the condition "the two discrete faithful characters of $\overline{X}(M)$ are contained in the same $\mathbb{C}$-component of $\overline{X}(M)$" is hard to check, there is no known example of a hyperbolic knot exterior in S^3 for which this condition is not satisfied.

We give two applications of Theorem 1.2, one on estimating degrees of A-polynomials and one on proving the AJ conjecture for a certain class of knots, which is the main motivation of this paper.

1.2. ***A*-polynomial.** When a knot manifold M is the exterior of a knot K in a homology 3-sphere W, we denote the A-polynomial of K in variables $\mathfrak{M}$ and $\mathfrak{L}$ by $A_{\rm K,W}(\mathfrak{M},\mathfrak{L})$, as defined in [CCGLS]. When $W=S^3$, we simply write $A_{\rm K}(\mathfrak{M},\mathfrak{L})$ for $A_{{\rm K},S^3}(\mathfrak{M},\mathfrak{L})$. Note that $A_{\rm K,W}(\mathfrak{M},\mathfrak{L})\in\mathbb{Z}[\mathfrak{M},\mathfrak{L}]$ has no repeated factors and always contains the factor $\mathfrak{L}-1$. Let the *non-abelian A-polynomial* be defined by

$$\widehat{A}_{\rm K,W}(\mathfrak{M},\mathfrak{L}):=\frac{A_{\rm K,W}(\mathfrak{M},\mathfrak{L})}{\mathfrak{L}-1}.$$

We call the maximum power of $\mathfrak{M}$ (respectively of $\mathfrak{L}$) in $\widehat{A}_{\rm K,W}(\mathfrak{M},\mathfrak{L})$ the $\mathfrak{M}$-degree (respectively the $\mathfrak{L}$-degree) of $\widehat{A}_{\rm K,W}(\mathfrak{M},\mathfrak{L})$.

When M is a finite volume hyperbolic 3-manifold, the trace field of M is defined to be the field generated by the values of a discrete faithful character of M over the base field $\mathbb{Q}$. It is known that the trace field of M is a number field, i.e. a finite degree extension of $\mathbb{Q}$, with the extension degree at least two.

Theorem 1.4. *Suppose that M is a hyperbolic knot manifold which is the exterior of a knot K in a homology 3-sphere W. Let d be the extension degree of the trace field of M over $\mathbb{Q}$. If the two discrete faithful characters of $\overline{X}(M)$ are contained in the same $\mathbb{C}$-component of $\overline{X}(M)$, then both the $\mathfrak{M}$-degree and the $\mathfrak{L}$-degree of $\widehat{A}_{K,W}(\mathfrak{M},\mathfrak{L})$ are at least d. In particular both the $\mathfrak{M}$-degree and the $\mathfrak{L}$-degree of $\widehat{A}_{K,W}(\mathfrak{M},\mathfrak{L})$ are at least 2.*

1.3. **AJ conjecture.** Suppose M is the exterior of a knot in a homology 3-sphere. All the reducible characters in $X(M)$ (resp. $\overline{X}(M)$) form a unique $\mathbb{C}$-component of $X(M)$ (resp. $\overline{X}(M)$), which we denote by $X^{\rm red}(M)$ (resp. $\overline{X}^{\rm red}(M)$). We use $X^{\rm irr}(M)$ (resp. $\overline{X}^{\rm irr}(M)$) to denote the union of the rest of the $\mathbb{C}$-components of $X(M)$ (resp. $\overline{X}(M)$). We caution that our definition of $X^{\rm irr}(M)$ (resp. $\overline{X}^{\rm irr}(M)$) may not be the exact complement of $X^{\rm red}(M)$ (resp. $\overline{X}^{\rm red}(M)$) in $X(M)$ (resp. $\overline{X}(M)$) and it may still contain finitely many reducible characters. All $X^{\rm red}(M), X^{\rm irr}(M), \overline{X}^{\rm red}(M), \overline{X}^{\rm irr}(M)$ are varieties defined over $\mathbb{Q}$.

For a knot K in S^3, its recurrence polynomial $\alpha_{\rm K}(t,\mathfrak{M},\mathfrak{L})\in\mathbb{Z}[t,\mathfrak{M},\mathfrak{L}]$ is derived from the colored Jones polynomials of K, see [G, GL, Le2]. The AJ-conjecture raised in [G] (see also [FGL]) anticipates a striking relation between the colored Jones polynomials of K and the A-polynomial of K. It states that for every knot $K\subset S^3$, $\alpha_{\rm K}(1,\mathfrak{M},\mathfrak{L})$ is equal to the A-polynomial $A_{\rm K}(\mathfrak{M},\mathfrak{L})$ of K, up to a factor depending on $\mathfrak{M}$ only. The following theorem generalizes [LT, Theorem 1] and is the main result of this paper (see Section 4 for detailed definitions of terms mentioned here and for more background description).

Theorem 1.5. *Let K be a knot in S^3 whose exterior M is hyperbolic. Suppose the following conditions are satisfied:*
(1) $\overline{X}^{\rm irr}(M)=\overline{X}^{\rm rg}(M)$ and the two discrete faithful characters of $\overline{X}(M)$ are contained in the

same $\mathbb{C}$*-component of* $\overline{X}(M)$*,*
(2) the $\mathfrak{L}$*-degree of the recurrence polynomial* $\alpha_K(t,\mathfrak{M},\mathfrak{L})$ *of* K *is larger than one,*
(3) the localized skein module $\overline{\mathcal{S}}$ *of* M *is finitely generated.*
Then the AJ-conjecture holds for K*.*

In [LT, Theorem 1], it is required that $X^{\rm irr}(M) = X^{\rm rg}(M)$ and both are $\mathbb{C}$-irreducible, which is obviously stronger than our condition (1) of Theorem 1.5. In general, irreducibility over $\mathbb{C}$ is difficult to check. We also remove the condition required in [LT, Theorem 1] that the universal SL_2-character ring of M is reduced.

It was known that condition (2) of Theorem 1.5 is satisfied by any nontrivial adequate knot (in particular any nontrivial alternating knot) in S^3 (see [Le2]) and condition (3) of Theorem 1.5 is satisfied by all 2-bridge knots (see [Le2]) and all pretzel knots of the form $(-2, 3, 2n+1)$ (see [LT]). Concerning condition (1) of Theorem 1.5, we have the following.

Theorem 1.6. *Let* K *be a* 2*-bridge knot in* S^3 *with a hyperbolic exterior* M*.*
(1) The two discrete faithful characters of $\overline{X}(M)$ *are contained in the same* $\mathbb{C}$*-component of* $\overline{X}(M)$*,*
(2) All the four discrete faithful $SL_2(\mathbb{C})$*-characters are contained in the same* $\mathbb{C}$*-component of* $X(M)$*, and* $X^{\rm rg}(M)$ *is irreducible over* $\mathbb{Q}$*.*

Therefore we have the following corollary which generalizes [LT, Theorem 2 (b)].

Corollary 1.7. *Let* K *be a* 2*-bridge knot in* S^3 *with a hyperbolic exterior* M*. If* $X^{\rm irr}(M) = X^{\rm rg}(M)$*, then the AJ-conjecture holds for* K*.*

Note that a two-bridge knot has hyperbolic exterior if and only if it is not a torus knot, and for all torus knots the AJ conjecture is known to hold [Hi, Tr].

Since $X^{\rm rg}(M) \subset X^{\rm irr}(M)$ and $X^{\rm rg}(M)$ is defined over $\mathbb{Q}$, if $X^{\rm irr}(M)$ is $\mathbb{Q}$-irreducible, then $X^{\rm rg}(M) = X^{\rm irr}(M)$. For a two-bridge knot, the variety $X^{\rm irr}(M)$ is the zero locus of the Riley polynomial which is a polynomial in two variable, see [Ri]. Hence, we have the following.

Corollary 1.8. *Let* K *be a* 2*-bridge knot in* S^3*. If* $X^{\rm irr}(M)$ *is* $\mathbb{Q}$*-irreducible, or if the Riley polynomial of* K *is irreducible over* $\mathbb{Q}$*, then the AJ conjecture holds for* K*.*

In [LT, Section A1] it was proved that the Riley polynomial of the two bridge knot $\mathfrak{b}(p,q)$ is $\mathbb{Q}$-irreducible if p is a prime. Here we use the notation of [BuZ] for two bridge knots: $\mathfrak{b}(p,q)$ is the two bridge knot such that the double branched covering of S^3 along $\mathfrak{b}(p,q)$ is the lens space $L(p,q)$. Note that both p,q are odd numbers, co-prime with each other, and $1 \le q \le p-2$, and $\mathfrak{b}(p,q)$ is hyperbolic if and only if $q \neq 1$. When $q = 1$, $\mathfrak{b}(p,1)$ is a torus knot, and the AJ conjecture for it holds. Thus we have

Corollary 1.9. *The AJ conjecture holds for all two bridge knots* $\mathfrak{b}(p,q)$ *with odd prime* p*.*

Among all 544 two-bridge knots $\mathfrak{b}(p,q)$ with $p < 100$, 48 of them are hyperbolic and have $\mathbb{Q}$-reducible Riley polynomial (calculated by Vu Huynh). Here is the list of the 48 knots (their

(p,q)-values):
$(15,11)$, $(21,13)$, $(27,17)$, $(27,5)$, $(33,23)$, $(33,5)$, $(35,29)$, $(39,25)$, $(39,7)$, $(45,29)$, $(45,19)$, $(45,7)$, $(51,35)$, $(55,21)$, $(57,37)$, $(57,5)$, $(63,55)$, $(63,41)$, $(63,11)$, $(63,5)$, $(65,51)$, $(69,47)$, $(69,19)$, $(69,11)$, $(75,59)$, $(75,49)$, $(75,29)$, $(75,13)$, $(77,43)$, $(81,53)$, $(81,13)$, $(81,7)$, $(85,69)$, $(85,47)$, $(85,9)$, $(87,59)$, $(87,7)$, $(87,5)$, $(91,27)$, $(93,61)$, $(93,11)$, $(93,5)$, $(95,39)$, $(95,9)$, $(99,89)$, $(99,65)$, $(99,29)$, $(99,17)$.
Thus the AJ conjecture holds for all 544 two-bridge knots $\mathfrak{b}(p,q)$ with $p<100$, except for the 48 listed above.

K. Murasugi [Mur] conjectured that the $\mathfrak{M}$-degree of $\widehat{A}_K(\mathfrak{M},\mathfrak{L})$ is at least twice the $\mathfrak{L}$-degree of $\widehat{A}_K(\mathfrak{M},\mathfrak{L})$ for any nontrivial knot K in S^3. We prove Murasugi's conjecture for 2-bridge knots in part (1) of the following theorem. Part (2) of the same theorem will follow from a similar argument together with an application of Theorem 1.2.

Theorem 1.10. *Let $K=\mathfrak{b}(p,q)$ be a nontrivial 2-bridge knot.*
(1) The $\mathfrak{M}$-degree of $\widehat{A}_K(\mathfrak{M},\mathfrak{L})$ is at least twice the $\mathfrak{L}$-degree of $\widehat{A}_K(\mathfrak{M},\mathfrak{L})$.
(2) When K is hyperbolic and p is prime, the $\mathfrak{L}$-degree of $\widehat{A}_K(\mathfrak{M},\mathfrak{L})$ is exactly $(p-1)/2$.

Plan of paper. In §2, we prove Theorems 1.1, 1.2 and 1.4. The proof of Theorem 1.1 applies the theory of volumes of representations developed in [H], [CCGLS], [D], [F], plus the consideration of the Aut($\mathbb{C}$)-action on varieties. Theorem 1.2 follows quickly from Theorem 1.1 under the consideration of the $H_1(M;\mathbb{Z}_2)$-action on $X(M)$. Theorem 1.4 follows from Theorem 1.2 together with the fact observed in [SZ] that the $(\mathrm{Aut}(\mathbb{C})\times H_1(M;\mathbb{Z}_2))$-orbit of a discrete faithful $SL_2(\mathbb{C})$-character of $X(M)$, which of course is contained in $X^{\mathrm{rg}}(M)$, contains at least $2d$ elements. §2 also contains some related results, notably Theorem 2.4 which is a refinement of Theorem 1.1, and Proposition 2.7 which gives a property of A-polynomial that will be applied in the proof of Theorem 1.5 in §4 (see also Remark 2.8). To prove our main result Theorem 1.5 we need to first prepare some properties concerning the representation schemes and character schemes of knot manifolds in §3. In §4, we illustrate how the approach of [LT] can be applied to reduce Theorem 1.5 to Proposition 4.1. This proposition will then be proved in §5, where Theorem 1.2 and results from §3 are applied. In last section, we prove Theorems 1.6 and 1.10, applying results from [Ta, Section 5] [BZ3] [BZ2] [K] as well as Theorem 1.2.

Acknowledgements. We would like to thank Stefano Francaviglia, Kunio Murasugi, Tomotada Ohtsuki, Adam Sikora, Anh Tran for helpful discussions, and especially Joan Porti for providing the proof of Proposition 3.4, and Vu Huynh for producing the list of hyperbolic 2-bridge knots $\mathfrak{b}(p,q)$ with $p<100$ and with $\mathbb{Q}$-reducible Riley polynomial.

2. PROOFS OF THEOREMS 1.1, 1.2 AND 1.4

2.1. **Preliminaries.** Let Aut($\mathbb{C}$) denote the group of all field automorphisms of the complex field $\mathbb{C}$. Let $\tau\in\mathrm{Aut}(\mathbb{C})$ denote the complex conjugation.

