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On denominators of the Kontsevich integral and the universal perturbative invariant of 3-manifolds

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Abstract. The integrality of the Kontsevich integral and perturbative invariants is discussed. It is shown that the denominator of the degree n part of the Kontsevich integral of any knot or link is a divisor of $(2!3!\dots n!)^4(n+1)!$. We prove this by establishing the existence of a Drinfeld's associator in the space of chord diagrams with special denominators. We also show that the denominator of the degree n part of the universal perturbative invariant of homology 3-spheres is not divisible by any prime greater than $2n+1$.

0. Introduction

The aim of this paper is to study the denominators of the Kontsevich integral and the universal perturbative invariant (see [LMO]) of homology 3-spheres. The Kontsevich integral is a very interesting knot invariant (see [Ko1]) which contains in itself all finite type invariants, as well as the Jones polynomial and its numerous generalizations (see [BL,BN1]). The universal perturbative invariant contains in itself all finite type invariants of integral homology 3-spheres (see [Le]).

The Kontsevich integral has values in a graded algebra of chord diagrams and is expressed by a formula involving iterated integrals. These iterated integrals, when calculated separately, are mostly transcendental numbers, and can be expressed as rational linear combinations of multiple zeta values. However, if we collect terms together using the relations between chord diagrams, then J. Mu-

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rakami and the author showed that the coefficients are rational, see [LM2]. The proof follows Drinfeld's work on quasi-Hopf algebras [Dr2].

In many problems concerning the Kontsevich integrals and quantum and perturbative invariants of 3-manifolds, one needs to know the prime factors of the denominators of the Kontsevich integral. Examples of such problems are the conjecture 7.3 in [LMO] about the relation between perturbative and quantum invariants of 3-manifolds and the conjecture of Lawrence in [Law] about the integrality and p-adic convergence of perturbative invariants (see also [LR]). Here we show that the denominator of the degree n part of the Kontsevich integral is a divisor of $(2!\dots n!)^4(n+1)!$, hence it can not have any prime factor greater than $n+1$. This result plays important role in Ohtsuki's proof (see [Oh2]) of the conjecture 7.3 of [LMO] for the sl_2 case. The main results of this paper will help to investigate finite type invariants of knots and 3-manifolds with values in fields of positive characteristic. For example, our main results play important role in a recent work of L. Freidel [Fre], where mod p invariants (p prime) of 3-manifolds are defined.

In the joint work [LMO] with Murakami and Ohtsuki, using the Kontsevich integral and the Kirby calculus, we constructed an invariant Ω of 3-manifolds with values in a graded algebra of 3-valent graphs. We call it a universal perturbative invariant. Later the author in [Le] showed that Ω is a universal finite type invariant of integral homology 3-spheres, hence it plays the role of the Kontsevich integral for homology 3-spheres. For the theory of finite type invariants of homology 3-spheres see [Oh1,GO]. Here we show that the denominator of the degree n part of Ω of rational homology 3-spheres does not have prime factor greater than $2n+1$. This result is closely related to the integrality property of quantum invariants which says that quantum invariants (see [Tu]) of homology 3-spheres at prime root of unity are cyclotomic integer. This had been conjectured by Kontsevich, and was proved in various cases by Murakami, Takata-Yokota, Masbaum-Wenzl (see [Mur, TY, MW]).

The idea of the proof of the main result can be explained as follows. First we reduce the proof to establishing the existence of an *associator* (introduced by Drinfeld), whose denominator has some specific properties. Associators are solution of a system of equations, important among them are the so-called pentagon and hexagon equations. The well-known Knizhnik-Zamolodchikov associator, found by Drinfeld [Dr1], is not good, since its coefficients are not even rational. An explicit formula for this associator is given in [LM2]. So we search for another associator following Drinfeld's perturbative method in [Dr2] which was adapted to the chord diagram case by Bar-Natan [BN2]. In this method, one first finds an associator up to degree n, and then tries to extend it to degree $n+1$. Drinfeld observed that the obstruction to the extension is in the cohomology of a certain complex. He then showed that the cohomology is equal to 0, hence there is no obstruction at all. In Drinfeld's and Bar-Natan's papers, the mentioned cohomology groups vanish, over the rationals. We follow Drinfeld's method, trying to solve the hexagons and pentagons equations and keeping track of the denominators. This leads to the problem of calculating the cohomology groups over the integers. It turns out that this cohomology group is a torsion group, annihilated by $(n+1)!$ in degree n. This is a main technical result, and is rather difficult. Using this result we can estimate the denominator in each step.

There is, however, another difficulty to overcome. In Drinfeld's and Bar-Natan's papers, in order to solve the hexagon and pentagon equation, one has to assume some freedom for the so-called R-matrix. But in order to get the Kontsevich integral, the R-matrix must be fixed and equal to the simplest one. The purely combinatorial method (in Drinfeld's and Bar Natan's papers) to solve the hexagon and pentagon equations does *not* work if the R-matrix is fixed. We remedy this by means of Lemma 3.1 whose proof utilizes the uniqueness of the associator up to gauge transformations (see [LM2]) and the existence of a special associator (proved in [Dr2]; the proof used analysis). Lemma 3.1 not only helps us to overcome the difficulty, but also enables us to reduce the number of steps in the search for associators (and hence reduce the denominators).

The paper is organized as follows. In §1 we recall basic definitions of Chinese character diagrams (chord diagrams). In §2 we discuss the cobar complex of Chinese character diagrams. Associators are discussed in §3. We proved the main result about the existence of an associator with special denominators in §4. The results about the denominators of the Kontsevich integral and Ω are proved in §5 and §6. Variations of the definition of the universal perturbative invariant Ω are discussed in §6. Finally in §7 we prove some technical results.

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1. Chinese character diagrams

1.1. Preliminaries. A *uni-trivalent graph* is a graph every vertex of which is either univalent or trivalent. In this paper we always assume that each connected component of a uni-trivalent graph contains at least one univalent vertex. A uni-trivalent graph is *vertex-oriented* if at each trivalent vertex a cyclic order of edges is fixed. Trivalent vertices are also called *internal vertices*, and univalent vertices – *external*. A vertex-oriented uni-trivalent graph is also known a *Chinese character*.

Let X be a compact oriented 1-dimensional manifold. A *Chinese character diagram* with support X is the manifold X together with a vertex-oriented uni-trivalent graph whose univalent vertices are on X. A Chinese character diagram is *g-connected* if the graph is connected.

In all figures the components of X will be depicted by solid lines, the graph by dashed lines, and the orientation at every internal vertex is given by the counterclockwise direction. For this reason the graph is also called the dashed graph.

Chinese characters and Chinese character diagrams are regarded up to homeomorphisms preserving components of the support and orientations at vertices.

Let $\tilde{\mathscr{A}}(X)$ be the vector space over $\mathbb{Q}$ spanned by Chinese character diagrams with support X. Let $\mathscr{A}(X)$ be the quotient space of $\tilde{\mathscr{A}}(X)$ by dividing out the STU relation shown in Figure 1.

The *degree* of a Chinese character diagram is half the number of vertices of the dashed graph. Since the relation STU respects the degree, there is a grading on $\mathscr{A}(X)$ induced by this degree. We also use $\mathscr{A}(X)$ to denote the completion of $\mathscr{A}(X)$ with respect to the degree.

We define a co-multiplication $\hat{\Delta}$ in $\mathscr{A}(X)$ as follows. A *Chinese character sub-diagram* of a Chinese character diagram D with dashed graph G is any Chinese character diagram obtained from D by removing some (possibly empty) connected components of G. The *complement Chinese character sub-diagram* of a Chinese character sub-diagram D' is the Chinese character sub-diagram obtained by removing components of G which are in D'. We define

$$\hat{\Delta}(D) = \sum D' \otimes D'' \ .$$

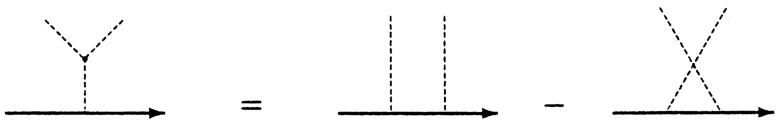

Fig. 1. The STU relation

Here the sum is over all Chinese character sub-diagrams D' of D, and D'' is the complement of D'. This co-multiplication is co-commutative.

Suppose that X, X' have distinguished components ℓ, ℓ', and that X consists of loop components only. Let $D \in \mathscr{A}(X)$ and $D' \in \mathscr{A}(X')$ be two Chinese character diagrams. From each of ℓ, ℓ' we remove a small arc which does not contain any vertices. The remaining part of ℓ is an arc which we glue to ℓ' in the place of the removed arc such that the orientations are compatible. The new Chinese character diagram is called *the connected sum of D, D' along the distinguished components*. It does not depend on the locations of the removed arcs, which follows from the STU relation and the fact that all components of X are loops. The proof is the same as in case $X = X' = S^1$ as in [BN1].

In case when $X = X' = S^1$, the connected sum defines a multiplication which turns $\mathscr{A}(S^1)$ into an algebra.

1.2. Algebra structure. In special cases we can equip $\mathscr{A}(X)$ with an algebra structure. Suppose that X is n numbered oriented lines on the plane pointing downwards. The space $\mathscr{A}(X)$ will be denoted by $\mathscr{P}_n$; and a connected component of X will be called a *string*. If D_1 and D_2 are two Chinese character diagrams in $\mathscr{P}_n$, let $D_1 \times D_2$ be the Chinese character diagram obtained by placing D_1 on top of D_2. The unit of this algebra is the Chinese character diagram without dashed graph. Let $\mathscr{P}_0 = \mathbb{Q}$. It is known that the algebra $\mathscr{P}_1$ is commutative (see [BN1]).