Each element $\phi\in\mathrm{Aut}(\mathbb{C})$ extends to a unique ring automorphism of the ring $\mathbb{C}[x_1,\dots,x_n]$ by $\phi(x_i)=x_i$ for $i=1,\dots,n$. Each element $\phi\in\mathrm{Aut}(\mathbb{C})$ acts naturally on the complex affine space $\mathbb{C}^n$ coordinate-wise by

$$\phi(a_1, ..., a_n) := (\phi(a_1), ..., \phi(a_n)).$$

As a ring automorphism, ϕ maps an ideal of $\mathbb{C}[x_1, ..., x_n]$ to an ideal, a primary ideal to a primary ideal and a prime ideal to a prime ideal. If $I \subset \mathbb{C}[x_1, ..., x_n]$ is an ideal defined over $\mathbb{Q}$, i.e. I is generated by elements in $\mathbb{Z}[x_1, ..., x_n]$, then $\phi(I) = I$. If $V(I) \subset \mathbb{C}^n$ is the zero locus defined by an ideal $I \subset \mathbb{C}[x_1, ..., x_n]$, then $\phi(V(I)) = V(\phi(I))$. We call $\phi(V(I))$ a Galois conjugate of $V(I)$. As a map from $\mathbb{C}^n$ to itself, ϕ maps a variety to a variety, an irreducible variety to an irreducible variety preserving its dimension. Furthermore if V is a variety defined over $\mathbb{Q}$, then for any $\mathbb{C}$-component V_1 of V, the $\mathrm{Aut}(\mathbb{C})$-orbit of V_1 is the $\mathbb{Q}$-component of V containing V_1. In particular if V is $\mathbb{Q}$-irreducible, then its $\mathbb{C}$-components are the $\mathrm{Aut}(\mathbb{C})$-orbit of one of them and thus all have the same dimension. (cf. [BZ2, Section 5]).

A variety is *1-equidimensional* if every its $\mathbb{C}$-component has dimension 1. A rational map $f : V_1 \to V_2$ between two 1-equidimensional varieties is said to have degree d if there is an open dense subset $V_2' \subset V_2$ such that $f^{-1}(V_2')$ is dense in V_1 and $f^{-1}(x)$ has exactly d elements for each $x \in V_2'$. When V_1, V_2 are $\mathbb{C}$-irreducible, this definition is the same as the well-known definition of a degree d map in algebraic geometry [S]. It is known that a rational map between two $\mathbb{C}$-irreducible varieties is birational if and only if it has degree 1.

2.2. **Proof of Theorem 1.1.** By Mostow-Prasad rigidity, $\overline{X}(M)$ has two discrete faithful characters, which are related by the τ-action. Let $\overline{\chi}_0$ be one of the two discrete faithful characters, then $\tau(\overline{\chi}_0)$ is the other one. By [P, Corollary 3.28] (which is also valid in $PSL_2(\mathbb{C})$-setting), each of $\overline{\chi}_0$ and $\tau(\overline{\chi}_0)$ is a smooth point of $\overline{X}(M)$. In particular each of them is contained in a unique $\mathbb{C}$-component of $\overline{X}(M)$, which has dimension 1 (a curve) by a result of Thurston (see [CGLS, Proposition 1.1.1]).

We may assume that $\overline{X}_1$ is the $\mathbb{C}$-component of $\overline{X}(M)$ which contains $\overline{\chi}_0$. It follows obviously that $\overline{X}^{\mathrm{rg}}(M)$ (whose definition is given in Section 1) is the $\mathrm{Aut}(\mathbb{C})$-orbit of $\overline{X}_1$, and thus is irreducible over $\mathbb{Q}$. Furthermore each $\mathbb{C}$-component of $\overline{X}^{\mathrm{rg}}(M)$ is a curve. Hence we have proved part (1) of Theorem 1.1

By [D, Theorem 3.1], $\iota^* : \overline{X}_1 \to \overline{Y}_1$ is a birational map. For each $j = 2, ..., l$, there is $\phi_j \in \mathrm{Aut}(\mathbb{C})$ such that $\overline{X}_j = \phi_j(\overline{X}_1)$. Since ι^* is defined over $\mathbb{Q}$, we have the following commutative diagram of maps:

$$\begin{tikzcd}
\overline{X}_1 \arrow[r, "\iota^*"] \arrow[d, "\phi_j"'] & \overline{Y}_1 \arrow[d, "\phi_j"] \\
\overline{X}_j \arrow[r, "\iota^*"] & \overline{Y}_j .
\end{tikzcd}$$

As ϕ_j is a bijection and $\iota^* : \overline{X}_1 \to \overline{Y}_1$ is a degree one map, $\iota^* : \overline{X}_j \to \overline{Y}_j$ is a degree one map and thus is a birational map for each j. This proves part (2) of Theorem 1.1.

Now we proceed to prove part (3) of Theorem 1.1. By our assumption, both $\overline{\chi}_0$ and $\tau(\overline{\chi}_0)$ are contained in $\overline{X}_1$.

Proposition 2.1. *For each $j = 2, ..., l$, $\overline{Y}_j$ and $\overline{Y}_1$ are two distinct curves.*

Proof. Suppose otherwise that $\overline{Y}_1 = \overline{Y}_j$ for some $j \geq 2$. We will get a contradiction from this assumption. The argument goes by applying the theory of volumes of representations.

We first recall some of the results from [D] concerning volumes of representations. For any (connected) closed 3-manifold W and any representation $\overline{\rho} \in \overline{R}(W)$, the volume $v(\overline{\rho})$ of $\overline{\rho}$ is defined, and if in addition $\overline{\rho}$ is irreducible, the volume function v descends down to be defined on $\overline{\chi}_{\overline{\rho}}$ so that $v(\overline{\chi}_{\overline{\rho}}) = v(\overline{\rho})$. What's important in this theory is the Gromov-Thurston-Goldman Volume Rigidity (proved in [D] as Theorem 6.1), which states that when W is a closed hyperbolic 3-manifold and $\overline{\chi} \in \overline{X}(W)$ is an irreducible character, then $|v(\overline{\chi})| = vol(W)$ iff $\overline{\chi}$ is a discrete faithful character. For a hyperbolic knot manifold M, the volume function v is well defined, in our current notation, for each $PSL_2(\mathbb{C})$-representation $\overline{\rho}$ of $\pi_1(M)$ whose character $\overline{\chi}_{\overline{\rho}}$ lies in $\overline{X}_1$ ([D, Lemma 2.5.2]). Similarly if $\overline{\chi}_{\overline{\rho}} \in \overline{X}_1$ is an irreducible character, then $v(\overline{\chi}_{\overline{\rho}}) = v(\overline{\rho})$. So v is defined at all but finitely many points of $\overline{X}_1$. Furthermore if $\overline{Y}_1^\nu$ is a normalization of $\overline{Y}_1$ and $f_1 : \overline{Y}_1 \rightarrow \overline{Y}_1^\nu$ a birational map, then the volume function v factors through $\overline{Y}_1^\nu$ in the sense that there is a function $v_1 : \overline{Y}_1^\nu \rightarrow \mathbb{R}$ such that if $\overline{\chi}_{\overline{\rho}} \in \overline{X}_1$ is an irreducible character and if f_1 is defined at $\iota^*(\overline{\chi}_{\overline{\rho}})$, then

$$v(\overline{\chi}_{\overline{\rho}}) = v_1(f_1(\iota^*(\overline{\chi}_{\overline{\rho}}))).$$

That is, we have the following commutative diagram of maps (at points where all maps are defined):

$$\begin{tikzcd}
 & & \overline{Y}_1^\nu \arrow[dll, "v_1"'] \\
\mathbb{R} & \overline{X}_1 \arrow[l, "v"'] \arrow[r, "\iota^*"] & \overline{Y}_1. \arrow[u, "f_1"]
\end{tikzcd}$$

This is [D, Theorem 2.6]. Moreover if $\overline{\chi}_{\overline{\rho}} \in \overline{X}_1$ is an irreducible character such that $\bar{\rho}$ factors through the fundamental group of a Dehn filling $M(\gamma)$ of M for some slope γ on ∂M, then the volume of $\overline{\rho}$ with respect to M is equal to the volume of $\overline{\rho}$ with respect to the closed manifold $M(\gamma)$ ([D, Lemma 2.5.4]). We note that in the above cited results of [D] the volume $v(\bar{\rho})$ is the absolute value of the integral over M (or $M(\gamma)$) of certain 3-form associated to $\bar{\rho}$ but all these results remain valid when $v(\bar{\rho})$ is defined to be the mentioned integral without taking the absolute value. It is this latter version of volume function that we are using here and subsequently.

In [F], [D, Lemma 2.5.2] is generalized and it is shown there that the volume function v is well defined at every $PSL_2(\mathbb{C})$-representation of a finite volume hyperbolic 3-manifold, and also in [F] the volume rigidity is extended to all hyperbolic link manifolds, which states that the volume of a representation of a hyperbolic link manifold attains its maximal value in absolute value precisely when the representation is discrete faithful and the maximal value in absolute value is the volume of the hyperbolic link manifold. That means that in our current case, the volume function v is defined at any irreducible character of $\overline{X}(M)$ without the restriction that the character lies in a $\mathbb{C}$-component of $\overline{X}(M)$ which contains a discrete faithful character. We should also note that the definition of the volume of a representation defined in [F] is consistent with that defined in [D] in case of a knot manifold. More specifically for a knot manifold M and a representation $\bar{\rho} \in \overline{R}(M)$, the volume $vol(\bar{\rho})$ of $\bar{\rho}$ is defined through a so called pseudodeveloping map for $\bar{\rho}$ which is defined in [D] and the independence of $vol(\bar{\rho})$ from the choice of the pseudodeveloping map is proved in [D] when χ_ρ is contained in a $\mathbb{C}$-component of $\overline{X}(M)$ which contains a discrete faithful character and proved in [F] without any restriction. One can then check that the results of [D] which we recalled in the preceding paragraph can be extended to the following theorem.

Theorem 2.2. *(1) If $\overline{Y}_j^\nu$ is a normalization of $\overline{Y}_j$ and $f_j : \overline{Y}_j \to \overline{Y}_j^\nu$ is a birational map, then there is a function $v_j : \overline{Y}_j^\nu \to \mathbb{R}$ which makes the following diagram of maps commutes (at points where all the maps are defined):*

$$\begin{tikzcd}
 & & \overline{Y}_j^\nu \arrow[dll, "v_j"'] \\
\mathbb{R} & \overline{X}_j \arrow[l, "v"'] \arrow[r, "\iota^*"] & \overline{Y}_j \arrow[u, "f_j"]
\end{tikzcd}$$

(2) If $\overline{\chi} \in \overline{X}(M)$ is an irreducible character which factors through a Dehn filling $M(\gamma)$, i.e. $\overline{\chi} \in \overline{X}(M(\gamma))$, then the volume of $\overline{\chi}$ with respect to M is the same volume with respect to $M(\gamma)$. If in addition that $M(\gamma)$ is hyperbolic, then $|v(\overline{\chi})| = vol(M(\gamma))$ iff $\overline{\chi}$ is a discrete faithful character of $M(\gamma)$.

(3) For any irreducible character $\overline{\chi} \in \overline{X}(M)$, $|v(\overline{\chi})| \le vol(M)$, and the equality holds exactly at the two discrete faithful characters of M.

Remark 2.3. For the proof of part (1) of the theorem, following that of [D, Theorem 2.6], one needs the property that the curve $\overline{X}_j \subset \overline{X}(M)$ lifts to a curve in $X(M)$. But that follows from the fact that $\overline{X}_1$ lifts (by Thurston) to a curve in $X(M)$, say X_1, and then $\phi_j(X_1)$ is a lift of $\overline{X}_j = \phi_j(\overline{X}_1)$.

We now continue to prove Proposition 2.1. Take a sequence of distinct slopes $\{\gamma_k\}$ in ∂M, and let $M(\gamma_k)$ be the closed 3-manifold obtained by Dehn filling M with the slope γ_k. By Thurston's hyperbolic Dehn filling Theorem, we may assume that $M(\gamma_k)$ is hyperbolic and that the core circle of the filling solid torus is a geodesic, for each k. Note that $\overline{X}(M(\gamma_k)) \subset \overline{X}(M)$ for each k. Also for each k, $\overline{X}(M(\gamma_k))$ contains precisely two discrete faithful characters, which we denote by $\overline{\chi}_k$ and $\tau(\overline{\chi}_k)$. Again by Thurston's hyperbolic Dehn filling theorem, we may assume that $\overline{\chi}_k \to \overline{\chi}_0$ in $\overline{X}(M)$ with respect to the classical topology of $\overline{X}(M)$, up to replacing $\overline{\chi}_k$ by $\tau(\overline{\chi}_k)$ for some k's. It follows that $\overline{\chi}_k$ is contained in $\overline{X}_1$ for all sufficiently large k.

Note that for any irreducible character $\overline{\chi} \in \overline{X}(M)$, $v(\overline{\chi}) = -v(\tau(\overline{\chi}))$ (see, e.g. [F, Proposition 4.16]). Without loss of generality we may assume that $v(\overline{\chi}_0) = vol(M) > 0$ and so $v(\tau(\overline{\chi}_0)) = -vol(M)$. It follows that $v(\overline{\chi}_k) = vol(M(\gamma_k)) > 0$ and $v(\tau(\overline{\chi}_k)) = -vol(M(\gamma_k)) < 0$, at least for all sufficiently large k.

As $\overline{\chi}_0$ and $\tau(\overline{\chi}_0)$ are smooth points of $\overline{X}(M)$, it follows that $\tau(\overline{X}_1) = \overline{X}_1$ and that $\tau(\overline{\chi}_k)$ approaches $\tau(\overline{\chi}_0)$ as $k \to \infty$ since τ is a continuous map. Therefore $\overline{X}_1$ is the only $\mathbb{C}$-component of $\overline{X}(M)$ which contains $\overline{\chi}_k$ and $\tau(\overline{\chi}_k)$ for all sufficiently large k.

Since $\overline{Y}_1 = \overline{Y}_j$ and the map $\iota^* : \overline{X}_j \to \overline{Y}_1$ is an almost onto map, there are two sequences of points $\{\overline{\chi}'_k\}$ and $\{\overline{\chi}''_k\}$ in $\overline{X}_j$ such that $\iota^*(\overline{\chi}'_k) = \iota^*(\overline{\chi}_k)$ and $\iota^*(\overline{\chi}''_k) = \iota^*(\tau(\overline{\chi}_k))$ for almost all k. We may also assume that $\overline{\chi}'_k$ and $\overline{\chi}''_k$ are irreducible for almost all k since there are at most finitely many reducible characters in $\overline{X}_j$.

Let $\overline{Y}^{\nu}_1$ be a normalization of $\overline{Y}_1$, and let $f_1 : \overline{Y}_1 \to \overline{Y}^{\nu}_1$ be a birational map. As f_1 is defined on $\overline{Y}_1$ except for possibly finitely many points, we may assume that f_1 is well defined at $\iota^*(\overline{\chi}'_k) = \iota^*(\overline{\chi}_k)$ and $\iota^*(\overline{\chi}''_k) = \iota^*(\tau(\overline{\chi}_k))$ for all large k.

Let v_1 and v_j be the functions on $\overline{Y}^{\nu}_1$ provided by part (1) of Theorem 2.2 with respect to the map $f_1 : \overline{Y}_1 \to \overline{Y}^{\nu}_1$. Note that away from finitely many points in $\overline{Y}_1$, $v_1 \circ f_1$ and $v_j \circ f_1$ are smooth functions and have the same differential, up to sign. (cf. the proof of [D, Theorem 2.6] for this assertion. Briefly, on $(\mathbb{C}^\times)^2$ there is a real valued 1-form:

$$\omega = -\frac{1}{2}(\log|\mathfrak{L}| d\, arg(\mathfrak{M}) - \log|\mathfrak{M}| d\, arg(\mathfrak{L}))$$

which is defined in [CCGLS]. This 1-from is invariant under the involutions σ and ϵ_1^* on $(\mathbb{C}^\times)^2$ defined in Subsection 2.5 and thus descends to a 1-form ω' on $\overline{X}(\partial M)$. For each j, $d(v_j \circ f_j)$ is equal to the restriction of ω' over an open dense subset of $\overline{Y}_j$, up to sign.) It follows that

$$v_j \circ f_1 = \delta(v_1 \circ f_1) + c$$

for some $\delta \in \{1, -1\}$ and some constant c, in the complement of finitely many points in $\overline{Y}_1$. Let U denote this complement. Then we may assume that $\iota^*(\overline{\chi}'_k) = \iota^*(\overline{\chi}_k)$ and $\iota^*(\overline{\chi}''_k) = \iota^*(\tau(\overline{\chi}_k))$ are contained in U for all large k.

Hence

$$v(\overline{\chi}'_k) = v_j(f_1(\iota^*(\overline{\chi}'_k))) = \delta v_1(f_1(\iota^*(\overline{\chi}_k))) + c = \delta v(\overline{\chi}_k) + c$$

and

$$v(\overline{\chi}''_k) = v_j(f_1(\iota^*(\overline{\chi}''_k))) = \delta v_1(f_1(\iota^*(\tau(\overline{\chi}_k)))) + c = \delta v(\tau(\overline{\chi}_k)) + c.$$

So

$$v(\overline{\chi}'_k) - v(\overline{\chi}''_k) = \delta[v(\overline{\chi}_k) - v(\tau(\overline{\chi}_k))] = \delta 2 v(\overline{\chi}_k) = \delta 2 vol(M(\gamma_k))$$

and thus

$$|v(\overline{\chi}'_k)| + |v(\overline{\chi}''_k)| \geq 2vol(M(\gamma_k))$$

for sufficiently large k. Because $\iota^*(\overline{\chi}'_k) = \iota^*(\overline{\chi}_k)$ and $\iota^*(\overline{\chi}''_k) = \iota^*(\tau(\overline{\chi}_k))$, $\overline{\chi}'_k$ and $\overline{\chi}''_k$ are both characters of $\overline{X}(M(\gamma_k))$. To see this in detail, let $\overline{\rho}_k, \overline{\rho}'_k \in \overline{R}(M)$ be representations with $\overline{\chi}_k$ and $\overline{\chi}'_k$ as characters respectively. Note that $\overline{\rho}_k$ is a discrete faithful representation of $\pi_1(M(\gamma_k))$ and so $\overline{\rho}_k(\gamma_k) = 1$. Let η_k be a simple essential loop in ∂M such that $\{\gamma_k, \eta_k\}$ form a basis of $\pi_1(\partial M)$. Then η_k is isotopic in $M(\gamma_k)$ to the core circle of the filling solid torus in forming $M(\gamma_k)$ from M. As we have assumed that the core circle is a geodesic in the hyperbolic 3-manifold $M(\gamma_k)$, $\overline{\rho}_k(\eta_k)$ is a hyperbolic element of $PSL_2(\mathbb{C})$. In particular its trace square is not equal to 4. Now since $\overline{\chi}'_k(\gamma_k) = \overline{\chi}_k(\gamma_k)$ and $\overline{\chi}'_k(\eta_k) = \overline{\chi}_k(\eta_k)$, $\overline{\rho}'_k(\gamma_k)$ is a parabolic element or the identity element and $\overline{\rho}'_k(\eta_k)$ is a hyperbolic element of $PSL_2(\mathbb{C})$. But these two elements commute, $\overline{\rho}'_k(\gamma_k)$ has to be the identity element. Hence $\overline{\chi}'_k \in \overline{X}(M(\gamma_k))$. Similarly one can show that $\overline{\chi}''_k \in \overline{X}(M(\gamma_k))$. But neither $\overline{\chi}'_k$ nor $\overline{\chi}''_k$ is a discrete faithful character of $\overline{X}(M(\gamma_k))$ by our construction, we get a contradiction with the volume rigidity theorem for closed hyperbolic 3-manifolds. $\diamondsuit$

To finish the proof of Theorem 1.1 (3), we just need to show that $\overline{Y}_j$, $j \geq 2$ are mutually distinct. Suppose that $\overline{Y}_{j_1} = \overline{Y}_{j_2}$ for some $j_1, j_2 \geq 2$. There is $\phi \in \mathrm{Aut}(\mathbb{C})$ such that $\phi(\overline{X}_{j_1}) = \overline{X}_1$. As ϕ commutes with ι^*, $\phi(\overline{Y}_{j_1}) = \overline{Y}_1$. We also have $\phi(\overline{Y}_{j_2}) = \phi(\overline{Y}_{j_1}) = \overline{Y}_1$. So by Proposition 2.1, $\phi(\overline{X}_{j_2}) = \overline{X}_1$ as well. Hence $\overline{X}_{j_1} = \overline{X}_{j_2}$, i.e. $j_1 = j_2$.

2.3. **A refinement of Theorem 1.1.** Let M be a hyperbolic knot manifold. Suppose $\overline{X}_1, \ldots, \overline{X}_k$ are all $\mathbb{C}$-components of $\overline{X}(M)$ and $\overline{Y}_j$ is the Zariski closure of $\iota^*(\overline{X}_j)$ in $\overline{X}(\partial M)$ for $i = 1, \ldots, k$. It is known that $\overline{Y}_j$ has dimension either 1 or 0.

In proving $\overline{Y}_j \neq \overline{Y}_1$ in the previous subsection, the fact that $\overline{X}_j$ is in the Aut($\mathbb{C}$)-orbit of $\overline{X}_1$ is used only to show that $\overline{X}_j$ lifts to $X(M)$ (see Remark 2.3). When M is a hyperbolic knot manifold which is the exterior of a knot in a homology 3-sphere, every $PSL_2(\mathbb{C})$-representation of $\pi_1(M)$ lifts to a SL_2-representation. Hence, the proof of part (3) of Theorem 1.1 also proves the following.

Theorem 2.4. *Let M be a hyperbolic knot manifold which is the exterior of a knot in a homology 3-sphere. Let $\overline{X}_1, \ldots, \overline{X}_k$ be the $\mathbb{C}$-components of the PSL_2-character varieties $\overline{X}(M)$, and $\overline{Y}_j$ be the Zariski closure of $\iota^*(X_j)$, $j = 1, \ldots, k$. Suppose the two discrete faithful characters of $\overline{X}(M)$ are contained in $\overline{X}_1$. Then $\overline{Y}_j \neq \overline{Y}_1$ for all $j \geq 2$.*

2.4. **Proof of Theorem 1.2.** Let (μ, λ) be the standard meridian-longitude basis for $\pi_1(\partial M) \subset \pi_1(M)$. Let $\mathbb{Z}_2 = \{1, -1\}$. Since $H^1(M, \mathbb{Z}_2) = \mathbb{Z}_2$, there is a unique non-trivial group homomorphism $\varepsilon : \pi_1(M) \to \mathbb{Z}_2$. One has $\varepsilon(\mu) = -1$ and $\varepsilon(\lambda) = 1$. The homomorphism ϵ induces an involution ϵ^* on $R(M)$ and on $X(M)$, defined by $\varepsilon^*(\rho)(\gamma) = \varepsilon(\gamma)\,\rho(\gamma)$ for $\rho \in R(M)$ and $\epsilon^*(\chi_\rho) = \chi_{\epsilon^*(\rho)}$ for $\chi_\rho \in X(M)$.

Obviously ϵ^* is a bijective regular involution on $X(M)$, and it is defined over $\mathbb{Q}$. The quotient space of $X(M)$ by this involution gives rise a regular map Φ^* from $X(M)$ into $\overline{X}(M)$. Let $\Phi : SL_2(\mathbb{C}) \to PSL_2(\mathbb{C})$ be the canonical quotient homomorphism. Then Φ^* is exactly the map induced by Φ.

On the other hand, since $H_1(M; \mathbb{Z}_2) = \mathbb{Z}_2$, every $PSL_2(\mathbb{C})$-representation $\overline{\rho}$ of M lifts to a $SL_2(\mathbb{C})$ representation ρ of M in the sense that $\overline{\rho} = \Phi \circ \rho$ (cf. e.g. [BZ1, Page 756]). Hence Φ^* is an onto map on $X(M)$.

Similarly if $\epsilon_1 : \pi_1(\partial M) \to \mathbb{Z}_2 = \{1, -1\}$ is the homomorphism defined by $\epsilon_1(\mu) = -1$ and $\epsilon_1(\lambda) = 1$, it induces an involution ϵ_1^* on $X(\partial M)$. Let Φ_1^* be the corresponding quotient map from $X(\partial M)$ into $\overline{X}(\partial M)$. Then Φ_1^* is also a regular and surjective map. We have the following commutative diagrams of regular maps:

$$\begin{tikzcd} X(M) \arrow[r, "\iota^*"] \arrow[d, "\Phi^*"'] & X(\partial M) \arrow[d, "\Phi_1^*"] \\ \overline{X}(M) \arrow[r, "\iota^*"] & \overline{X}(\partial M) \end{tikzcd} \tag{2.4.1}$$

and the upper ι^* satisfies the identity

$$\epsilon_1^* \circ \iota^* = \iota^* \circ \epsilon^*. \tag{2.4.2}$$

In particular we have

$$\begin{tikzcd} X^{\mathrm{rg}}(M) \arrow[r, "\iota^*"] \arrow[d, "\Phi^*"'] & Y \arrow[d, "\Phi_1^*"] \\ \overline{X}^{\mathrm{rg}}(M) \arrow[r, "\iota^*"] & \overline{Y}. \end{tikzcd} \tag{2.4.3}$$

where $\overline{Y}$ is the Zariski closure of $\iota^*(\overline{X}^{\mathrm{rg}}(M))$ and Y is the Zariski closure of $\iota^*(X^{\mathrm{rg}}(M))$. All the varieties in Diagram (2.4.3) are 1-equidimensional. Each of Φ^* and Φ_1^* is a degree 2 map, while the lower ι^* has degree 1 by Theorem 1.1. It follows that there are an open dense subset $(\overline{X}^{\mathrm{rg}})'$ of $\overline{X}^{\mathrm{rg}}(M)$ and an open dense subset $\overline{Y}'$ of $\overline{Y}$ such that the restriction $\iota^* : (X^{\mathrm{rg}})' \to \overline{Y}'$ is a bijection and for every $x \in (\overline{X}^{\mathrm{rg}})'$ and $y \in \overline{Y}'$, $(\Phi^*)^{-1}(x)$ has exactly two distinct elements and so does $(\Phi_1^*)^{-1}(y)$. Besides, $(X^{\mathrm{rg}})' := (\Phi^*)^{-1}((\overline{X}^{\mathrm{rg}})')$ is open dense in $X^{\mathrm{rg}}(M)$ and $Y' = (\Phi_1^*)^{-1}(\overline{Y}')$ is open dense in Y. Suppose $x \in (\overline{X}^{\mathrm{rg}})'$, $y = \iota^*(x)$, $\{x_1, x_2\} = (\Phi^*)^{-1}(x)$, and $\{y_1, y_2\} = (\Phi_1^*)^{-1}(y)$. The commutativity of the above diagram means $\iota^*(x_1)$ is one of y_1, y_2, say $\iota^*(x_1) = y_1$. Then the identity (2.4.2) implies $\iota^*(x_2) = y_2$. This shows that ι^* is a bijection from the open dense subset $(X^{\mathrm{rg}})'$ of $X^{\mathrm{rg}}(M)$ onto the open dense subset Y' of Y. Hence, $\iota^* : X^{\mathrm{rg}}(M) \to Y$ is a degree one map. Now Theorem 1.2 follows from Theorem 1.1.