The algebra and co-algebra structure are compatible, and $\mathscr{P}_n$ becomes a Hopf algebra. It is not hard to see that every primitive element, i.e. element x such that $\hat{\Delta}(x) = 1 \otimes x + x \otimes 1$, is a linear combination of g-connected Chinese character diagrams. We now introduce a couple of operators acting on $\mathscr{P}_n$.

Suppose D is a Chinese character diagram in $\mathscr{P}_n$ with the dashed graph G. Replace the i-th string by two strings, the left and the right, very close to the old one, and renumber all the strings from left to right. Attach the graph G to the new set of strings in the same way as in D; this would cause no problem if there is no univalent vertex on the i-th string of D. If there is a univalent vertex of G on the i-th string then it yields two possibilities, attaching to the left or to the right string. Summing up all 2^m, where m is the number of univalent vertices of D on the i-th string, possible Chinese character diagrams of this type, we get $\Delta_i(D) \in \mathscr{P}_{n+1}$. Using linearity we can define $\Delta_i : \mathscr{P}_n \to \mathscr{P}_{n+1}$, for $i = 1, 2, \dots, n$.

Define ε_i by $\varepsilon_i(D) = 0$ if the Chinese character diagram $D \in \mathscr{P}_n$ has a univalent vertex on the i-th string. Otherwise let $\varepsilon_i(D)$ be the Chi-

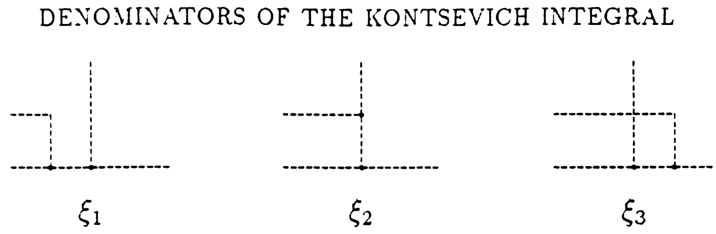

Fig. 2.

nese character diagram in $\mathscr{P}_{n-1}$ obtained by removing the i-th string and renumbering the remaining strings from left to right. We continue ε_i to a linear map from $\mathscr{P}_n$ to $\mathscr{P}_{n-1}$.

1.3. Chinese characters. An *n-marked Chinese character* ξ is a Chinese character whose external vertices are colored by $1, 2, \ldots, n$. Two n-marked Chinese characters are considered the same if there is a homeomorphism between them which preserves the colors. Let $\mathscr{B}_n$ be the vector space over $\mathbb{Q}$ spanned by all n-marked Chinese characters subject to the following identities:

(1) the antisymmetry identity: $\xi_1 + \xi_2 = 0$, for every two Chinese characters ξ_1 and ξ_2 identical everywhere except for the orientation at one internal vertex.

(2) the Jacobi identity (also known as the IHX identity): $\xi_1 = \xi_2 + \xi_3$, for every three Chinese characters identical outside a ball in which they differ as shown in Figure 2.

Now we define a linear mapping $\chi : \mathscr{B}_n \to \mathscr{P}_n$ as follows. Suppose an n-marked Chinese character ξ has k_i external vertices of color i. There are $k_i!$ ways to put vertices of color i on the i-th string and each of the $k_1! \ldots k_n!$ possibilities gives us a Chinese character diagram in $\mathscr{P}_n$. Summing up all such elements and dividing by $1/(k_1! k_2! \ldots k_n!)$, we get $\chi(\xi)$. It is well-known that χ is an isomorphism between the vector spaces $\mathscr{B}_n$ and $\mathscr{P}_n$. A proof for the case $n = 1$ is presented in [BN1]; the statement itself is Kontsevich's. The proof can be easily generalized to any n. For an explicit description of χ^{-1}, see §7.3.

2. The cobar complex of Chinese character diagrams

2.1. The general complex. Put

$$C^n(\mathscr{P}) = \mathscr{P}_n \ ,$$

$$s_i^n = \varepsilon_i \colon C^n \to C^{n-1} \qquad \text{for } i = 1, \ldots, n \ .$$

Let us define

$$d_i^n \colon C^n \to C^{n+1}$$

by $d_i^n = \Delta_i$ if $1 \le i \le n$, and

$$d_0^n(x) = 1 \otimes x, \qquad d_{n+1}^n = x \otimes 1 \ ,$$

where $1 \otimes x$ (respectively, $x \otimes 1$) is obtained from x by adding a string to the left (respectively, right) of x and renumbering all the strings from left to right.

It was noticed in [BN2] that (C^n, d_i^n, s_i^n) form a co-simplicial set. It is natural to consider the following chain complex

$$0 \to C^0 \xrightarrow{d} C^1 \xrightarrow{d} C^2 \xrightarrow{d} \cdots C^n \xrightarrow{d} C^{n+1} \cdots$$

where

$$d(x) = \sum_{i=0}^{n+1} (-1)^i d_i^n \ .$$

We call it the cobar complex of Chinese character diagrams. The cohomology of this complex and its subcomplexes will play important role. Complexes of this type for the Lie algebra case were first considered by Cartier, see [Car, Kas].

2.2. Subcomplexes. The symmetric group $\mathfrak{S}_n$ acts on $C^n = \mathscr{P}_n$ on the left by permuting the strings of the support. An element $x \in \mathscr{P}_n$ is said to be *symmetric* if

$$x + (-1)^{n(n+1)/2} \sigma(x) = 0 \ ,$$

where σ is the permutation sending 1 to n, 2 to $n-1$, 3 to $n-2$, etc.

Let $C^n_{\text{sym}}(\mathscr{P})$ be the subspace of all symmetric elements of $C^n(\mathscr{P})$. It is easy to see that (C^*_{sym}, d) is a chain subcomplex of (C^*, d) .

Let us define the Harrison sub-complex of $\mathscr{P}_n$ (see [BN2]). Let p, q be positive integers. A permutation σ of $\{1, 2, \ldots, p+q\}$ is called a (p, q)-shuffle if σ preserves the order of $\{1, \ldots, p\}$ and the order of $\{p+1, \ldots, p+q\}$. Define: $\text{sh}_{p,q} \in \mathbb{Q}[\mathfrak{S}_{p+q}]$ by

$$\text{sh}_{p,q} = \sum_{\text{all}(p,q)-\text{shuffles } \sigma} \text{sign}(\sigma)\sigma \ .$$

Let $C^n_{\text{Harr}}(\mathscr{P})$ be the subset of $C^n(\mathscr{P})$ consisting of elements $x \in C^n$ such that $\text{sh}_{p,q}(x) = 0$ whenever $p + q = n$. As in the usual theory of

Harrison cohomology, one can show that (C^*_{Harr}, d) is a subcomplex of (C^n, d).

Another way to define the Harrison complex is to use the Eulerian idempotents. It was proved in [GS] that there exist idempotents $\mathrm{e}_n^{(l)} \in \mathbb{Q}[\mathfrak{S}_n]$, for $l = 1, 2, \ldots, n$ such that

$$(\mathrm{e}_n^{(l)})^2 = \mathrm{e}_n^{(l)} \ ,$$

$$\mathrm{e}_n^{(l)}\mathrm{e}_n^{(k)} = 0 \quad \text{if } k \neq l, \text{ and}$$

$$\mathrm{e}_n^{(1)} + \cdots \mathrm{e}_n^{(n)} = 1 \ .$$

These idempotents, as elements of $\mathbb{Q}[\mathfrak{S}_n]$, acts on $\mathscr{P}_n$ as projections. The important point is that all these idempotents commutes with the co-boundary operator d. Hence for each fixed l, we have a subcomplex $(\mathrm{e}_n^{(l)} C^n, d)$ of (C^n, d), if we put $\mathrm{e}_n^{(l)} = 0$ for $l > n$.

The subcomplex $(\mathrm{e}_n^{(1)} C^n, d)$ is exactly the above defined Harrison subcomplex. We record here the formula of $\mathrm{e}_n^{(1)}$:

$$\mathrm{e}_n^{(1)} = \sum_{\sigma \in \mathfrak{S}_n} \frac{(-1)^{r(\sigma)}}{n \binom{n-1}{r(\sigma)}} (-1)^{\sigma} \sigma \ , \tag{2.1}$$

where $r(\sigma)$ is the number of $k \in \{1, 2, \ldots, n-1\}$ such that $\sigma(k) > \sigma(k+1)$.

2.3. Non-degeneracy, integral lattices. An element $x \in \mathscr{P}_n$ is said to be *non-degenerate* if $\varepsilon_i(x) = 0$ for every $i = 1, 2, \ldots, n$. We will be interested in g-connected, non-degenerate Chinese character diagrams.

Let $\mathscr{Q}_n^{\mathbb{Z}}$ be the set of all elements in $\mathscr{P}_n$ which are linear combinations of g-connected non-degenerate Chinese character diagrams with *integer* coefficients. Then $\mathscr{Q}_n^{\mathbb{Z}}$ is a free $\mathbb{Z}$-module, and let $\mathscr{Q}_n$ be the vector space spanned by $\mathscr{Q}_n^{\mathbb{Z}}$:

$$\mathscr{Q}_n = \mathscr{Q}_n^{\mathbb{Z}} \otimes \mathbb{Q} \ .$$

From the STU relation one can easily prove the following.