Remark 2.5. Let M be a hyperbolic knot manifold which is the exterior of a knot in a homology 3-sphere. We saw in the above proof that Φ^* is surjective on $X(M)$. Since $(\Phi^*)^{-1}(\overline{X}^{\mathrm{irr}}(M)) = X^{\mathrm{irr}}(M)$ and $(\Phi^*)^{-1}(\overline{X}^{\mathrm{rg}}(M)) = X^{\mathrm{rg}}(M)$, we conclude that $X^{\mathrm{irr}}(M) = X^{\mathrm{rg}}(M)$ if and only in $\overline{X}^{\mathrm{irr}}(M) = \overline{X}^{\mathrm{rg}}(M)$.

2.5. ***A*-polynomial and its symmetry.** We briefly recall the definition of the A-polynomial for a knot K in a homology 3-sphere W, as defined in [CCGLS]. Let M be the exterior of K and let $\{\mu, \lambda\}$ be the standard meridian-longitude basis for $\pi_1(\partial M)$.

Let $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ and $\sigma : (\mathbb{C}^\times)^2 \to (\mathbb{C}^\times)^2$ be the involution defined by $\sigma(\mathfrak{M}, \mathfrak{L}) = (\mathfrak{M}^{-1}, \mathfrak{L}^{-1})$. We can identify $X(\partial M)$ with $(\mathbb{C}^\times)^2/\sigma$ as follows. For $(\mathfrak{M}, \mathfrak{L}) \in (\mathbb{C}^\times)^2$ let $\chi_{(\mathfrak{M},\mathfrak{L})} \in X(\partial M)$ be the character of the representation

$$\rho : \pi_1(\partial M) \to SL_2(\mathbb{C}), \quad \rho(\mu) = \begin{pmatrix} \mathfrak{M} & 0 \\ 0 & \mathfrak{M}^{-1} \end{pmatrix}, \rho(\lambda) = \begin{pmatrix} \mathfrak{L} & 0 \\ 0 & \mathfrak{L}^{-1} \end{pmatrix}.$$

Then the map $(\mathfrak{M}, \mathfrak{L}) \to \chi_{(\mathfrak{M},\mathfrak{L})}$ descends to an isomorphism from $(\mathbb{C}^\times)^2/\sigma$ onto $X(\partial M)$, which we use to identify $(\mathbb{C}^\times)^2/\sigma$ with $X(\partial M)$. Let $\mathrm{pr}_\sigma : (\mathbb{C}^\times)^2 \to (\mathbb{C}^\times)^2/\sigma \equiv X(\partial M)$ be the natural projection.

Let $\varepsilon_1^* : (\mathbb{C}^\times)^2 \to (\mathbb{C}^\times)^2$ be the involution defined by $\varepsilon_1^*(\mathfrak{M}, \mathfrak{L}) = (-\mathfrak{M}, \mathfrak{L})$. Then ε_1^* commutes with σ and descends to an involution of $(\mathbb{C}^\times)^2/\sigma$, which coincides with the ε_1^* of Section 2.4. Thus, we can identify $\overline{X}(\partial M)$ with $((\mathbb{C}^\times)^2/\sigma)/\varepsilon_1^* = (\mathbb{C}^\times)^2/\langle\sigma, \varepsilon_1^*\rangle$, and Φ_1^* with the natural projection $(\mathbb{C}^\times)^2/\sigma \to (\mathbb{C}^\times)^2/\langle\sigma, \varepsilon_1^*\rangle$. Here $\langle\sigma, \varepsilon_1^*\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is the group generated by σ and ε_1^*. Let $\mathrm{pr} : (\mathbb{C}^\times)^2 \to (\mathbb{C}^\times)^2/\langle\sigma, \varepsilon_1^*\rangle$ be the natural projection.

The involution σ naturally induces an algebra involution, also denoted by σ, acting on the algebra $\mathbb{C}[\mathfrak{M}^{\pm 1}, \mathfrak{L}^{\pm 1}]$. That is, $\sigma(P)(\mathfrak{M}, \mathfrak{L}) = P(\mathfrak{M}^{-1}, \mathfrak{L}^{-1})$ for $P \in \mathbb{C}[\mathfrak{M}^{\pm}, \mathfrak{L}^{\pm 1}]$. A polynomial $P \in \mathbb{C}[\mathfrak{M}, \mathfrak{L}]$ is said to be *balanced* if $\sigma(P) = \delta\mathfrak{M}^a\mathfrak{L}^b P$ for certain $\delta \in \{-1, 1\}$ and $a, b \in \mathbb{Z}$. For any subring $\mathcal{R} \subset \mathbb{C}[\mathfrak{M}, \mathfrak{L}]$, we say that $P \in \mathbb{C}[\mathfrak{M}, \mathfrak{L}]$ is *balanced-irreducible in* $\mathcal{R}$ if $P \in \mathcal{R}$ and P is balanced but is not the product of two non-constant balanced polynomials in $\mathcal{R}$.

Suppose $Z \subset X(\partial M)$ is 1-equidimensional variety. The Zariski closure $\tilde{Z}$ of $\mathrm{pr}_\sigma^{-1}(Z)$ in $\mathbb{C}^2$ is a 1-equidimensional variety. The ideal of all polynomials in $\mathbb{C}[\mathfrak{M}, \mathfrak{L}]$ vanishing on $\tilde{Z}$ is principal, and is generated by a polynomial $P_Z \in \mathbb{C}[\mathfrak{M}, \mathfrak{L}]$, defined up to a non-zero constant factor. The σ-invariance of $\mathrm{pr}_\sigma^{-1}(Z)$ implies that P_Z is balanced. If Z is $\mathbb{C}$-irreducible, then P_Z is balanced-irreducible in $\mathbb{C}[\mathfrak{M}, \mathfrak{L}]$. If Z is defined over $\mathbb{Q}$, then one can choose $P_Z \in \mathbb{Z}[\mathfrak{M}, \mathfrak{L}]$ and it is defined up to sign. If Z is $\mathbb{Q}$-irreducible, then P_Z is balanced-irreducible in $\mathbb{Z}[\mathfrak{M}, \mathfrak{L}]$.

Similarly, suppose $\overline{Z} \subset \overline{X}(\partial M)$ is 1-equidimensional variety, one defines $P_{\overline{Z}} \in \mathbb{C}[\mathfrak{M}, \mathfrak{L}]$ as the generator of the ideal of all polynomials in $\mathbb{C}[\mathfrak{M}, \mathfrak{L}]$ vanishing on $\mathrm{pr}^{-1}(\overline{Z})$. The $\langle\sigma, \varepsilon_1^*\rangle$-invariance of $\mathrm{pr}^{-1}(\overline{Z})$ implies that $P_{\overline{Z}}$ is balanced and belongs to $\mathbb{C}[\mathfrak{M}^2, \mathfrak{L}]$. If $\overline{Z}$ is $\mathbb{C}$-irreducible, then $P_{\overline{Z}}$ is balanced-irreducible in $\mathbb{C}[\mathfrak{M}^2, \mathfrak{L}]$. If $\overline{Z}$ is defined over $\mathbb{Q}$, then one can choose $P_{\overline{Z}} \in \mathbb{Z}[\mathfrak{M}^2, \mathfrak{L}]$ and it is defined up to sign. If $\overline{Z}$ is $\mathbb{Q}$-irreducible, then $P_{\overline{Z}}$ is balanced-irreducible in $\mathbb{Z}[\mathfrak{M}^2, \mathfrak{L}]$.

Now let $\overline{Z}$ be the union of all one-dimensional $\mathbb{C}$-components of the Zariksi closure of $\iota^*(\overline{X}(M))$ in $\overline{X}(\partial M) = (\mathbb{C}^\times)^2/\langle\sigma, \varepsilon_1^*\rangle$. It is known that $\overline{Z}$ is defined over $\mathbb{Q}$. The polynomial $P_{\overline{Z}} \in \mathbb{Z}[\mathfrak{M}^2, \mathfrak{L}]$ is the A-polynomial $A_{\mathrm{K,W}}(\mathfrak{M}, \mathfrak{L})$. If Z is the union of all one-dimensional $\mathbb{C}$-components of the Zariski closure of $\iota^*(X(M))$ in $X(\partial M)$. Then $Z = (\Phi^*)^{-1}(\overline{Z})$. Thus $P_Z = P_{\overline{Z}}$ is the A-polynomial.

Remark 2.6. *To define the A-polynomial* $A_{K,W}(\mathfrak{M}, \mathfrak{L})$*, one just needs to consider in* $SL_2(\mathbb{C})$*-setting, i.e. in terms of* P_Z*, as how it's done in* [CCGLS]*. For our purpose (e.g. for a convenience in proving Proposition* 2.7*) we also present the same A-polynomial from* $PSL_2(\mathbb{C})$ *point of view, i.e. in terms of* $P_{\overline{Z}}$*.*

2.6. **Proof of Theorem 1.4.** Since M is the exterior of a knot K in a homology 3-sphere, its trace field is equal to its invariant trace field. Let χ_0 be a discrete faithful character of $X(M)$. It is proved in [SZ] that the $\mathrm{Aut}(\mathbb{C})$-orbit of χ_0 has d distinct elements, which we denote by χ_i, $i = 0, 1, ..., d-1$, and $\epsilon^*(\chi_i)$, $i = 0, 1, ..., d-1$, is another set of d distinct elements, which is disjoint from the former set. These $2d$ characters are obviously contained in $X^{\mathrm{rg}}(M)$. Furthermore they are irreducible faithful characters whose values on elements of $\pi_1(\partial M)$ are 2 or -2.

For $\gamma \in \pi_1(M)$, let f_γ be the regular function on $X(M)$ defined by $f_\gamma(\chi_\rho) = [\mathrm{trace}(\rho(\gamma))]^2 - 4$. Then by the discussion above, for each peripheral element $\gamma \in \pi_1(\partial M)$, f_γ has at least $2d$ zero points: $\chi_i, \epsilon^*(\chi_i)$, $i = 0, ..., d-1$.

Now let $X_1, ..., X_l$ be the $\mathbb{C}$-components of $X^{\mathrm{rg}}(M)$. By [BZ2, Section 5], f_γ is non-constant on each X_j for every nontrivial element $\gamma \in \pi_1(\partial M)$ and the degree of f_γ on X_j remains the same for $j = 1, \dots, l$. It is shown in [SZ] that

$$\sum_{j=1}^{l} \mathrm{degree}(f_\gamma)|_{X_j} \geq 2d. \tag{2.6.1}$$

Perhaps we need to note that $\mathrm{degree}(f_\gamma)|_{X_j}$ is equal to the Culler-Shalen norm of $\gamma \in \pi_1(\partial M)$ defined by the curve X_j, and the inequality (2.6.1) is given in [SZ] in terms the Culler-Shalen norm.

Let $P_j = P_{Y_j} \in \mathbb{C}[\mathfrak{M}, \mathfrak{L}]$ (see the definition of P_Z in Subsection 2.5), where Y_j is the Zariski closure of $\iota^*(X_j)$. Theorem 1.2 part (2) and [BZ2, Proposition 6.6] together imply that the $\mathfrak{M}$-degree of $P_j(\mathfrak{M}, \mathfrak{L})$ is equal to $\frac{1}{2}\mathrm{degree}(f_\lambda)\Big|_{X_j}$ and the $\mathfrak{L}$-degree of $P_j(\mathfrak{M}, \mathfrak{L})$ is equal to $\frac{1}{2}\mathrm{degree}(f_\mu)\Big|_{X_j}$. We note at this point that although the definition of A-polynomial defined in [BZ2] is a bit different from that given in [CCGLS], when the degree of the map $\iota^*|_{X_j}$ is one the factor $P_j(\mathfrak{M}, \mathfrak{L})$ contributed by X_j is the same polynomial either as defined in [CCGLS] or as defined in [BZ2]. Since we do have that $\iota^*|_{X_j}$ is a degree one map, [BZ2, Proposition 6.6] applies.

Moreover it follows from Theorem 1.2 part (3) that all factors $P_j(\mathfrak{M}, \mathfrak{L})$, $j = 1, ..., l$, are mutually distinct. Hence the $\mathfrak{M}$-degree and the $\mathfrak{L}$-degree of $A_{\mathrm{K,W}}(\mathfrak{M}, \mathfrak{L})$ are larger than or equal to $\sum_{j=1}^{l} \frac{1}{2}\mathrm{degree}(f_\lambda)\Big|_{X_j}$ and $\sum_{j=1}^{l} \frac{1}{2}\mathrm{degree}(f_\mu)\Big|_{X_j}$ respectively, which are bigger than or equal to d by (2.6.1). This completes the proof of Theorem 1.4.

2.7. A-polynomial and balanced-irreduciblity.

The following will be used in the proof of Theorem 1.5.

Proposition 2.7. *Suppose that M is a hyperbolic knot manifold which is the exterior of a knot K in a homology 3-sphere W. Assume that the two discrete faithful characters are in the same $\mathbb{C}$-component of the $PSL_2(\mathbb{C})$-character varietiy $\overline{X}(M)$ and $\overline{X}^{\mathrm{irr}}(M) = \overline{X}^{\mathrm{rg}}(M)$. Then the non-abelian A-polynomial $\widehat{A}_{K,W}(\mathfrak{M}, \mathfrak{L})$ is non-constant, does not contain any $\mathfrak{M}$-factor or $\mathfrak{L}$-factor, and is balanced-irreducible in $\mathbb{Z}[\mathfrak{M}^2, \mathfrak{L}]$.*

Here an $\mathfrak{M}$-factor (resp. $\mathfrak{L}$-factor) means a non-constant element of $\mathbb{Z}[\mathfrak{M}]$ (resp. $\mathbb{Z}[\mathfrak{L}]$).

Proof. Let $\overline{Y}$ be the Zariski closure of $\iota^*(\overline{X}^{\mathrm{rg}})$ in $\overline{X}(\partial M)$. Since $\overline{X}$ is $\mathbb{Q}$-irreducible, it follows from Theorem 1.1 that $\overline{Y}$ is $\mathbb{Q}$-irreducible. Therefore $P_{\overline{Y}}$ is balanced-irreducible in $\mathbb{Z}[\mathfrak{M}^2, \mathfrak{L}]$

(see Section 2.5). When $\overline{X}^{\mathrm{irr}}(M) = \overline{X}^{\mathrm{rg}}(M)$, $P_{\overline{Y}}$ is the whole $\widehat{A}_{\mathrm{K,W}}(\mathfrak{M}, \mathfrak{L})$. Hence $\widehat{A}_{\mathrm{K,W}}(\mathfrak{M}, \mathfrak{L})$ is balanced-irreducible in $\mathbb{Z}[\mathfrak{M}^2, \mathfrak{L}]$.

Let $X_1, \ldots, X_l$ be the $\mathbb{C}$-components of $X^{\mathrm{rg}}(M)$. As pointed out in the proof of Theorem 1.4, for $j = 1, \ldots, l$, both of the $\mathfrak{M}$-degree and the $\mathfrak{L}$-degree of $P_{X_j}(\mathfrak{M}, \mathfrak{L})$ are positive. As P_{X_j} is balanced-irreducible, it follows that P_{X_j} cannot contain any $\mathfrak{M}$-factor or $\mathfrak{L}$-factor. In particular $P_{X_j}(\mathfrak{M}, \mathfrak{L}) \neq \mathfrak{L} - 1$. Hence $X^{\mathrm{rg}}(M)$ contributes the factor $\widehat{A}_{\mathrm{K,W}}(\mathfrak{M}, \mathfrak{L}) = A_{\mathrm{K,W}}(\mathfrak{M}, \mathfrak{L})/(\mathfrak{L} - 1)$ which is a non-constant and does not contain any $\mathfrak{M}$-factor or $\mathfrak{L}$-factor. ♢

Remark 2.8. The above proof, combined with Theorem 2.4, actually yields the following stronger statement. Suppose M is a hyperbolic knot manifold which is the exterior of a knot K in a homology sphere W such that the two $PSL_2(C)$ discrete faithful characters are contained in the same $\mathbb{C}$-component of $\overline{X}(M)$. Then $\widehat{A}_{K,W}(\mathfrak{M}, \mathfrak{L})$ is balanced-irreducible in $\mathbb{Z}[\mathfrak{M}^2, \mathfrak{L}]$ if and only if X^{rg} contains every $\mathbb{C}$-component of $X(M)$ whose image under $\iota^* : X(M) \to X(\partial M)$ is one dimensional.

3. Representation schemes and character schemes

3.1. **Reduced and essentially reduced schemes.** Concerning the proof of Theorem 1.5, we need to consider the scheme counterparts of the $SL_2(\mathbb{C})$ representation variety and character variety of a group Γ. Let's first prepare some facts about an affine scheme Spec(R) for a ring R of the form $R = \mathbb{C}[x_1, ..., x_n]/I$ where I is a proper ideal of $\mathbb{C}[x_1, ..., x_n]$. The ideal I admits an irredundant primary decomposition, i.e.

$$I = \bigcap_{j=1}^{m} Q_j$$

for some positive integer m such that each Q_j is a primary ideal and $\sqrt{Q_i} \neq \sqrt{Q_j}$ for $i \neq j$. The radical $P_j = \sqrt{Q_j}$ is a prime ideal. Recall that Q_j is called an isolated component of I if P_j is minimum in the inclusion relation among $P_1, ..., P_m$, and if Q_j is not isolated, it is called an embedded component of I. The set $\{P_1, ..., P_m\}$ is uniquely determined by I, as well as the set of all isolated components Q_j of I. We may assume that Q_j, $j = 1, ..., k$, are the isolated components of I.

Let $V(I) \subset \mathbb{C}^n$ be the zero locus of an ideal $I \subset \mathbb{C}[x_1, ..., x_n]$, which is a variety. Note that $V(I) = V(\sqrt{I})$. The coordinate ring $\mathbb{C}[V]$ of $V = V(I)$ is given by $\mathbb{C}[V] = \mathbb{C}[x_1, ..., x_n]/\sqrt{I}$ which is also equal to the quotient ring of $R = \mathbb{C}[x_1, ..., x_n]/I$ divided by its nilradical $\sqrt{(0)}$, i.e.

$$\mathbb{C}[V] = R/\sqrt{(0)}.$$

The variety $V = V(I)$ can be naturally identified with the set of closed points of the scheme Spec(R). The zero loci $V_j = V(Q_j) = V(P_j)$, $j = 1, ..., k$, are all irreducible $\mathbb{C}$-components of V. Let $R_j = \mathbb{C}[x_1, ..., x_n]/Q_j$, $j = 1, ..., k$. Then Spec(R_j), for each $j = 1, \ldots, k$, is an irreducible component of Spec(R), called the *component corresponding to* V_j.

Recall that a ring is called *reduced* if it does not contain any non-zero nilpotent elements. For the ring R above, it is reduced if and only if $I = \sqrt{I}$. Similarly the ring R_j is reduced if and only if $Q_j = \sqrt{Q_j} = P_j$ (or equivalently R_j is an integral domain). If all R_j, $j = 1, ..., k$, are reduced (i.e. $Q_j = P_j$ for all isolated components of I), we call the ring R *essentially reduced.* Correspondingly we call an affine scheme $\mathrm{Spec}(R)$ reduced if its defining ring R is reduced, and call it *essentially reduced* if each irreducible component $\mathrm{Spec}(R_j)$ of $\mathrm{Spec}(R)$ is reduced.

Let $\mathfrak{m} \in \mathrm{Spec}(R)$ be a closed point, which we shall also identify with a maximal ideal of R as well as with a point in V. Let $T_{\mathfrak{m}}(\mathrm{Spec}(R))$ denote the Zariski tangent space of the scheme $\mathrm{Spec}(R)$ at the point $\mathfrak{m}$ and let $R_{\mathfrak{m}}$ be the localization of R at the maximal ideal $\mathfrak{m}$. Note that $R_{\mathfrak{m}}$ is a local ring and is the stalk of the scheme $\mathrm{Spec}(R)$ at the point $\mathfrak{m}$. If the dimension of $T_{\mathfrak{m}}(\mathrm{Spec}(R))$ is equal to the (Krull) dimension of the local ring $R_{\mathfrak{m}}$, then $\mathfrak{m}$ is a *smooth point* of the scheme $\mathrm{Spec}(R)$ (called a regular point or a simple point in some textbooks), and the following conclusions follow: $\mathfrak{m}$ is contained in a unique irreducible component of $\mathrm{Spec}(R)$, say $\mathrm{Spec}(R_j)$, $R_{\mathfrak{m}} = (R_j)_{\mathfrak{m}}$ is an integral domain, which implies that R_j is an integral domain and thus R_j is reduced and $Q_j = P_j$ (see e.g. [Mum] [S]). We summarize this discussion into the following lemma in a form that is more convenient for us to apply.

Lemma 3.1. *Let $R = \mathbb{C}[x_1, .., x_n]/I$ for a proper ideal I. Let $\mathfrak{m} \in \mathrm{Spec}(R)$ be a closed point and let V_j be an irreducible component of the variety $V = V(I)$ which contains $\mathfrak{m}$. Suppose that dim $T_{\mathfrak{m}}(\mathrm{Spec}(R))$ = dim V_j, then $\mathfrak{m}$ is a smooth point of the scheme $\mathrm{Spec}(R)$, V_j is the unique irreducible component of V which contains $\mathfrak{m}$, and the isolated component Q_j of I which defines V_j is a prime ideal.*

Proof. Let Q_j be the isolated component of I which defines V_j and let $R_j = \mathbb{C}[x_1, ..., x_n]/Q_j$. Then $\mathfrak{m} \in \mathrm{Spec}(R_j) \subset \mathrm{Spec}(R)$. As we always have

$$\dim T_{\mathfrak{m}}(\mathrm{Spec}(R)) \geq \dim R_{\mathfrak{m}} = \dim\,(R_j)_{\mathfrak{m}} = \dim R_j = \dim R_j/\sqrt{(0)} = \dim V_j,$$

the assumption $\dim T_{\mathfrak{m}}(\mathrm{Spec}(R)) = \dim V_j$ implies the equality $\dim T_{\mathfrak{m}}(\mathrm{Spec}(R)) = \dim R_{\mathfrak{m}}$ and thus all the conclusions of the lemma follow from the discussion preceding the lemma. ♢

Remark 3.2. Recall that every element $\phi \in \mathrm{Aut}(\mathbb{C})$ induces an action on $\mathbb{C}[x_1, ..., x_n]$. If in the above lemma the ideal I is defined over $\mathbb{Q}$, then every element $\phi \in \mathrm{Aut}(\mathbb{C})$ will keep I invariant, sending isolated components of I to isolated components, and sending scheme reduced components Q_j (i.e. $Q_j = \sqrt{Q_j}$ is prime) of I to scheme reduced components. Hence the $\mathrm{Aut}(\mathbb{C})$-orbit of a scheme reduced isolated component of I is a set of scheme reduced isolated components of I whose intersection is an ideal defined over $\mathbb{Q}$.

3.2. **Character scheme.** Given a finitely presented group Γ, let $\mathfrak{A}(\Gamma)$ be *the universal $SL_2(\mathbb{C})$ representation ring of* Γ, which is a finitely generated $\mathbb{C}$-algebra (as given by [LM, Proposition 1.2], replacing GL_n there by SL_2 and k there by $\mathbb{C}$). The $SL_2(\mathbb{C})$ *representation scheme* $\mathfrak{R}(\Gamma)$ of Γ is defined to be the scheme $\mathrm{Spec}(\mathfrak{A}(\Gamma))$, i.e. $\mathfrak{R}(\Gamma) = \mathrm{Spec}(\mathfrak{A}(\Gamma))$. The set of closed points of $\mathfrak{R}(\Gamma)$ can be identified with the $SL_2(\mathbb{C})$ representation variety $R(\Gamma)$ of Γ. The coordinate ring $\mathbb{C}[R(\Gamma)]$ of $R(\Gamma)$ can be obtained as the quotient of $\mathfrak{A}(\Gamma)$ by its nilradical $\sqrt{(0)}$, i.e.

$$\mathbb{C}[R(\Gamma)] = \mathfrak{A}(\Gamma)/\sqrt{(0)}.$$

Induced by the matrix conjugation, the group $SL_2(\mathbb{C})$ acts naturally on $\mathfrak{A}(\Gamma)$. Let

$$\mathfrak{B}(\Gamma) = \mathfrak{A}(\Gamma)^{SL_2(\mathbb{C})}$$

be the subring of invariant elements of $\mathfrak{A}(\Gamma)$ under this action, which is finitely-generated as a $\mathbb{C}$-algebra (by the Hilbert-Nagata theorem [DC]). Then $\mathfrak{B}(\Gamma)$ is called *the universal* $SL_2(\mathbb{C})$ *character ring of* Γ and the scheme

$$\mathfrak{X}(\Gamma) := \mathrm{Spec}(\mathfrak{B}(\Gamma))$$

is called *the* $SL_2(\mathbb{C})$ *character scheme of* Γ. The set of closed points of $\mathfrak{X}(\Gamma)$ can be identified with the character variety $X(\Gamma)$ of Γ and the coordinate ring $\mathbb{C}[X(\Gamma)]$ of $X(\Gamma)$ is $\mathfrak{B}(\Gamma)$ divided by its zero radical, i.e.

$$\mathbb{C}[X(\Gamma)] = \mathfrak{B}(\Gamma)/\sqrt{(0)}.$$

Let $\rho \in \mathfrak{R}(\Gamma) = \mathrm{Spec}(\mathfrak{A}(\Gamma))$ be a closed point. Then identified as a point in $R(\Gamma)$, $\rho : \Gamma \rightarrow SL_2(\mathbb{C})$ is a $SL_2(\mathbb{C})$ representation of Γ. Similarly the character $\chi_\rho \in X(\Gamma)$ of $\rho \in R(\Gamma)$ shall also be considered as a closed point in the character scheme $\mathfrak{X}(\Gamma) = \mathrm{Spec}(\mathfrak{B}(\Gamma))$. Let $sl_2(\mathbb{C})$ be the Lie algebra of $SL_2(\mathbb{C})$, $Ad : SL_2(\mathbb{C}) \rightarrow \mathrm{Aut}(sl_2(\mathbb{C}))$ the adjoint representation, and $sl_2(\mathbb{C})_\rho$ the Γ-module $sl_2(\mathbb{C})$ given by $Ad \circ \rho : \Gamma \rightarrow \mathrm{Aut}(sl_2(\mathbb{C}))$. Then a fundamental observation made in [W] states that the space of group 1-cocycles $Z^1(\Gamma, sl_2(\mathbb{C})_\rho)$ of Γ with coefficients in $sl_2(\mathbb{C})_\rho$ is naturally isomorphic to the Zariski tangent space $T_\rho(\mathfrak{R}(\Gamma))$ of the scheme $\mathfrak{R}(\Gamma)$ at the point ρ, and when ρ is an irreducible representation and is a smooth point of $\mathfrak{R}(\Gamma)$, the group 1-cohomology $H^1(\Gamma, sl_2(\mathbb{C})_\rho)$ is isomorphic to the Zariski tangent space $T_{\chi_\rho}(\mathfrak{X}(\Gamma))$ of the scheme $\mathfrak{X}(\Gamma)$ at the point χ_ρ (cf [LM, Lemma 2.18]).