Lemma 2.1. *If x,y are in $\mathscr{Q}_n^{\mathbb{Z}}$, then so is the commutator $xy - yx$.*

Definition. *Suppose V is a vector space over $\mathbb{Q}$ which contains a fixed integral lattice $V^{\mathbb{Z}}$, i.e. a $\mathbb{Z}$-module such that $V = V^{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$. We say that an element $x \in V$ has denominator N if Nx is in $V^{\mathbb{Z}}$.*

We will consider $\mathscr{Q}_n^{\mathbb{Z}}$ the integral lattice of $\mathscr{Q}_n$.

Note that the operator d preserves $\mathcal{Q}_n$ and $\mathcal{Q}_n^{\mathbb{Z}}$, hence we can speak about the subcomplexes $(C^*(\mathcal{Q}), d)$ and $(C^*(\mathcal{Q}^{\mathbb{Z}}), d)$, where

$$C^n(\mathcal{Q}) := \mathcal{Q}_n, \quad \text{and}$$

$$C^n(\mathcal{Q}^{\mathbb{Z}}) := \mathcal{Q}_n^{\mathbb{Z}} \ .$$

The latter is a complex over $\mathbb{Z}$.

One can also consider the subcomplexes $(C^*_{\text{sym}}(\mathcal{Q}), d)$, $(C^*_{\text{Harr}}(\mathcal{Q}), d)$, $(C^*_{\text{sym}}(\mathcal{Q}^{\mathbb{Z}}), d)$ and $(C^*_{\text{Harr}}(\mathcal{Q}^{\mathbb{Z}}), d)$ by taking the intersections with $C^*_{\text{sym}}(\mathcal{P})$ and $C^*_{\text{Harr}}(\mathcal{P})$.

It is known that (see [BN2]) the even-dimensional cohomology groups of $(C^*_{\text{sym}}(\mathcal{P}), d)$ vanish. This fact is fundamental in solving the pentagon equation (see below) in [BN2,Dr2]. The proof can be modified easily to show that the even-dimensional cohomology groups of $(C^*_{\text{sym}}(\mathcal{Q}), d)$ vanish.

Actually, to solve the pentagon equation, one needs only the result for the four-dimensional cohomology group. For the purpose of this paper we need to calculate the four-dimensional cohomology group of the $\mathbb{Z}$-complex $(C^*_{\text{sym}}(\mathcal{Q}^{\mathbb{Z}}), d)$, which must have rank 0 as a $\mathbb{Z}$-module, but may have some non-trivial torsion part.

Note that d preserves the degree of Chinese character diagrams, hence the complex $(C^n_{\text{sym}}(\mathcal{Q}^{\mathbb{Z}}), d)$ can be decomposed further by degree. The result, whose proof will be given in §7, is

Proposition 2.2. The degree *m part of the four-dimensional cohomology group of the complex* $(C^n_{\text{sym}}(\mathcal{Q}^{\mathbb{Z}}), d)$ *is annihilated by* $2(m+1)!\,[m!(m-1)!]^2$.

3. The Drinfeld Associator

3.1. Associators. Drinfeld defined associators and R-matrix for quasi-triangular quasi-Hopf algebras, see [Dr1, Dr2]. We recall here the definition, adapted for the case of Chinese character algebras in [LM1, LM2] (see also [BN2]). Note that we don't have any quasi-Hopf algebra here. The algebra $\mathcal{P}_n$ will play the role of the n-th power of a quasi-Hopf algebra. Associators are used to construct invariants of framed links (see §5).

Let $r \in \mathcal{P}_2$ be the Chinese character diagram whose dashed graph is a line connecting the two strings of the support. Let $r^{ij} \in \mathcal{P}_3$ be the Chinese character diagram whose dashed graph is a line connecting the i-th and j-th strings of the support. Define

$$R = \exp(r/2) \in \mathscr{P}_2 \quad \text{and}$$

$$R^{ij} = \exp(r^{ij}/2) \in \mathscr{P}_3 \ .$$

Definition. *An associator is an element $\Phi \in \mathscr{P}_3$ satisfying the following equations*:

$$\Delta_3(\Phi) \times \Delta_1(\Phi) = (1 \otimes \Phi) \times \Delta_2(\Phi) \times (\Phi \otimes 1) \ , \tag{A1}$$

$$\Delta_1(R) = \Phi^{312} \times R^{13} \times (\Phi^{132})^{-1} \times R^{23} \times \Phi \ , \tag{A2}$$

$$\Phi^{-1} = \Phi^{321} \ , \tag{A3}$$

$$\varepsilon_1(\Phi) = \varepsilon_2(\Phi) = \varepsilon_3(\Phi) = 1 \ . \tag{A4}$$

Here Φ^{ijk} is the element of $\mathscr{P}_3$ obtained from Φ by permuting the strings: the first to the i-th, the second to the j-th, the third to the k-*th*. Equation (A1) holds in $\mathscr{P}_4$, equations (A2, A3) in $\mathscr{P}_3$, and equation (A4) in $\mathscr{P}_2$. There are some redundancy in this system of equations. Equation (A1) is known as the pentagon equation, (A2) – the hexagon equation. Due to (A3), one has that $(\Phi^{132})^{-1} = \Phi^{231}$.

Remark. Our definition of associator is stricter than that of [BN2, Dr2] since the R-matrix is $\exp(r/2)$.

An element in $\mathscr{A}(X)$ is said to be *even* if it is a linear combination of Chinese character diagrams of even degree. It follows from Drinfeld [Dr2] that there exists an *even* associator Φ_e. This even associator possesses a lot of symmetry which has many topological applications, see [LM3].

3.2. Symmetric twisting. Suppose that $F \in \mathscr{P}_2$ satisfies the following conditions:

(T1) $\varepsilon_1(F) = \varepsilon_2(F) = 1$, (hence $F = 1 +$ (higherorder terms)),

(T2) F is symmetric.

Then there exists the inverse F^{-1} in $\mathscr{P}_2$. If Φ is an element of $\mathscr{P}_3$, then the element

$$\Phi^F := [1 \otimes F]\Delta_2(F)\Phi\Delta_1(F^{-1})[F^{-1} \otimes 1]$$

is said to be obtained from Φ by *twisting via F*, or by a *gauge transformation* (see [Dr1]). Note that Drinfeld defined gauge transformations in a more general situation. Here we require F to be symmetric (condition T2), so that the gauge transformation does not change the co-product.

If $\Phi \in \mathscr{P}_3$ is an associator, i.e. a solution of (A1–A4), then it is not difficult to check that Φ^F is also an associator. Condition (T2) guarantees that Φ^F satisfies (A3).

In [LM2], following Drinfeld's treatment of the Lie algebra case, it was proved that any two associators Φ, Φ' are related by a twist F. The proof also shows that if both Φ, Φ' are even, then F can be chosen to be even. Moreover, if Φ, Φ' are "even associators up to degree $2k$", i.e. they satisfy the equations (A1–A4) up to degree $2k$, then they are related, up to degree $2k$, by a twist via an even F.

The next lemma is very important. It helps us to solve the pentagon and hexagon equations, using Drinfeld's perturbative method, even in the case when the R-matrix is fixed (equal to $\exp(r/2)$). Moreover it significantly simplifies the process of solving the equations.

Lemma 3.1. *Suppose that $\Phi \in \mathscr{P}_3$ is an even element which satisfies the hexagon and pentagon equations* (A1), (A2) *up to* (*and including*) *degree $2k$. Then Φ also satisfies the same equations up to degree $2k+1$.*

Proof. The pentagon equation is obviously satisfied, since there is nothing of odd degree. For the hexagon equation, we need the fact that there exits an even associator Φ_e. Since both Φ and Φ_e are associator up to degree $2k$, there is an even element $F \in \mathscr{P}_2$, satisfying (T1) and (T2), such that $\Phi_e^F = \Phi$ up to degree $2k$. Since both Φ_e^F and Φ don't have terms of odd degree, they are the same up to degree $2k+1$. The element Φ_e^F is still an associator, hence it satisfies the hexagon equation. It follows that Φ satisfies the hexagon equation up to degree $2k+1$. □

Remark. In the proof we used the existence of Φ_e, which was established by Drinfeld using analysis. It is still an open problem to prove this lemma using only algebra.

4. Solving Equations A1–A4 for associators

4.1. The existence of a special associator. To solve the pentagon and hexagon equations we will follow [Dr2, BN2]. Note that the combinatorial method in [Dr2, BN2] does not apply to the case when the R-matrix is equal to $\exp(r/2)$. It is Lemma 3.1 that helps us to remedy this, and besides, makes the solving procedure simpler by eliminating one step. Let

$$d_n = (2!3!\dots n!)^4(n+1)!$$

Theorem 4.1. *There exists an even associator* $\Phi \in \mathscr{P}_3$ *of the form* $\Phi = \exp(\phi)$, *where* ϕ *is a linear combination of even, non-degenerate, g-connected, symmetric Chinese character diagrams. In other words,* ϕ *is*

$$\text{even and in } C^3_{\text{sym}}(\mathscr{Q}) \ . \tag{4.1}$$

Moreover, the degree $2m$ *part of* ϕ *has denominator* d_{2m}.

This is the main result. Recall that $\phi \in C^3(\mathscr{Q}) = \mathscr{Q}_3$ is symmetric means that $\phi^{321} = -\phi$. The proof of this theorem will occupy the rest of this section.