For a compact manifold W we use $\mathfrak{A}(W)$, $\mathfrak{B}(W)$, $\mathfrak{R}(W)$ and $\mathfrak{X}(W)$ to denote $\mathfrak{A}(\pi_1(W))$, $\mathfrak{B}(\pi_1(W))$, $\mathfrak{R}(\pi_1(W))$ and $\mathfrak{X}(\pi_1(W))$ respectively. When M is a hyperbolic knot manifold, let $\mathfrak{X}^{\mathrm{rg}}(M) \subset \mathfrak{X}(M) = \mathrm{Spec}(\mathfrak{B}(M))$ be the counterpart of $X^{\mathrm{rg}}(M) \subset X(M)$, that is, $\mathfrak{X}^{\mathrm{rg}}(M)$ is the union of the components of $\mathfrak{X}(M)$ corresponding to the $\mathbb{C}$-components of $X^{\mathrm{rg}}(M)$.

Proposition 3.3. *Let M be a hyperbolic knot manifold. Then $\mathfrak{X}^{\mathrm{rg}}(M)$ is essentially reduced.*

Proof. Let χ_ρ be the character of a discrete faithful representation of $\pi_1(M)$ and let X_1 be a $\mathbb{C}$-component of $X(M)$ containing χ_ρ. It is known that dim $X_1 = 1$ [CGLS, Proposition 1.1.1]. It is also known that dim $H^1(\pi_1(M), sl_2(\mathbb{C})_\rho) = 1$ (see [P]). Since ρ is an irreducible representation, the 1-coboundary $B^1(\pi_1(M), sl_2(\mathbb{C})_\rho)$ is 3-dimensional and thus the dimension of $Z^1(\pi_1(M), sl_2(\mathbb{C})_\rho)$ is 4 which is equal to the dimension of the $\mathbb{C}$-component R_1 of $R(M)$ which maps onto X_1 under the canonical surjective regular map tr : $R(M) \rightarrow X(M)$. That is we have dim $T_\rho(\mathfrak{R}(M))$ = dim $Z^1(\pi_1(M), sl_2(\mathbb{C})_\rho))$ = dim R_1 which means by Lemma 3.1 that ρ is a smooth point of the scheme $\mathfrak{R}(M)$. In turn we have dim $T_{\chi_\rho}(\mathfrak{X}(M))$ = dim $H^1(\pi_1(M), sl_2(\mathbb{C})_\rho)$ = dim X_1 which means by Lemma 3.1 again that χ_ρ is a smooth point of the scheme $\mathfrak{X}(M)$, that χ_ρ is contained in a unique irreducible component $\mathfrak{X}_1$ of $\mathfrak{X}(M)$ which is the scheme counterpart of the component X_1 and that $\mathfrak{X}_1$ is reduced. By Remark 3.2, the Aut($\mathbb{C}$)-orbit of $\mathfrak{X}_1$ consists of reduced components. As $\mathfrak{X}^{\mathrm{rg}}(M)$ consists of such orbits, each of its components is reduced. ◇

If M is the exterior of a knot in S^3, its set of abelian representations forms a unique component R_0 of $R(M)$ and $\dim R_0 = 3$. In fact R_0 is isomorphic, as a variety, to $SL_2(\mathbb{C})$. The image X_0 of R_0 in $X(M)$ under the quotient map tr is a component of $X(M)$ and $\dim X_0 = 1$. The proof of the following proposition is due to Joan Porti.

Proposition 3.4. *Let M be the exterior of a knot in a homology 3-sphere and let $\mathfrak{X}_0$ be the unique irreducible component of $\mathfrak{X}(M)$ corresponding to X_0. Then $\mathfrak{X}_0$ is reduced.*

Proof. Note that the meridian element μ generates the first homology of $H_1(M;\mathbb{Z}) = \mathbb{Z}$. Thus an abelian representation ρ of $\pi_1(M)$ is determined by the matrix $\rho(\mu)$. Now take a diagonal representation ρ of $\pi_1(M)$ and assume that $\rho(\mu) = \begin{pmatrix} \mathfrak{M} & 0 \\ 0 & \mathfrak{M}^{-1} \end{pmatrix}$ such that $\mathfrak{M} \neq \pm 1$ and $\mathfrak{M}^2$ is not a root of the Alexander polynomial of K. As $\dim X_0 = 1$, we just need to show, by Lemma 3.1, that for the diagonal representation ρ given above, we have $\dim T_{\chi_\rho}(\mathfrak{X}(M)) = 1$.

It is shown in the proof of [HP, Lemma 4.8] that $H^1(\pi_1(M), sl_2(\mathbb{C})_\rho) = H^1(\pi_1(M), \mathbb{C}_0)$ where $\mathbb{C}_0 = \mathbb{C}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is a trivial $\pi_1(M)$-module. Hence $\dim H^1(\pi_1(M), sl_2(\mathbb{C})_\rho) = 1$. For the given diagonal representation ρ, $B^1(\pi_1(M), sl_2(\mathbb{C})_\rho)$ is two dimensional. Hence we have $\dim Z^1(\pi_1(M), sl_2(\mathbb{C})_\rho) = 3$, which implies that the representation ρ is a smooth point of $\mathfrak{R}(M)$ since the component $R_0 = \mathrm{tr}^{-1}(X_0)$ is of dimension 3.

Now by [Si, Theorem 53 (3)], we have

$$\dim T_{\chi_\rho}(\mathfrak{X}(M)) = \dim T_0(H^1(\pi_1(M), sl_2(\mathbb{C})_\rho)//S_\rho)$$

where S_ρ is, in our current case, the group of diagonal matrices and it acts on $H^1(\pi_1(M), sl_2(\mathbb{C})_\rho$, in our current case, trivially (as the cohomology $H^1(\pi_1(M), sl_2(\mathbb{C})_\rho) = H^1(\pi_1(M), \mathbb{C}_0)$ is realized by cocycles taking values in diagonal matrices). Thus $\dim T_0(H^1(\pi_1(M), sl_2(\mathbb{C})_\rho)//S_\rho) = \dim H^1(\pi_1(M), \mathbb{C}_0) = 1$. $\diamondsuit$

Combining Propositions 3.3 and 3.4, we have

Corollary 3.5. *Let M be the exterior of a hyperbolic knot in a homology 3-sphere such that $X^{\mathrm{irr}}(M) = X^{\mathrm{rg}}(M)$. Then $\mathfrak{X}(M)$ is essentially reduced.*

4. Proof of Theorem 1.5 – a reduction

In this section we briefly review some background material and give an outline of the approach taken in [LT], from which we can specify the issues that we need to deal with in order to extend [LT, Theorem 1] to our current theorem, that is, we reduce Theorem 1.5 to Proposition 4.1.

4.1. **Recurrence polynomial.** For a knot K in S^3, let $J_{\mathrm{K},n}(t) \in \mathbb{Z}[t^{\pm 1}]$ denote the *n-colored Jones polynomial* of K with the zero framing, which is the sl_2-quantum invariant of the knot colored by the n-dimensional representations [RT]. We use the normalization so that for the unknot U,

$$J_{U,n}(t) = \frac{t^{2n} - t^{-2n}}{t^2 - t^{-2}}.$$

By defining $J_{\mathrm{K},-n}(t) := -J_{\mathrm{K},n}(t)$ and $J_{K,0} = 0$, one may treat $J_{\mathrm{K},n}(t)$ as a discrete function

$$J_{\mathrm{K},-}(t) : \mathbb{Z} \to \mathbb{Z}[t^{\pm 1}].$$

The quantum torus

$$\mathcal{T} = \mathbb{C}[t^{\pm 1}] \left\langle \mathfrak{M}^{\pm 1}, \mathfrak{L}^{\pm 1} \right\rangle / (\mathfrak{L}\mathfrak{M} - t^2\mathfrak{M}\mathfrak{L})$$

acts on the set of all functions $f : \mathbb{Z} \to \mathbb{C}[t^{\pm 1}]$ by

$$\mathfrak{M}f := t^{2n} f, \quad (\mathfrak{L}f)(t) := f(n+1).$$

Now the set

$$\mathcal{A}_{\mathrm{K}} := \{\alpha \in \mathcal{T} \mid \alpha J_{\mathrm{K},n}(t) = 0\},$$

is obviously a left ideal of $\mathcal{T}$, called the *recurrence ideal* of K. By [GL] $\mathcal{A}_{\mathrm{K}}$ is not the zero ideal for every knot K in S^3. The ring $\mathcal{T}$ can be extended to a principal left ideal domain $\widetilde{\mathcal{T}}$ by adding inverses of all polynomials in t and $\mathfrak{M}$. The extended left ideal $\widetilde{\mathcal{A}}_{\mathrm{K}} := \widetilde{\mathcal{T}}\mathcal{A}_{\mathrm{K}}$ is then generated by a single nonzero polynomial in $\widetilde{\mathcal{T}}$, which can be chosen to be of the form

$$\alpha_{\mathrm{K}}(t, \mathfrak{M}, \mathfrak{L}) = \sum_{i=0}^{m} a_i(t, \mathfrak{M})\mathfrak{L}^i,$$

with smallest total degrees in $t, \mathfrak{M}, \mathfrak{L}$ and with $a_0(t, \mathfrak{M}), ..., a_m(t, \mathfrak{M}) \in \mathbb{Z}[t, \mathfrak{M}]$ being coprime in $\mathbb{Z}[t, \mathfrak{M}]$. The polynomial $\alpha_{\mathrm{K}}(t, \mathfrak{M}, \mathfrak{L})$ is uniquely determined up to a sign and is called *the recurrence polynomial of* K. When the framing of K is 0, then $J_{K,n}(t) \in t^{2n-2}\mathbb{Z}[t^{\pm 4}]$ (see eg. [Le1], with our t equal to $q^{1/4}$ there). From here, it is not difficult to show that $\alpha_{\mathrm{K}}(t, \mathfrak{M}, \mathfrak{L})$ has only even powers in t and even powers in $\mathfrak{M}$, i.e. $a_i(t, \mathfrak{M}) \in \mathbb{Z}[t^2, \mathfrak{M}^2]$ (see [Le3, Proposition 5.6]. It follows that $\alpha_{\mathrm{K}}(1, \mathfrak{M}, \mathfrak{L}) = \alpha_{\mathrm{K}}(-1, \mathfrak{M}, \mathfrak{L})$.

Now the AJ-conjecture asserts that for every knot K in S^3, $\alpha_{\mathrm{K}}(\pm 1, \mathfrak{M}, \mathfrak{L})$ is equal to the A-polynomial of K, up to a factor of a polynomial in $\mathfrak{M}$, see [G] and also [FGL, Le2, LT, Le3].

4.2. **Kauffman bracket skein module.** For an oriented 3-manifold W, we let $\mathcal{S}(W)$ denote the Kauffman bracket skein module of W over $\mathbb{C}[t^{\pm 1}]$, which is the quotient module of the free $\mathbb{C}[t^{\pm 1}]$-module generated by the set of isotopy classes of framed links in W modulo the well known Kauffman skein relations, see e.g. [PS, Le2, LT]. A fundamental fact is that when $\mathcal{S}(W)$ is specialized at $t = -1$ (which we denote by $\mathfrak{s}(W)$, i.e. $\mathfrak{s}(W) = \mathcal{S}(W)/(t+1)$), it acquires a ring structure and is naturally isomorphic as a ring to the universal character ring of $\pi_1(W)$, i.e.

$$\mathfrak{s}(W) = \mathfrak{B}(W).$$

So $\mathfrak{s}(W)/\sqrt{(0)}$ is isomorphic to the coordinate ring of $X(W)$ (see [B] [PS]). For the exterior M of a knot K in S^3, we shall simply write $\mathcal{S}$ for $\mathcal{S}(M)$ and $\mathfrak{s}$ for $\mathfrak{s}(M)$.

If F is an oriented surface, we define $\mathcal{S}(F) := \mathcal{S}(F \times [0,1])$. Then $\mathcal{S}(F)$ has a natural algebra structure, where the product of two framed links L_1, L_2 is obtained by placing L_1 atop L_2. For a torus T^2, $\mathcal{S}(T^2)$ can be identified, as an $\mathbb{C}[t^{\pm 1}]$-algebra, with

$$\mathcal{T}^\sigma := \{f \in \mathcal{T}; \sigma(f) = f\}$$

where $\sigma : \mathcal{T} \to \mathcal{T}$ is the involution defined by $\sigma(\mathfrak{M}) = \mathfrak{M}^{-1}$ and $\sigma(\mathfrak{L}) = \mathfrak{L}^{-1}$ (see [FG]).

If M is the exterior of knot K in S^3, there is a natural map

$$\Theta : \mathcal{S}(\partial M) = \mathcal{T}^\sigma \to \mathcal{S} = \mathcal{S}(M) \tag{4.2.1}$$

induced by the inclusion $\partial M \hookrightarrow M$. Then $\mathcal{P} := ker(\Theta)$ is called the *quantum peripheral ideal of* K and by [FGL] and [G2], $\mathcal{P} \subset \mathcal{A}_{\rm K}$ (see also [LT, Corollary 1.2]).