If ϕ satisfies (4.1), then $\Phi = \exp(\phi)$ is even and satisfies (A3) and (A4). There are only the hexagon and pentagon equations to worry about.

For a graded algebra A let $\text{Grad}_m A$ (resp. $\text{Grad}_{\leq m}$) be the subspace spanned by elements of grading m (of grading $\leq m$).

We will solve the pentagon and hexagon equations up to degree $2m$, and then show that the solution can be extended so that it solves these equations up to degree $2m+2$.

Suppose that there exists $\Phi_{2m} = \exp(\phi_{2m})$ satisfying (A1) and (A2) up to degree $2m$, where $\phi_{2m} \in \text{Grad}_{\leq 2m}\mathscr{P}_3$ satisfies (4.1) and the part of degree $2k$ of ϕ_{2m} has denominator d_{2k}. We know that for $m = 2$ such a Φ_{2m} exists, see [BN2].

We will find $\phi_{2m+2} = \phi_{2m} + \varphi$, where φ is of degree $2m+2$ and satisfying (4.1) such that $\Phi_{2m+2} := \exp(\phi_{2m+2})$ satisfies the pentagon and hexagon equations up to degree $2m+2$. In addition, φ, and hence ϕ_{2m+2}, has denominator d_{2m+2}.

By Lemma 3.1, Φ_{2m} also satisfies the pentagon and hexagon equations up to degree $2m+1$. Now focus on degree $2m+2$.

4.2. The hexagon equation. Let ψ be the degree $2m+2$ mistake in the hexagon equation when using Φ_{2m} in place of Φ, i.e.

$$1 + \psi = \Phi^{312}_{2m} \times (R)^{13} \times (\Phi^{-1}_{2m})^{132} \times (R)^{23} \times \Phi_{2m} \times \Delta_1(R^{-1}) \tag{4.2}$$

(the equation is taken modulo degree $\geq 2m+3$)

It is conceivable that with the knowledge of the denominators of the terms of the right hand side, one can estimate the denominator of ψ. In fact, in §7 we will show that

Lemma 4.2. *The mistake* ψ *is an element in* $\mathscr{Q}_3$ *having denominator* $[(2m+2)!]^2 d_{2m}$.

Suppose $\Phi' = \exp(\phi_{2m} + u)$, where u satisfies (4.1) and is of degree $2m+2$. If we replace Φ_{2m} by Φ' in Equation (4.1), then it's easy to see that the new mistake ψ' is

$$\psi' = \psi + u + u^{312} + u^{231} \ . \tag{4.3}$$

Hence if $\psi + u + u^{312} + u^{231} = 0$, then Φ' solves the hexagon equation up to degree $2m+2$. The following fact was proved in [BN2, Dr1].

Lemma 4.3. *The mistake ψ is totally antisymmetric, i.e. for every permutation σ of* 3 *numbers* $\{1,2,3\}$ *one has* $\sigma(\psi) = \mathrm{sign}(\sigma)\psi$.

In particular $\psi \in C^3_{\mathrm{sym}}(\mathcal{Q}_3)$. Let $u = -\psi/3$, then

$$\psi + u + u^{312} + u^{231} = 0 \ .$$

Hence

$$\bar{\Phi} := \exp(\phi_{2m} - \psi/3)$$

solves the hexagon equation up to degree $2m+2$. Note that u satisfies (4.1), is of degree $2m+2$, and has denominator $3 \times [(2m+2)!]^2 d_{2m}$.

4.3. The pentagon equation. Let μ be the mistake of degree $2m+2$ in the pentagon equation if Φ is replaced by $\bar{\Phi}$, i.e.

$$1 + \mu = \Delta_1(\bar{\Phi}^{-1}) \times \Delta_3(\bar{\Phi}^{-1}) \times (1 \otimes \bar{\Phi}) \times \Delta_2(\bar{\Phi}) \times (\bar{\Phi} \otimes 1) \tag{4.4}$$

(this equation is taken modulo degrees $\geq 2m+3$). Again, one can easily estimate the denominator of μ. In §7 we will show

Lemma 4.4. *The mistake μ is in $C^4(\mathcal{Q}) = \mathcal{Q}_4$ having denominator* $3[2m+2)!]^2 d_{2m}$.

If we replace $\bar{\Phi}$ by

$$\bar{\Phi}' = \exp(\phi_{2m} - \psi/3 + v),$$

where v satisfies (4.1) and is of degree $2m+2$, then the new mistake μ' is

$$\mu' = \mu - \Delta_1(v) - \Delta_3(v) + \Delta_2(v) + v \otimes 1 + 1 \otimes v = \mu + dv.$$

The new mistake of the hexagon equation, by (4.3), is $v^{123} + v^{312} + v^{231}$.

Hence, we need to find v of degree $2m+2$ satisfying (4.1) and the following equations

$$dv + \mu = 0 \ , \tag{4.5}$$

$$v^{123} + v^{312} + v^{231} = 0 \ . \tag{4.6}$$

The condition (4.1) says that $v \in C^3(\mathcal{Q})$ and v is symmetric. That v is symmetric and satisfying (4.6) is equivalent to: v is symmetric and annihilated by (1,2)- and (2,1)-shuffles. So, we need to find v in the intersection of $C^3_{\mathrm{sym}}(\mathcal{Q})$ and $C^3_{\mathrm{Harr}}(\mathcal{Q})$ and satisfying $dv = -\mu$.

The following has been proved in [BN2].

Lemma 4.5. *The mistake μ is in the intersection of $C^4_{\mathrm{sym}}(\mathcal{Q})$ and $C^4_{\mathrm{Harr}}(\mathcal{Q})$, and $d\mu = 0$.*

So $\mu \in C^4_{\mathrm{sym}}(\mathcal{Q})$ and $d\mu = 0$. From Proposition 2.2 it follows that there exists $v' \in C^3_{\mathrm{sym}}(\mathcal{Q})$ having denominator $2[(2m+1)!(2m+2)!]^2(2m+3)!$ times that of μ, such that $dv' = -\mu$. This v' may not be in C^3_{Harr}, i.e. may not satisfy (4.6). Put

$$v = \mathrm{e}_3^{(1)}(v') \ .$$

Explicitly one has (see Equation (2.1))

$$v = \frac{2}{3}v' - \frac{1}{3}(v')^{312} - \frac{1}{3}(v')^{231} \ .$$

Then v is in both $C^3_{\mathrm{sym}}(\mathcal{Q})$ and $C^3_{\mathrm{Harr}}(\mathcal{Q})$. The commutativity of $\mathrm{e}_n^{(1)}$ and d shows that

$$dv = d(\mathrm{e}_3^{(1)}v') = \mathrm{e}_4^{(1)}dv' = \mathrm{e}_4^{(1)}(-\mu) = -\mu.$$

So v is an element satisfies all (4.1), (4.5) and (4.6). Note that v has denominator $6 \times 3[(2m+1)!]^2[(2m+2)!]^4(2m+3)!d_{2m}$, which is a divisor of d_{2m+2} when $m \geq 2$. This completes the induction step, and hence the proof of Theorem 4.1.

5. Denominators of the Kontsevich integral

5.1. The Kontsevich integral. We briefly recall here the (modification of the) Kontsevich integral for framed links and framed tangles (see [LM1, LM2]). First we recall the definition of framed tangles.

We fix an oriented 3-dimensional Euclidean space $\mathbb{R}^3$ with coordinates (x, y, t). *A tangle* is a smooth one-dimensional compact oriented manifold $L \subset \mathbb{R}^3$ lying between two horizontal planes

$\{t = a\}, \{t = b\}, a < b$ such that all the boundary points are lying on two lines $\{t = a, y = 0\}, \{t = b, y = 0\}$, and at every boundary point L is orthogonal to these two planes. These lines are called the top and the bottom lines of the tangle.

A normal vector field on a tangle L is a smooth vector field on L which is nowhere tangent to L (and, in particular, is nowhere zero) and which is given by the vector $(0, -1, 0)$ at every boundary point. *A framed tangle* is a tangle enhanced with a normal vector field. Two framed tangles are isotopic if they can be deformed by a 1-parameter family of diffeomorphisms into one another within the class of framed tangles.

Framed oriented links are special framed tangles when there is no boundary point. The empty link, or empty tangle, by definition, is the empty set.

One assigns a symbol $+$ or $-$ to all the boundary points of a tangle according to whether the tangent vector at this point directs downwards or upwards. Then on the top boundary line of a tangle diagram we have a word of symbols consisting of $+$ and $-$. Similarly on the bottom boundary line there is another word of symbols $+$ and $-$.

A *non-associative word* on $+, -$ is an element of the free magma generated by $+, -$ (see the definition of a free magma in [Ser]). For a non-associative word one w one defines its length as the number of its letters.

A *q-tangle* (or *non-associative* tangle) is a tangle together with two non-associative words w_t, w_b such that if we ignore the non-associative structure, from w_t, w_b we get the words on the top and bottom lines.

If T_1, T_2 are framed q-tangles such that $w_b(T_1) = w_t(T_2)$ we can define the product $T = T_1 T_2$ by placing T_1 on top of T_2. In this case, if $\xi_1 \in \mathscr{A}(T_1), \xi_2 \in \mathscr{A}(T_2)$ are Chinese character diagrams, then the *product* $\xi_1\xi_2$ is a chord diagram in $\mathscr{A}(T)$ obtained by placing ξ_1 on top of ξ_2.

For any two framed q-tangles T_1, T_2 with the same top and bottom lines, we can define their *tensor product* $T_1 \otimes T_2$ by putting T_2 to the right of T_1. The non-associate structure of the boundaries are the natural composition of those of T_1, T_2. Similarly, if $\xi_1 \in \mathscr{P}(T_1), \xi_2 \in \mathscr{P}(T_2)$ are chord diagrams, then one defines $\xi_1 \otimes \xi_2 \in \mathscr{P}(T_1 \otimes T_2)$ by the same way.

It is easy to see that every framed q-tangle T can be obtained from *elementary q-tangles*, using the product and tensor product. Here an elementary q-tangles is one of the following:

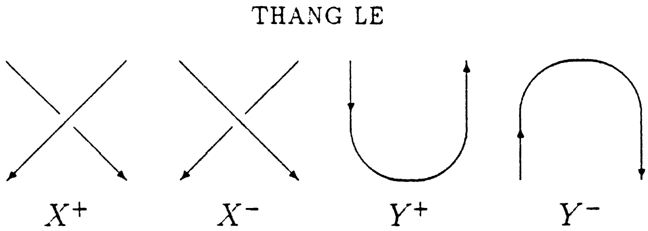

Fig. 3.

a) the trivial framed q-tangle consisting of one vertical line pointing downwards, the framing everywhere is given by the vector $(0, -1, 0)$.

b) one of the framed q-tangle depicted in Figures 3. Again the framing is given by the vector $(0, -1, 0)$.

c) the framed q-tangle T_{w_1,w_2,w_3} which has trivial underlying framed tangle, but the non-associative word of the top is $w_1(w_2w_3)$, the one of the bottom is $(w_1w_2)w_3$. Here w_1, w_2, w_3 are three arbitrary non-associative words.

d) any of the above with reversing orientation on some of the components.

We will define an invariant $\hat{Z}(T) \in \mathscr{A}(T)$ for every framed q-tangle such that

$$\hat{Z}(T_1T_2) = \hat{Z}(T_1)\hat{Z}(T_2) \ ,$$

$$\hat{Z}(T_1 \otimes T_2) = \hat{Z}(T_1) \otimes \hat{Z}(T_2)$$

With these requirements, one needs only to define $\hat{Z}$ of the elementary framed q-tangles. For a trivial framed q-tangle T let $\hat{Z}(T) = T$, the Chinese character diagram in $\mathscr{A}(T)$ without dashed graph. Put

$$\hat{Z}(X^{\pm}) = 1 + \frac{\pm 1}{2}(x)^{(1)} + \cdots + \frac{(\pm 1)^n}{n!2^n}x^{(n)} + \cdots \ ,$$

where $x^{(n)}$ is the Chinese character diagram in $\mathscr{A}(X^{\pm})$ whose dashed graph consists of n parallel horizontal dashed lines connecting the two solid strings of the support. Let

$$\hat{Z}(T_{w_1w_2w_3}) = \Delta^{(|w_1|)} \otimes \Delta^{(|w_2|)} \otimes \Delta^{(|w_3|)}(\Phi) \ .$$

Here $|w|$ is the length of the word w and $\Delta^{(1)} = id$, $\Delta^{(2)} = \Delta$, $\Delta^{(n)} = \Delta_1 \circ \Delta_1 \circ \cdots \circ \Delta_1$ ($n-1$ times). The operation $\Delta^{(n)}$ replaces one string of the support by n strings. The right hand side of the above equation means that we apply $\Delta^{(|w_1|)}$ to the first string, $\Delta^{(|w_2|)}$ to the second, and $\Delta^{(|w_3|)}$ to the third string of the support of Φ.

If T' is obtained from T by reversing the orientation of a component C, then we put $\hat{Z}(T') = S_C[\hat{Z}(T)]$, where $S_C : \mathscr{A}(T) \to \mathscr{A}(T')$ is the linear map defined as follows. Suppose $D \in \mathscr{A}(T)$ is a Chinese character diagram with m univalent vertices on C. Reversing the orientation of C, then multiplying by $(-1)^m$, from D we get $S_C(D)$.

Finally, $\hat{Z}(Y^{\pm}) = \sqrt{v}$ where $v \in \mathscr{P}_1$ is obtained from $S_{C_2}\Phi$ by identifying the terminating point of the first string with the beginning point of the second string, and the terminating point of the second with the beginning point of the third string. Here C_2 is the second string.

These requirements define $\hat{Z}(T)$ uniquely. It is known (see [LM1, LM2, BN2]) that $\hat{Z}$ is well-defined and is an isotopy invariant of framed q-tangles. In fact, $\hat{Z}(T)$ is a universal finite type invariant of framed q-tangles. For more properties of $\hat{Z}$, see [LM2].

For a knot K, the natural projection from $\mathscr{A}(S^1)$ to $\mathscr{A}(S^1)/\approx$ takes $\hat{Z}(K)$ to the Kontsevich integral of K (see [LM1]). Here $\approx$ is the equivalence relation generated by: any Chinese character diagram with an isolated dashed chord is equivalent to 0. The original Kontsevich integral is given by an explicit formula (see [Ko1, BN1]). Note that if the R-matrix is not equal to $\exp(r/2)$, then the corresponding invariant of knots, in general, does not project to the original Kontsevich integral. This is the reason why we have to stick to the case $R = \exp(r/2)$.

It was proved in [LM2] that $\hat{Z}(T)$ does not depend on the associator Φ if T is a link.

5.2. Denominators of the Kontsevich integral. Let $\mathscr{A}^{\mathbb{Z}}(X)$ be the set of elements in $\mathscr{A}(X)$ which are linear combinations of Chinese character diagrams with *integer coefficients*. We consider $\mathscr{A}^{\mathbb{Z}}(X)$ as the *integral lattice* of $\mathscr{A}(X)$, and we can speak about denominators of elements of $\mathscr{A}(X)$. Since $\mathscr{Q}_n \subset \mathscr{P}_n$, and each has its own integral lattice, there may be confusion about denominator of an element $x \in \mathscr{Q}_n \subset \mathscr{P}_n$. However, if $x \in \mathscr{Q}_n$ has denominator N with respect to the integral lattice of the smaller vector space $\mathscr{Q}_n$, then x also has denominator N with respect to the integral lattice of the bigger space $\mathscr{P}_n$. In this section, when confusion might arise, we always assume that denominators are considered with respect to the lattice of the bigger space.

Lemma 5.1. *Suppose a q-tangle T is decomposed as $T = T_1T_2\cdots T_k$, and that $x_i \in \mathscr{A}(T_i), i = 1, \ldots, k$ satisfy*

$$\textit{the denominator of the degree m part has denominator } d_m \quad . \tag{5.1}$$

Then the element $(x_1 \cdots x_k)/k! \in \mathscr{A}(T)$ *also satisfies* 5.1 *In particular, the product* $x_1 \cdots x_k$ *satisfies* 5.1.

Proof. An element of degree n in $(x_1 \cdots x_k)/k!$ has denominator $k! d_{n_1} \cdots d_{n_k}$, where $n_1 + \cdots + n_k = n$. By Corollary 7.8 (proved in §7) $k! d_{n_1} \cdots d_{n_k}$ is a divisor of d_n. □

By Theorem 4.1, $\Phi = \exp(\phi) = \sum_k \phi^k/k!$. Since ϕ satisfies (5.1), the previous lemma shows that Φ also satisfies (5.1). So the values of $\hat{Z}$ of elementary q-tangles satisfy (5.1). Hence from Lemma 5.1 we have

Theorem 5.2. *For every framed q-tangle T, the degree m part of* $\hat{Z}(T)$ *has denominator* $d_m = [2! \cdots m!]^4 (m+1)!$. *In particular, the degree m part of the Kontsevich integral of any knot has denominator* d_m.

5.3. String links. *A framed string link* is a framed tangle containing no loops such that the endpoint of any component on the top line projects vertically to the other endpoint of the same component. In what follows, every framed string link will be considered as a framed q-tangle, where the non-associative words of the top and the bottom are supposed to be the same.

Let L be framed string link with l components represented by a diagram on the plane with framing vector $(0, -1, 0)$ everywhere. Closing the framed string link in the same way as one closes the braids, we obtain an l component link. We assume that the framing of the closing part is given by the vector $(0, -1, 0)$ everywhere. We define the linking number of two components of L to be the linking number of their closures. Similarly, the self-linking number of a component is the self-linking number of its closure. The linking numbers and self-linking numbers form the linking matrix.

Definition. *An element* $x \in \mathscr{A}(X)$ *is said to have i-filter k if it is a linear sum of Chinese character diagrams, each has at least k internal vertices.*

Note that $x \in \mathscr{P}_l$ has i-filter k if and only if $\chi^{-1}(x) \in \mathscr{B}_l$ is a linear combination of Chinese characters with at least k internal vertices. If x has i-filter k, then certainly x has i-filter $k-1$.

Proposition 5.3. *Suppose L is a framed string link whose matrix of linking numbers is* 0. *Then* $\hat{Z}(L)$ *can be represented as a linear sum* $\sum_m z_m$, *where* z_m *has i-filter m and has denominator not divisible by any prime greater than* $m+2$.

Proof. It is known that $\hat{Z}(L) \in \mathscr{P}_l$ is a group-like element (see [LM3]). Hence

$$\hat{Z}(L) = \exp \xi \ ,$$

where ξ is primitive. The element ξ is a linear sum of g-connected Chinese character diagrams. Let $\xi = \xi_1 + \xi_2 + \cdots$, where ξ_k has degree k. Since the linking matrix is 0, we have $\xi_1 = 0$. The g-connectedness implies that ξ_k has i-filter $k - 1$.