4.3. **Dual construction of A-polynomial.** On the other hand, there is a dual construction of the A-polynomial of a knot K in S^3. Let $\mathfrak{t} := \mathbb{C}[\mathfrak{M}^{\pm 1}, \mathfrak{L}^{\pm 1}]$, which is the function ring of $(\mathbb{C}^\times)^2$, and let $\mathfrak{t}^\sigma := \{f \in \mathfrak{t}; \sigma(f) = f\}$, which is the function ring of $X(\partial M)$. The restriction map $\iota^* : X(M) \to X(\partial M)$ induces a ring homomorphism between coordinate rings

$$\theta : \mathbb{C}[X(\partial M)] = \mathfrak{t}^\sigma \to \mathbb{C}[X(M)]. \tag{4.3.1}$$

Let $\mathfrak{p} := ker(\theta)$, which is called *the classical peripheral ideal of the knot* K. Now extend $\mathfrak{t}$ naturally to the principal ideal domain $\tilde{\mathfrak{t}} := \mathbb{C}(\mathfrak{M})[\mathfrak{L}^{\pm 1}]$ where $\mathbb{C}(\mathfrak{M})$ is the fractional field of $\mathbb{C}[\mathfrak{M}]$. Then the extended ideal $\tilde{\mathfrak{p}} := \tilde{\mathfrak{t}}\mathfrak{p}$ of $\mathfrak{p}$ in $\tilde{\mathfrak{t}}$ is generated by a single polynomial which can be normalized to be of the form

$$B_{\rm K}(\mathfrak{M}, \mathfrak{L}) = \sum_{i=0}^{m} b_i(\mathfrak{M}) \mathfrak{L}^i,$$

with smallest total degree and with $b_0(\mathfrak{M}), ..., b_m(\mathfrak{M}) \in \mathbb{Z}[\mathfrak{M}]$ being coprime in $\mathbb{Z}[\mathfrak{M}]$. So $B_{\rm K}(\mathfrak{M}, \mathfrak{L})$ is uniquely defined up to a sign. The polynomial $B_{\rm K}(\mathfrak{M}, \mathfrak{L})$ is called *the B-polynomial of K* and is equal to the A-polynomial $A_{\rm K}(\mathfrak{M}, \mathfrak{L})$ divided by its $\mathfrak{M}$-factor (see [LT, Corollary 2.3]).

Note that the universal character ring of ∂M is reduced, so we have $\mathfrak{s}(\partial M) = \mathbb{C}[\partial M] = \mathfrak{t}^\sigma$. Specializing (4.2.1) at $t = -1$, we get

$$\theta : \mathfrak{t}^\sigma \to \mathfrak{s} = \mathfrak{s}(M) \tag{4.3.2}$$

in which the map θ is the same one given in (4.3.1).

4.4. **Localized skein module and reduction of Theorem 1.5.** Note that the inclusion map $\partial M \subset M$ also induces a left $\mathcal{S}(\partial M) = \mathcal{T}^\sigma$-module structure on $\mathcal{S} = \mathcal{S}(M)$. Let $D := \mathbb{C}[t^{\pm 1}, \mathfrak{M}^{\pm 1}]$, $D^\sigma := \{f \in D; \sigma(f) = f\}$ where σ is the involution defined by $\sigma(\mathfrak{M}) = \mathfrak{M}^{-1}$ and $\overline{D}$ the localization of D at $(1 + t)$, i.e.

$$\overline{D} := \{f/g; f, g \in D, g \notin (1 + t)D\}.$$

Then we may consider $\mathcal{S}$, as well as $\mathcal{T}^\sigma$, as a left D^σ-modules as D^σ is contained in $\mathcal{T}^\sigma$. Now let

$$(\overline{\mathcal{T}} \xrightarrow{\overline{\Theta}} \overline{\mathcal{S}}) := (\mathcal{T}^\sigma \xrightarrow{\Theta} \mathcal{S}) \otimes_{D^\sigma} \overline{D}, \quad (\bar{\mathfrak{t}} \xrightarrow{\bar{\theta}} \bar{\mathfrak{s}}) := (\mathfrak{t}^\sigma \xrightarrow{\theta} \mathfrak{s}) \otimes_{\mathbb{C}[\mathfrak{M}^{\pm 1}]^\sigma} \mathbb{C}(\mathfrak{M}).$$

We shall consider $\overline{\mathcal{S}}$ as a left $\overline{D}$-module and call it *the localized skein module of* M. The following commutative diagram:

$$\begin{tikzcd}
\overline{\mathcal{T}} \arrow[r, "\overline{\Theta}"] \arrow[d, "\epsilon"'] & \overline{\mathcal{S}} \arrow[d, "\epsilon"] \\
\overline{\mathfrak{t}} \arrow[r, "\overline{\theta}"] & \overline{\mathfrak{s}}
\end{tikzcd}$$

is obtained in [LT] as Lemma 3.2, where the vertical maps are the natural projections $\mathcal{M} \to \mathcal{M}/(t+1)$, for $\mathcal{M} = \overline{\mathcal{T}}$ and $\mathcal{M} = \overline{\mathcal{S}}$. We claim that the proof of Theorem 1.5 can be reduced to the proof of the following proposition:

Proposition 4.1. *Let* M *be the exterior of a hyperbolic knot in* S^3*. If* $X^{\mathrm{rg}}(M) = X^{\mathrm{irr}}(M)$ *(or equivalently,* $\overline{X}^{\mathrm{rg}}(M) = \overline{X}^{\mathrm{irr}}(M)$*), and the two discrete faithful characters of* $\overline{X}(M)$ *lie in the same component of* $\overline{X}(M)$*, then*
(1) the ring $\overline{\mathfrak{s}}$ *is reduced, and*
(2) the map $\bar{\theta}$ *is surjective.*

Assuming Proposition 4.1, we may finish the proof of Theorem 1.5 as follows. By Condition (1) of Theorem 1.5, we have Proposition 4.1. Combining Proposition 4.1 with Condition (3) of Theorem 1.5, we may apply [LT, Corollary 3.6] to have

$$\alpha_{\mathrm{K}}(-1, \mathfrak{M}, \mathfrak{L}) | B_{\mathrm{K}}(\mathfrak{M}, \mathfrak{L}) \in \mathbb{Z}[\mathfrak{M}^2, \mathfrak{L}].$$

Condition (1) of Theorem 1.5 and Proposition 2.7 together imply $A_{\mathrm{K}}(\mathfrak{M}, \mathfrak{L}) = (\mathfrak{L}-1)\widehat{A}_{\mathrm{K}}(\mathfrak{M}, \mathfrak{L}) = B_{\mathrm{K}}(\mathfrak{M}, \mathfrak{L})$ and $\widehat{A}_{\mathrm{K}}(\mathfrak{M}, \mathfrak{L})$ are balanced-irreducible in $\mathbb{Z}[\mathfrak{M}^2, \mathfrak{L}]$. It's known that $L-1$ is a factor $\alpha_{\mathrm{K}}(-1, \mathfrak{M}, \mathfrak{L})$ ([Le2, Proposition 2.3]). By Condition (2) of Theorem 1.5 and [LT, Lemma 3.9] we know that the $\mathfrak{L}$-degree of $\alpha_{\mathrm{K}}(-1, \mathfrak{M}, \mathfrak{L})$ is greater than or equal to 2. As $\alpha_{\mathrm{K}}(-1, \mathfrak{M}, \mathfrak{L})$ is also balanced (see the lemma below) and its coefficients are all integers, the polynomial $\widehat{\alpha}_{\mathrm{K}}(-1, \mathfrak{M}, \mathfrak{L}) := \alpha_{\mathrm{K}}(-1, \mathfrak{M}, \mathfrak{L})/(\mathfrak{L}-1)$ is also balanced, belongs to $\mathbb{Z}[\mathfrak{M}^2, \mathfrak{L}]$, and has $\mathfrak{L}$-degree ≥ 1. Since $\widehat{\alpha}_{\mathrm{K}}(-1, \mathfrak{M}, \mathfrak{L})$ divides $\widehat{A}_{\mathrm{K}}(\mathfrak{M}, \mathfrak{L})$ which is balanced-irreducible in $\mathbb{Z}[\mathfrak{M}^2, \mathfrak{L}]$, the two polynomials must be equal (up to sign). Hence $\alpha_{\mathrm{K}}(-1, \mathfrak{M}, \mathfrak{L}) = A_{\mathrm{K}}(\mathfrak{M}, \mathfrak{L})$ (up to sign).

Lemma 4.2. *Let* $\alpha_K(t, \mathfrak{M}, \mathfrak{L})$ *be the normalized recurrence polynomial of a knot* K *in* S^3*. Then* $\alpha_K(-1, \mathfrak{M}, \mathfrak{L})$ *is a balanced polynomial.*

Proof. By [G2, Theorem 1.4], the recurrence (left) ideal $\mathcal{A}_{\mathrm{K}}$ of K is invariant under the involution σ of the quantum torus $\mathcal{T}$ defined by $\sigma(\mathfrak{M}) = \mathfrak{M}^{-1}, \sigma(\mathfrak{L}) = \mathfrak{L}^{-1}$. Hence $\sigma(\alpha_{\mathrm{K}}(t, \mathfrak{M}, \mathfrak{L})) = \alpha_{\mathrm{K}}(t, \mathfrak{M}^{-1}, \mathfrak{L}^{-1})$ is contained in $\mathcal{A}_{\mathrm{K}}$. Suppose the $\mathfrak{L}$-degree of $\alpha_{\mathrm{K}}(t, \mathfrak{M}, \mathfrak{L})$ is m. Then using the relation $\mathfrak{L}\mathfrak{M} = t^2\mathfrak{M}\mathfrak{L}$, one can easily see that there is a monomial $t^{2a}\mathfrak{M}^b\mathfrak{L}^m$, for some integers a, b, with $b \geq 0$, such that $t^{2a}\mathfrak{M}^b\mathfrak{L}^m\alpha_{\mathrm{K}}(t, \mathfrak{M}^{-1}, \mathfrak{L}^{-1})$ is contained in $\mathbb{Z}[t, \mathfrak{M}, \mathfrak{L}]$ of $\mathfrak{L}$-degree m with relatively prime coefficients with respect to the variable $\mathfrak{L}$. It follows that $t^{2a}\mathfrak{M}^b\mathfrak{L}^m\alpha_{\mathrm{K}}(t, \mathfrak{M}^{-1}, \mathfrak{L}^{-1})$ is also a generator of $\tilde{\mathcal{A}}_{\mathrm{K}}$ and by the unique normalized form of such generator, we have

$$t^{2a}\mathfrak{M}^b\mathfrak{L}^m\alpha_{\mathrm{K}}(t, \mathfrak{M}^{-1}, \mathfrak{L}^{-1}) = \alpha_{\mathrm{K}}(t, \mathfrak{M}, \mathfrak{L})$$

up to sign. Hence $\mathfrak{M}^b\mathfrak{L}^m\alpha_{\mathrm{K}}(-1, \mathfrak{M}^{-1}, \mathfrak{L}^{-1}) = \alpha_{\mathrm{K}}(-1, \mathfrak{M}, \mathfrak{L})$ up to sign, i.e. $\alpha_{\mathrm{K}}(-1, \mathfrak{M}, \mathfrak{L})$ is balanced. $\diamondsuit$

5. PROOF OF PROPOSITION 4.1

Under the assumptions of Proposition 4.1, we know, by Corollary 3.5, that the character scheme $\mathfrak{X}(M)$ is essentially reduced, i.e. the universal character ring $\mathfrak{B}(M)$ is essentially reduced. We may assume that

$$\mathfrak{B}(M)=\mathbb{C}[x_1,...,x_n]/I$$

where I is an ideal $\mathbb{C}[x_1,...,x_n]$. We may also assume that the ideal I has an irredundant primary decomposition

$$I=\bigcap_{j=0}^{m}Q_j$$

such that $Q_0,Q_1,...,Q_k$ are the isolated components of I, $Q_{k+1},...,Q_m$ embedded components, with Q_0 defining the abelian component X_0 of $X(M)$, $Q_1,...,Q_k$ defining the components $X_1,...,X_k$ of $X^{\rm rg}(M)$ respectively. We have that $Q_0,Q_1,...,Q_k$ are prime ideals. As $X_0,X_1,...,X_k$ are all 1-dimensional, the zero locus of each $Q_j, j=k+1,...,m$, is a point, and thus $\sqrt{Q_j}=P_j$ is a maximal ideal, i.e. for some point $(a_1,...,a_n)$ in $X_0\cup X_1\cup...\cup X_k$, $P_j=(x_1-a_1,...,x_n-a_n)$.

Let $R_n=\mathbb{C}[x_1,...,x_n]$. Then $\mathfrak{B}(M)=R_n/I$. We may assume that the coordinate $x=x_1$ in $\mathfrak{B}(M)=\mathbb{C}[x_1,...,x_n]/I$ represents the function $x:\mathfrak{X}(M)\to\mathbb{C}$ given by $x(\chi_\rho)=\operatorname{trace}(\rho(\mu))$ where μ is a meridian of $\pi_1(M)$. Let $S=\mathbb{C}[x]\setminus\{0\}$, which is a multiplicative subset of $\mathbb{C}[x]$. If $\mathcal{M}$ is a $\mathbb{C}[x]$-module, let $S^{-1}\mathcal{M}$ denote the localization of $\mathcal{M}$ with respect to S. Note that every ideal J in R_n is a $\mathbb{C}[x]$-module and so is R_n/J.

Lemma 5.1. *For $j>k$ we have*

$$S^{-1}Q_j=S^{-1}R_n.$$

Proof. For $j>k$, $P_j=(x_1-a_1,\dots,x_n-a_n)$ is a maximal ideal. Since Q_j is primary and $\sqrt{Q_j}=P_j$, we have $P_j^d\subset Q_j$ for some integer $d>0$. It follows that $(x-a_1)^d\in Q_j$. Since $(x-a_1)^d\in S$, we have $1\in S^{-1}Q_j$. Hence $S^{-1}Q_j=S^{-1}R_n$. ◇

Hence $S^{-1}I=\bigcap_{j=0}^{m}S^{-1}Q_j=\bigcap_{j=0}^{k}S^{-1}Q_j$. As Q_j, $j\le k$, are prime, $S^{-1}I=\bigcap_{j=0}^{k}S^{-1}Q_j$ is a prime decomposition of the ideal $S^{-1}I$ in $S^{-1}R_n$. Therefore $S^{-1}(R_n/I)=S^{-1}R_n/S^{-1}I$ is a reduced ring. By definition,

$$S^{-1}(R_n/I)=\mathfrak{B}(M)\otimes_{\mathbb{C}[x]}\mathbb{C}(x)=\mathfrak{s}\otimes_{\mathbb{C}[x]}\mathbb{C}(x)=\mathfrak{s}\otimes_{\mathbb{C}[\mathfrak{M}+\mathfrak{M}^{-1}]}\mathbb{C}(\mathfrak{M}+\mathfrak{M}^{-1}).$$

Taking tensor product of this with $\mathbb{C}(\mathfrak{M})$, we have

$$(\mathfrak{s}\otimes_{\mathbb{C}[x]}\mathbb{C}(x))\otimes_{\mathbb{C}(x)}\mathbb{C}(\mathfrak{M})=\mathfrak{s}\otimes_{\mathbb{C}[\mathfrak{M}^{\pm1}]^\sigma}\mathbb{C}(\mathfrak{M})=\bar{\mathfrak{s}}$$

which is still reduced. This proves part (1) of Proposition 4.1.