We have $\xi = \ln \hat{Z}(L)$. From the formula of expansion of $\ln(1 + x)$ and Lemma 7.7 (proved in 7), it follows that ξ_k has denominator d_k.

Return to the formula

$$\hat{Z}(T) = \exp(\xi_2 + \xi_3 + \cdots) \ .$$

Expanding the right hand side, we get a sum of terms of the form $\xi_{n_1} \cdots \xi_{n_k}/k!$. This term has i-filter $(n_1 - 1) + (n_2 - 1) + \cdots (n_k - 1)$, and has denominator not divisible by any prime greater than the maximum q of $\{k, n_1 + 1, n_2 + 1, \dots, n_k + 1\}$. Let $m = q - 2$. Then

$$m \leq (n_1 - 1) + (n_2 - 1) + \cdots (n_k - 1) \ ,$$

since each n_i is ≥ 2. Hence this term has i-filter m and denominator not divisible by any prime greater than $m + 2$. □

6. The universal perturbative invariant of homology 3-spheres

6.1. The invariant. In [LMO] we used the Kontsevich integral and the Kirby calculus to construct an invariant Ω of 3-manifolds with values in the algebra of 3-valent graphs. We briefly recall here the definition of Ω.

Let $\mathscr{D}_n$ be the vector space (over $\mathbb{Q}$) generated by vertex-oriented 3-valent graphs with $2n$ vertices, subject to the anti-symmetry and the Jacobi relations. We don't allow loop components in these 3-valent graphs, so each component contains at least one, and hence two, vertices. If loops are allowed as components of vertex oriented 3-valent graphs, we denote the vector space by $\mathscr{D}_n^\circ$. For $n = 0$ let $\mathscr{D}_n = \mathbb{Q}$. Set

$$\mathscr{D} = \prod_{n=0}^{\infty} \mathscr{D}_n \ .$$

The $\mathscr{D}_n$ is, by definition, the subspace of degree n. Similarly let $\mathscr{D}^\circ = \prod_{n=0}^{\infty} \mathscr{D}_n^\circ$. In $\mathscr{D}$ we define the product of two 3-valent graph as their disjoint union. The unit is the empty graph which is in $\mathscr{D}_0$.

We define first a couple of linear operators. Suppose $x \in \mathscr{B}_l$ is an l-marked Chinese character. If the number of vertices of any color is different from $2n$, or if the degree of x is greater than $(l+1)n$, put $j'_n(x) = 0$. Otherwise, partitioning the $2n$ vertices of the same color into n pairs and identifying points of each pair, from x we get a 3-valent graph which may contains some loops and which depends on the partition. Summing over all possible partitions, we get $j'_n(x) \in \mathscr{D}^\circ$.

There is a surjective linear map

$$\mathrm{pr} : \mathscr{P}_l \to \mathscr{A}(\sqcup^l S^1)$$

defined by identifying the two endpoints of each solid string. For $x \in \mathscr{A}(\sqcup^l S^1)$, choose $y \in \mathscr{P}_l$ such that $x = \mathrm{pr}(y)$. Define

$$j'_n(x) = j'_n(\chi^{-1}(y)).$$

It is easy to see that j'_n is well-defined. Note that j'_n lowers the degree by nl, and the images of j'_n is in $\mathrm{Grad}_{\leq n} \mathscr{D}^\circ$. Another definition of j'_n is given in 6.3.

Finally, we define $\iota_n(x)$ by replacing every loop in $j'_n(x)$ by $-2n$; the result is in $\mathrm{Grad}_{\leq n}\mathscr{D}$.

For a framed link L let $\check{Z}(L)$ be obtained by taking connected sum of $\hat{Z}(L)$ with v along every component of L. Recall that v is the value of $\hat{Z}$ of the trivial knot with framing 0. It was proved in [LMO] that $\iota_n(\check{Z}(L))$, where L is a framed link, does not depend on the orientations of components of L, and does not change under the second Kirby move. Hence the element

$$\Omega_n(L) = \frac{\iota_n(L)}{\iota_n(U_+)^{\sigma_+} \iota_n(U_-)^{\sigma_-}} \in \mathrm{Grad}_{\leq n}\mathscr{D}$$

is an invariant of the 3-manifold M obtained from S^3 by surgery along L. Here $U_\pm$ are the trivial knots with framing ± 1, and $\sigma_\pm$ are the numbers of positive and negative eigenvalues of the linking matrix of L.

This invariant takes value in $\mathrm{Grad}_{\leq n}\mathscr{D}$, for a fixed n. Without loss of information, we can combine all the invariants Ω_n into one by putting

$$\Omega(M) = 1 + \sum_{n=1}^{\infty} \mathrm{Grad}_n(\Omega_n(M)) \ .$$

For rational homology 3-spheres, a more convenient form of this invariant is (see [LMO,Oh2])

$$\hat{\Omega}(M) = 1 + \sum_{n=1}^{\infty} \frac{\mathrm{Grad}_n(\Omega_n(M))}{|H_1(M,\mathbb{Z})|^n} \ ,$$

because $\hat{\Omega}$ behaves well under connected sum: $\hat{\Omega}(M\#M') = \hat{\Omega}(M) \times \hat{\Omega}(M')$.

The invariant Ω has the following important property: for integral homology 3-spheres, it's degree n part is a universal invariant of degree $3n$ (see [Le]). For the theory of finite type invariants of integral homology 3-spheres, see [Oh1]. The degrees of finite type invariants of homology 3-spheres are always multiples of 3 (see [GO]). From $\Omega(M)$, together with weight system coming from Lie algebras (see [BN1]) or from symplectic geometry (see [Kap, Ko2]) one can construct invariants of 3-manifolds with values in the space of formal power series in one variable.

6.2. Denominators of Ω. Note that if D is a Chinese character diagram with less than $2n$ external vertices on one solid component, then $j'_n(D) = 0$. Since j'_n annihilates any Chinese character diagram in $\mathscr{A}(\sqcup^l S^1)$ of degree $> (l+1)n$, it follows that if D has $> 4n$ external vertices on one solid component, then $j'_n(D) = 0$.

An element x in $\mathscr{D}$ or in $\mathscr{D}^\circ$ is said to have denominator N if Nx is a linear combination of 3-valent graphs with *integer coefficients*.

Proposition 6.1. Suppose that L *is a framed link with diagonal linking matrix. Then* $j'_n(\check{Z}(L))$ *has denominator not divisible by any prime greater than* $2n+1$.

Proof. Recall that if T' is obtained from T by increasing the framing of a component by 1, then $\hat{Z}(T')$ is obtained from $\hat{Z}(T)$ by taking connected sum with $\exp(\theta/2)$ along that component, where θ is the Chinese character diagram in $\mathscr{A}(S^1)$ whose dashed graph is a dashed line (see [LM2]).

Suppose that L is obtained from a framed string link T by closing. In [LM2] we proved that

$$\hat{Z}(L) = pr[\hat{Z}(T) \times \Delta^{(l)}(v)] \ .$$

Changing the framing of each component to 0, from T we get T'. Then T' has 0 linking matrix. One has

$$\hat{Z}(T) = \hat{Z}(T') \times (\mathrm{e}^{k_1\theta/2} \otimes \cdots \otimes \mathrm{e}^{k_l\theta/2}) \ ,$$

where $k_1, \ldots, k_l$ are the framings of the components of T. Write $\hat{Z}(T') = \sum_m z_m$ as in Proposition 5.3. Similarly, the element v, being the Kontsevich integral of the unknot, can be written in the form $v = \sum_q y_q$, where y_q has i-filter q and denominator not divisible by any prime greater than $q+2$. We have

$$\hat{Z}(L) = pr\left[\left(\sum_m z_m\right) \times \Delta^{(l)}\left(\sum_q y_q\right) \times (\mathrm{e}^{k_1\theta/2} \otimes \cdots \otimes \mathrm{e}^{k_l\theta/2})\right] \ .$$

If we expand the right hand side, then we get a sum of terms of the form

$$pr\left[z_m \times \Delta^{(l)}(y_q) \times \left(\frac{\theta^{n_1}}{n_1!2^{n_1}} \otimes \cdots \otimes \frac{\theta^{n_l}}{n_l!2^{n_l}}\right)\right] \ .$$

Since j'_n annihilates any Chinese character with i-filter $2n+1$, we can restrict to the case with $m, q \leq 2n$. And since j'_n annihilates any Chinese character diagram with more than $4n$ external vertices on one components, we may assume that all the n_k are less than or equal to $2n$. This means the above term has denominator not divisible by any prime greater than $2n+1$. It remains to use Lemma 6.4 (proved below) to conclude that j'_n of this term has denominator not divisible by any prime greater than $2n+1$. □

Theorem 6.2. *Suppose that M is a rational homology 3-spheres, i.e. $H_1(M, \mathbb{Q}) = 0$. Then the degree n part of $\Omega(M)$ has denominator not divisible by any prime greater than $2n+1$.*

Proof. It's sufficient to show that $\Omega_n(M)$ has denominator not divisible by any prime greater than $2n+1$. Suppose M is obtained from S^3 by surgery along a framed link L with diagonal linking matrix. Then by the previous proposition, $j'_n(\check{Z}(L))$ has denominator not divisible by any prime greater than $2n+1$, hence so do $\iota_n(\check{Z}(L))$ and $\Omega_n(M)$.

For an arbitrary rational homology 3-sphere M, Ohtsuki [Oh2] showed there are lens spaces M_i of type $(k_i, 1)$, $i = 1, \ldots, l$ such that the connected of M and all the M_i is obtained from S^3 by surgery along a framed link with diagonal linking matrix.

Since (see [LMO])

$$\Omega_n(M\#M') = \Omega_n(M)\Omega_n(M') ,$$

it's sufficient to consider the case when M is obtained from S^3 by surgery along a framed link L with diagonal matrix. □

6.3. On the map j'_n. Let $\mathscr{A}(m)$, for positive number m, be the subspace of $\mathscr{B}_m$ spanned by m-marked Chinese characters which have exactly 1 vertex of each color $\{1, 2, \ldots, m\}$. Let $\mathscr{A}(0) = \mathbb{Q}$. The symmetric group $\mathfrak{S}_m$ acts on $\mathscr{A}(m)$ by permuting the colors. For an element τ in the symmetric group $\mathfrak{S}_{m-2}$ acting on the set $\{2, 3, \ldots, m-1\}$, let $T_\tau \in \mathscr{A}(m)$ be the graph shown in Figure 4.

We define $T_m \in \mathscr{A}(m)$ by

$$T_m = \sum_{\tau \in \mathfrak{S}_{m-2}} \frac{(-1)^{r(\tau)}}{(m-1)\binom{m-2}{r(\tau)}} T_\tau ,$$

where we denote by $r(\tau)$ the number of k which satisfies $\tau(k) > \tau(k+1)$. The coefficients here look very similar to those of the Eulerian idempotent $e^{(1)}_{m-1}$, see (2.1).

There is a shuffle product in the space $\prod_{m=0}^{\infty} \mathscr{A}(m)$ defined as follows. Suppose D is a Chinese character in $\mathscr{A}(m)$, D' – in $\mathscr{A}(m')$. Change the colors of external vertices of D': 1 to $m+1$, 2 to $m+2$, etc., m' to $m+m'$. The union of D and D' now is an element of $\mathscr{A}(m+m')$. Define

$$D \bullet D' := \sum_{(m,m')\text{-shuffles } \sigma \in \mathfrak{S}_{m+m'}} \sigma(D \cup D') .$$

Let $T = \sum_{m=0}^{\infty} T_m$. Let $T^{\bullet n}$ be the n-th power of T in the shuffle product. Denote by T^n_m the part of $(T^{\bullet n}/n!)$ with m external vertices. In other words,

$$T^n_m = \frac{1}{n!} \sum_{m_1 + \cdots + m_n = m} T_{m_1} \bullet \cdots \bullet T_{m_n} . \tag{6.1}$$

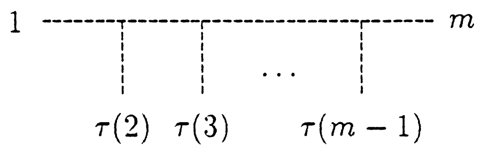

Fig. 4. The definition of T_τ

Note that if $m < 2n$, then, by definition, $T^n_m = 0$. The first non-trivial element $T^n_{2n} \in \mathscr{A}(2n)$ is the following. Partition $2n$ points $\{1, \dots, 2n\}$ into n pairs (there are $(2n-1)!!$ ways to do this), and then connect the two points of each pair by a dashed line, we get an element of $\mathscr{A}(2n)$. Summing up all such possible elements, we get T^n_{2n}.

In [LMO] it was proved that the T^n_m have the following important properties:

1) T^n_m is invariant under the cyclic permutation of the m external vertices.

2) For every n, m we have

$$T^n_m - (k \ \ k+1)(T^n_m) = T^n_{m-1} \star_k Y \ , \qquad (*)$$

where $(k \ \ k+1)$ is the permutation which interchanges k and $k+1$ $(1 \leq k \leq m-1)$, and $T^n_{m-1} \star_k Y$ denotes the element obtained from T_{m-1} by attaching a Y-shaped graph to the vertex k and then renumbering the vertices so that the remaining two vertices of Y are k and $k+1$. Equation $(*)$ is also presented in Figure 5.

For a fixed number n, we define a linear map $j_n : \mathscr{A}(\sqcup^l S^1) \to \mathscr{D}^\circ$ as follows. Suppose that $D \in \mathscr{A}(\sqcup^l S^1)$ is a Chinese character diagram and that C is a solid circle of D with m external vertices on it. Number the vertices, beginning at any vertex and following the orientation of C, by $1, 2, \dots, m$. Now remove the solid circle C, and glue the external vertices to the corresponding vertices of T^n_m. Do this with all solid circles of the Chinese character diagram D; and we get $j_n(D)$.

The well-definedness (because of the STU relation) of this map follows from equation $(*)$, and this equation can be regarded as the dual to the STU.

It follows from the definition that j_n lowers the degree of a chord diagram by ln, where l is the number of solid circles of D.

Lemma 6.3. *Suppose $x \in \mathscr{A}(\sqcup^l S^1)$ is a Chinese character diagram of degree less than or equal to $n(l+1)$. Then $j'_n(x) = j_n(x)$.*

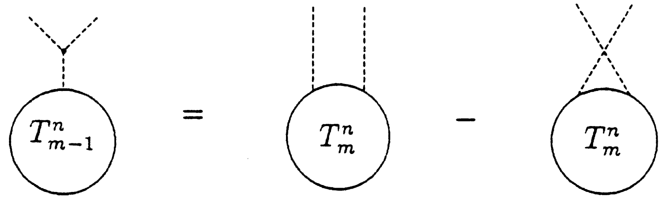

Fig. 5. The dual of the STU relation

Proof. Since χ is an isomorphism, we can assume that $x = pr(\chi(y))$, where y is an l-marked Chinese character. Suppose that y has exactly $2n$ vertices of each color. Then $j_n(x)$ is obtained by gluing T^n_{2n} to each group of vertices of the same color of y, which is, by definition, $j'_n(x)$. Hence in this case $j_n(x) = j'_n(x)$.

Now suppose one of the color, say k, of y has m vertex, where $m \neq 2n$. In this case $j'_n = 0$.

If $m < 2n$, then $j_n(x) = 0$, since $T^n_m = 0$. Hence we have $j_n(x) = j'_n(x)$.

Suppose $m > 2n$. Then T^n_m is a linear combination of Chinese character diagrams, each has at least one internal vertex, and each is a union of several Chinese character of the form T_τ (see Figure 4). Hence

$$\sum_{\sigma \in \mathfrak{S}_m} \sigma(T^n_m) = 0,$$

due to the anti-symmetry relation. On the other hand, the vertices of color k of the element $\chi(y)$ is invariant under actions of $\mathfrak{S}_m$. Hence when we glue T^n_m to the set of vertices of color k, we get 0. So in all cases $j_n(x) = j'_n(x)$. □

Proposition 6.4. *Suppose D is a Chinese character diagram in $\mathscr{A}(\sqcup^l S^1)$. Then $j'_n(D)$ has denominator not divisible by any prime greater than $2n+1$.*

Proof. We can assume that the degree of D is less than or equal to $n(l+1)$, since otherwise $j'_n(D) = 0$. By the previous proposition, $j'_n(D) = j_n(D)$ which is constructed using T^n_m, where m is between $2n$ and $4n$. In (6.1), $m_1 + \cdots + m_n = m$, hence the maximum of $\{m_1, \ldots, m_n\}$ is less than or equal to $2n+2$. From the formula of T_m it follows that the denominators of these T_{m_i} are not divisible by any prime greater than $2n+1$. □

7. Proof of Propositions 2.2, 4.2 and 4.4

7.1. The cobar complex of Chinese characters. The isomorphism χ between $\mathscr{P}_n$ and $\mathscr{B}_n$ carries the maps Δ_i, ε_i over to $\mathscr{B}_n$. These maps can be described as follows.

Suppose x is an n-marked Chinese character, with m vertices of color i. Here $i \leq n$ is a fixed number. There are 2^m ways of partition the set of vertices of color i into an ordered pair of subsets, the first and the second subsets. For each such partition, form an $(n+1)$-

marked Chinese character by first changing the color k to $k+1$ for every $k > i$, then coloring vertices in the first subset by i, in the second by $i+1$, and leave alone the vertices of color $< i$. Summing up, over all possible partitions, such $(n+1)$-marked Chinese characters, we get $\Delta_i(x)$.

If there is at least one vertex of color i, let $\varepsilon_i(x) = 0$. Otherwise $\varepsilon_i(x)$ is the $(n-1)$-marked Chinese character obtained from x by changing the color k to $k-1$ for every $k > i$.

It is easy to check that Δ_i, ε_i commute with χ.

For an n-marked Chinese character, the number of external vertices is called the *e-grading*, while half the number of all vertices is called the *degree*. Note that all the mappings Δ_i, ε_i preserve both the e-grading and degree, and χ preserves the degree.

The operator d acting on $\mathscr{P}_n$ is carried by χ over $\mathscr{B}_n$ to $d : \mathscr{B}_n \to \mathscr{B}_{n+1}$, where

$$d(\xi) = 1 \otimes \xi - \Delta_1(\xi) + \Delta_2(\xi) - \cdots + (-1)^n \Delta_n(\xi) + (-1)^{n+1} \xi \otimes 1 \ .$$

Here $1 \otimes \xi$ is the $(n+1)$-marked Chinese character obtained from x by changing the color k to $k+1$ (for every k), and $\xi \otimes 1$ – by leaving alone the colors.