From the above proof, we also get

$$\mathfrak{s}\otimes_{\mathbb{C}[x]}\mathbb{C}(x)=\mathbb{C}[X(M)]\otimes_{\mathbb{C}[x]}\mathbb{C}(x)$$

because $\mathbb{C}[X(M)]=R_n/I'$ with $I'=\bigcap_{j=0}^{k}Q_j$ and $S^{-1}I=S^{-1}I'$. The restriction of the function x on X_1 is non-constant and thus is non-constant on X_j for each $j=1,...,k$. It is easy to see that x is also non-constant on X_0. Hence a similar proof as that of [LT, Lemma 3.8] shows that

$$\mathbb{C}[X(M)] \otimes_{\mathbb{C}[x]} \mathbb{C}(x) = \prod_{j=0}^{k} \mathbb{C}[X_j] \otimes_{\mathbb{C}[x]} \mathbb{C}(x).$$

Note that $\mathbb{C}[X_j] \otimes_{\mathbb{C}[x]} \mathbb{C}(x)$ is isomorphic to the field of rational functions on X_j for each $j = 0, 1..., k$, (by [LT, Lemma 3.7]).

Recall that $\iota^* : X(M) \to X(\partial M)$ is the restriction map which induces the ring homomorphism $\theta : \mathbb{C}[X(\partial M)] \to \mathbb{C}[X(M)]$. Also recall that Y_j is the Zariski closure of $\iota^*(X_j)$ in $X(\partial M)$, $j = 0, 1, ..., k$. As x is non-constant on each Y_j, $\mathbb{C}[Y_j] \otimes_{\mathbb{C}[x]} \mathbb{C}(x)$ is isomorphic to the field of rational functions on Y_j for each $j = 0, 1..., k$. By Theorem 1.2, $\iota^* : X_j \to Y_j$ is a birational map for each $j = 1, ..., k$. When $j = 0$, $\iota^* : X_0 \to Y_0$ is also a birational map, which is an elementary fact. Hence the map ι^* induces an isomorphism

$$\mathbb{C}[Y_j] \otimes_{\mathbb{C}[x]} \mathbb{C}(x) \to \mathbb{C}[X_j] \otimes_{\mathbb{C}[x]} \mathbb{C}(x)$$

for each $j = 0, 1, ..., k$. As Y_j, $j = 0, 1, ..., k$, are distinct curves in $X(\partial M)$ by Theorem 1.2, ι^* induces the isomorphism

$$\prod_{j=0}^{k} \mathbb{C}[Y_j] \otimes_{\mathbb{C}[x]} \mathbb{C}(x) \to \prod_{j=0}^{k} \mathbb{C}[X_j] \otimes_{\mathbb{C}[x]} \mathbb{C}(x) = \mathbb{C}[X(M)] \otimes_{\mathbb{C}[x]} \mathbb{C}(x)$$

which implies that the map

$$\mathbb{C}[X(\partial M)] \otimes_{\mathbb{C}[x]} \mathbb{C}(x) \to \mathbb{C}[X(M)] \otimes_{\mathbb{C}[x]} \mathbb{C}(x)$$

induced by ι^* is surjective since $Y_0 \cup Y_1 \cup \cdots \cup Y_k$ is a subvariety of $X(\partial M)$. Taking tensor product of this map with $\mathbb{C}(\mathfrak{M})$ over $\mathbb{C}(x)$ and noting that $\mathbb{C}[X(\partial M)] = \mathfrak{t}^\sigma$ and $\mathbb{C}[X(M)] \otimes_{\mathbb{C}[x]} \mathbb{C}(x) = \mathfrak{s} \otimes_{\mathbb{C}[x]} \mathbb{C}(x)$, we get the map

$$\mathfrak{t}^\sigma \otimes_{\mathbb{C}[\mathfrak{M}^{\pm 1}]^\sigma} \mathbb{C}(\mathfrak{M}) \to \mathfrak{s} \otimes_{\mathbb{C}[\mathfrak{M}^{\pm 1}]^\sigma} \mathbb{C}(\mathfrak{M}) \tag{5.0.1}$$

which is still surjective. Now one can check that (5.0.1) is precisely the map

$$\overline{\theta} : \overline{\mathfrak{t}} \to \overline{\mathfrak{s}}.$$

Part (2) of Proposition 4.1 is proved.

6. Proof of Theorems 1.6 and 1.10

Let M be the exterior of a hyperbolic 2-bridge knot in S^3. We call a character $\bar{\chi}_{\bar{\rho}} \in \overline{X}(M)$ (resp. $\chi_\rho \in X(M)$) *dihedral* if it is the character of a dihedral representation, i.e. a representation whose image is a dihedral group (resp. a binary dihedral group). It was shown in [Ta, Section 5.3] (see also [BB, Appendix A]) that any dihedral character of $\overline{X}(M)$ (and of $X(M)$) is a smooth point and thus is contained in a unique $\mathbb{C}$-component of $\overline{X}(M)$ (resp. $X(M)$). It was also shown in [Ta, Section 5.3] that every $\mathbb{C}$-component of $\overline{X}^{\text{irr}}(M)$ contains a dihedral character.

Since a dihedral character of $\overline{X}^{\mathrm{irr}}(M)$ is real valued, it is a fixed point of the τ-action (the complex conjugation action given in Subsection 2.1). It follows that every $\mathbb{C}$-component of $\overline{X}^{\mathrm{irr}}(M)$ is invariant under the τ-action. Hence in particular the two discrete faithful characters of $\overline{X}(M)$ are contained in the same $\mathbb{C}$-component of $\overline{X}(M)$. Thus Theorem 1.6 (1) is proved.

By [BZ3, Lemma 5.5 (3)], any dihedral character in $X(M)$ is a fixed point of the ϵ-action (recall its definition in Subsection 2.4). It follows that every $\mathbb{C}$-component of $X^{\mathrm{irr}}(M)$ is invariant under the ϵ-action, as well as the τ-action, which implies that all the four discrete faithful characters of $X(M)$ are contained in the same $\mathbb{C}$-component of $X(M)$, say X_1. Therefore $X^{\mathrm{rg}}(M)$ is the $\mathrm{Aut}(\mathbb{C})$-orbit of X_1 and thus is $\mathbb{Q}$-irreducible. This proves part (2) of Theorem 1.6.

Now we proceed to prove Theorem 1.10. Let $K = \mathfrak{b}(p,q)$ be a nontrivial 2-bridge knot and M the knot exterior of K. If K is not hyperbolic, then it is a torus knot and its A-polynomial is $\widehat{A}_{\mathrm{K}}(\mathfrak{M},\mathfrak{L}) = \mathfrak{L}\mathfrak{M}^{2p}+1$, and Theorem 1.10 clearly holds. We will assume now that $K = \mathfrak{b}(p,q)$ is hyperbolic.

It was proved in [Ta] that $X^{\mathrm{irr}}(M)$ is positive dimensional (which actually holds for any nontrivial knot) and every $\mathbb{C}$-component X_0 in $X^{\mathrm{irr}}(M)$ is 1-dimensional and on X_0 the function f_μ is non-constant (due to the fact that every 2-bridge knot is a small knot and its meridian slope is not a boundary slope). Thus the restriction of X_0 to the boundary torus is one dimensional and thus contributes a balanced-irreducible factor $P_0(\mathfrak{M},\mathfrak{L})$ of $A_{\mathrm{K}}(\mathfrak{M},\mathfrak{L})$. We may assume that this factor is not $\mathfrak{L}-1$ since $\widehat{A}_{\mathrm{K}}(\mathfrak{M},\mathfrak{L})$ does not have this factor. Therefore we may assume that f_λ is also non-constant on X_0. Indeed as recalled above, X_0 contains a dihedral character χ_{ρ_0} and since any dihedral representation ρ_0 has to send λ to the identity matrix we have $f_\lambda(\chi_{\rho_0})=0$, which implies that if f_λ were constant on X_0 it would be constantly zero on X_0 and X_0 would contribute the factor $\mathfrak{L}-1$ to $A_{\mathrm{K}}(\mathfrak{M},\mathfrak{L})$.

Furthermore it was shown in [Ta] that over X_0,

$$\begin{aligned}\text{degree}(f_{\mu^2})\,|_{X_0} &= 2\text{degree}(f_\mu)\,|_{X_0} \\ &= \text{degree}(f_\mu)\,|_{X_0} + 2(\text{the number of dihedral characters in } X_0),\end{aligned} \tag{6.0.1}$$

due to the facts: every zero point of f_μ is a zero point of f_{μ^2}, each of f_μ and f_{μ^2} blows up at an ideal point of X_0, a point of X_0 is a zero point of f_{μ^2} but is not a point of f_μ if and only if it is a dihedral character, the zero degree of f_{μ^2} at a dihedral character in X_0 is 2. As any dihedral character in X_0 is also a zero point of f_λ of zero degree 2 (since f_λ is assumed non-constant on X_0) and every zero point of f_μ is a zero point of f_λ, it follows that

$$\text{degree}(f_\lambda)\,|_{X_0} \geq \text{degree}(f_{\mu^2})\,|_{X_0} = 2\text{degree}(f_\mu)\,|_{X_0}\,. \tag{6.0.2}$$

It also follows from (6.0.1) that

$$\text{degree}(f_\mu)\,|_{X_0} = 2(\text{the number of dihedral characters in } X_0). \tag{6.0.3}$$

On the other hand, the argument of [BZ2, Proposition 6.6] can be adapted to show that

$$\begin{aligned}\text{degree}(f_\mu)\,|_{X_0} &= 2d(\mathfrak{L}\text{-degree of } P_0(\mathfrak{M},\mathfrak{L})), \\ \text{degree}(f_\lambda)\,|_{X_0} &= 2d(\mathfrak{M}\text{-degree of } P_0(\mathfrak{M},\mathfrak{L})),\end{aligned} \tag{6.0.4}$$

where d is the degree of the boundary map ι^* restricted on X_0. Combining (6.0.2) and (6.0.4) we get

$$(\mathfrak{M}\text{-degree of } P_0(\mathfrak{M},\mathfrak{L})) \geq 2(\mathfrak{L}\text{-degree of } P_0(\mathfrak{M},\mathfrak{L})).$$

As the $\mathfrak{M}$-degree (resp. the $\mathfrak{L}$-degree) of $\widehat{A}_{\mathrm{K}}(\mathfrak{M},\mathfrak{L})$ is the sum of the $\mathfrak{M}$-degrees (resp. the $\mathfrak{L}$-degrees) of the balanced irreducible factors of $\widehat{A}_{\mathrm{K}}(\mathfrak{M},\mathfrak{L})$, part (1) of Theorem 1.10 follows.

Now assume that the given hyperbolic 2-bridge knot $\mathfrak{b}(p,q)$ has p prime. As recalled in the introduction section, the Riley polynomial of the knot is $\mathbb{Q}$-irreducible, i.e. $X^{\mathrm{irr}}(M)$ is $\mathbb{Q}$-irreducible. As the knot is assumed to be hyperbolic, we have $X^{\mathrm{irr}}(M) = X^{\mathrm{rg}}(M)$, which we assume to have k $\mathbb{C}$-components $X_1, ..., X_k$. By Theorems 1.2 and 1.6 the boundary restriction map ι^* has degree one on each of $X_1, ..., X_k$ and the images of $X_1, ..., X_k$ under ι^* are distinct curves in $X(\partial M)$, which imply that $X_1, ..., X_k$ contribute k distinct balanced-irreducible factors $P_1(\mathfrak{M},\mathfrak{L}), ..., P_k(\mathfrak{M},\mathfrak{L})$ to $\widehat{A}_{\mathrm{K}}(\mathfrak{M},\mathfrak{L})$ and

$$\widehat{A}_{\mathrm{K}}(\mathfrak{M},\mathfrak{L}) = P_1(\mathfrak{M},\mathfrak{L}) \cdots P_k(\mathfrak{M},\mathfrak{L}).$$

As mentioned in the proof of Theorem 1.4 that for each nontrivial element $\gamma \in \pi_1(\partial M)$, f_γ is non-constant on each X_j, $j = 1, ..., k$. Hence by (6.0.3) and (6.0.4) replacing X_0 there by X_j, $j = 1, ..., k$, we have

$$\begin{aligned} &\mathfrak{L}\text{-degree of } \widehat{A}_{\mathrm{K}}(\mathfrak{M},\mathfrak{L}) = \textstyle\sum_{j=1}^{k} \mathfrak{L}\text{-degree of } P_j(\mathfrak{M},\mathfrak{L}) \\ &= \tfrac{1}{2}\textstyle\sum_{j=1}^{k} \text{degree}(f_\mu)\,\big|_{X_j} \\ &= \textstyle\sum_{j=1}^{k} \text{the number of dihedral characters in } X_j \\ &= \text{the number of dihedral characters in } X(M). \end{aligned}$$

On the other hand the number of dihedral characters in $X(M)$ (for any 2-bridge knot $\mathfrak{b}(p,q)$) is precisely $(p-1)/2$ ([K, Theorem 10]). Part (2) of Theorem 1.10 is proved.

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