7.2. Subcomplexes. An element $x \in \mathscr{B}_n$ is *non-degenerate* if $\varepsilon_i(x) = 0$ for $i = 1, 2, \ldots, n$. Let $\mathscr{F}_n$ be the subspace of $\mathscr{B}_n$ spanned by connected non-degenerate Chinese characters. Let $\mathscr{F}_n^{\mathbb{Z}}$ be the subset of $\mathscr{F}_n$ consists of elements which are linear combinations of connected non-degenerated Chinese character with integer coefficients. Then

$$\mathscr{F}_n = \mathscr{F}_n^{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \ ,$$

and we regard $\mathscr{F}_n^{\mathbb{Z}}$ as the integral lattice of $\mathscr{F}_n$. Note that χ maps $\mathscr{F}_n$ isomorphically on $\mathscr{Q}_n$. Since χ preserves the degree, we can split this isomorphisms into smaller ones.

Let $\mathscr{F}_n^{\mathbb{Z}}(m)$ (respectively, $\mathscr{F}_n^{\mathbb{Z}}(m,k)$) be the subset of $\mathscr{F}_n^{\mathbb{Z}}$ consisting of elements of degree m (respectively, degree m and e-grading k). Every connected Chinese character of degree m has at most $m+1$ external vertices. Hence

$$\mathscr{F}_n^{\mathbb{Z}}(m) = \oplus_{k=1}^{m+1} \mathscr{F}_n^{\mathbb{Z}}(m,k) \ .$$

It's important that the above summation is up to $k = m+1$. One has

$$\mathscr{F}_n(m,k) = \mathscr{F}_n^{\mathbb{Z}}(m,k) \otimes \mathbb{Q} \ .$$

Let

$$C^n(\mathscr{F}^{\mathbb{Z}}(m)) = \mathscr{F}^{\mathbb{Z}}_n(m) \ ,$$

$$C^n(\mathscr{F}^{\mathbb{Z}}(m,k)) = \mathscr{F}^{\mathbb{Z}}_n(m,k) \ .$$

7.3. The inverse of χ. Let $\mathscr{H}(m)$ be the subspace of $\mathscr{B}_{m+1}$ spanned by $(m+1)$-marked Chinese characters which have one vertex of each color $1,2,\dots,m$. The other vertices must have color $m+1$; and we recolor these ones so that they have color 0. We define a shuffle product $\mathscr{H}(m_1)\otimes\mathscr{H}(m_2) \xrightarrow{\bullet} \mathscr{H}(m_1+m_2)$ as follows. Suppose D, D' are Chinese characters in $\mathscr{A}(m_1), \mathscr{A}(m_2)$, respectively. Change the colors of D': 1 to m_1+1, 2 to m_2+1, etc, and m_2 to m_1+m_2, leaving the 0 color alone. Now $D \cup D'$ is an element of $\mathscr{H}(m_1+m_2)$. Let

$$D \bullet D' = \sum_{(m_1,\, m_2)\text{-shuffles } \sigma\in\mathfrak{S}_{m_1+m_2}} \sigma(D\cup D') \ .$$

Lemma 7.1. *If $x \in \prod_{m=0}^{\infty}\mathscr{H}(m)$ has denominator 1, then so does $x^{\bullet n}/n!$.*

Proof. In the shuffle product $x^{\bullet n}$ we have repeated terms obtained by permuting the factor x. This cancels the denominator $n!$. □

Consider the element $T_{m+1} \in \mathscr{A}_{m+1}$ defined in 6.3. Changing the color $m+1$ to 0, from T_{m+1} we get $t_m \in \mathscr{H}_m$. Let

$$t = t_1 + t_2 + \cdots + t_m + \cdots \in \prod_{m=0}^{\infty}\mathscr{H}(m) \ .$$

Then $\mathrm{e}^t \in \prod_{m=0}^{\infty}\mathscr{H}(m)$, where we use the shuffle product. Let $t(m)$ be the part of e^t lying in $\mathscr{H}_m$ $(m \geq 1)$. In other words,

$$t(m) = \sum_{n=1}^{\infty}\left[\frac{1}{n!}\left(\sum_{m_1+\cdots+m_n=m} t_{m_1}\bullet\cdots\bullet t_{m_n}\right)\right] \ .$$

The inverse of χ can be expressed by e^t as follows. Suppose $D \in \mathscr{P}_l$ is a Chinese character diagram. Suppose on the i-th string there are m external vertices. Remove the i-th string, then glue the vertices of colors $1,2,\dots,m$ of $t(m)$ to the external vertices of the i-th string, and finally change the color of the other vertices of t_m from 0 to i. Do this with all the strings. The result is $\chi^{-1}(D)$. The well-definedness follows from $(*)$. The fact that $\chi(\chi^{-1}(D)) = D$ is easy to verify.

Proposition 7.2. a) *If x is a Chinese character in $\mathscr{B}_l$ having m external vertices, then $\chi(x)$ has denominator $m!$.*

If D is a Chinese character diagram in $\mathscr{P}_l$ with m external vertices then $\chi^{-1}(D)$ has denominator $m!$.

b) *If x is a Chinese character in $\mathscr{B}_l$ having m external vertices, where $l \geq 2$, then $\chi(x)$has denominator $(m-2)!(m-1)!$.*

If *D is a Chinese character diagram in $\mathscr{P}_l$ having m external vertices, where $l \geq 2$, then $\chi^{-1}(x)$ has denominator $(m-2)!(m-1)!$.*

Proof. a) The first statement follows directly from the definition of χ. Let us prove the second statement. As discussed above, χ^{-1} can be constructed explicitly using e^t. By Lemma 7.1 $t^n/n!$ has the same denominator as t. From the formula of $t_m = T_{m+1}$ in §6.3 it follows that has denominator $m!$.

b) follows from the proof of a) with the following observation: if both p, q are positive integers, then $p!q!$ is a divisor of $(p+q-1)!(p+q-2)!$. □

7.4. Some cohomology. Instead of the complex $(C^*(\mathscr{Q}^{\mathbb{Z}}(m)), d)$, we will study the complex $C^*(\mathscr{F}^{\mathbb{Z}}(m)), d)$ which is the direct sum of the complexes $C^*(\mathscr{F}^{\mathbb{Z}}(m,k)), d)$, with $k = 1, 2, \ldots, m+1$:

$$C^*(\mathscr{F}^{\mathbb{Z}}(m)), d) = \sum_{k=1}^{m+1} C^*(\mathscr{F}^{\mathbb{Z}}(m,k)), d) \tag{7.1}$$

For a fix number k consider the following $\mathbb{Z}$-complex $E(k)$. Let $C^n(E(k))$ be the $\mathbb{Z}$-module spanned by partitions $\theta_1, \ldots, \theta_n$ of $\{1, 2, \ldots, k\}$, such that each θ_i is a non-empty subset of $\{1, 2, \ldots, k\}$. So if $n > k$ then $C^n(E(k)) = 0$. Define

$$d : C^n(E(k)) \to C^{n+1}(E(k)) \quad \text{by}$$

$$d(\theta_1, \ldots, \theta_n) = (d\theta_1, \theta_2, \ldots, \theta_n) - (\theta 1, d\theta_2, \theta_3, \ldots, \theta_n) + \cdots + (-1)^{k-1}(\theta_1, \ldots, d\theta_n) \ ,$$

where for a non-empty set θ we set $d\theta = \sum(\theta', \theta'')$, the sum is over all possible partition of θ into an order pair θ', θ'' of non-empty subsets. Actually, $(C^*(E(k)), d)$ is the cochain complex of a triangulation of the k-dimensional punctured sphere (see, for example, [LM2], 9.2), and we have

Proposition 7.3. *One has that*

$$H^k(C^*(E(k))) = \mathbb{Z} \ ,$$

and

$$H^l(C^*(E(k))) = 0 \quad \text{for } l \neq k \ .$$

The symmetric group $\mathfrak{S}_k$ acts on the left on the complex $C^*(E(k))$ by permuting the numbers in the partitions. The action is compatible with the operator d. We will see that $(C^*(\mathscr{F}^{\mathbb{Z}}(m,k)), d)$ is isomorphic to the tensor product of the complex $(C^*(E(k)), d)$ and a $\mathfrak{S}_k$-module which we are going to describe.

Let $\Gamma^{\mathbb{Z}}(m,k)$ be the subset of $\mathscr{F}_k$ of $\mathbb{Z}$-linear combination of non-degenerate k-marked Chinese character of degree m with exactly one vertex of each color $\{1, 2, \ldots, k\}$. And let $\Gamma(m,k) = \Gamma^{\mathbb{Z}}(m,k) \otimes_{\mathbb{Z}} \mathbb{Q}$. The symmetric group $\mathfrak{S}_k$ acts on the right on $\Gamma^{\mathbb{Z}}(m,k)$ by permuting the colors of external vertices.

Lemma 7.4. *For fixed m, k, there is an isomorphism between complexes $(C^*(\mathscr{F}^{\mathbb{Z}}(m,k)), d)$ and $\Gamma(m,k) \otimes_{\mathfrak{S}_k} (C^*(E(k)), d)$.*

Proof. Consider a k-marked Chinese character ξ in $\Gamma^{\mathbb{Z}}(m,k)$. The k external vertices are colored by $\{1, 2, \ldots, k\}$. We map the element $\xi \otimes (\theta_1, \ldots, \theta_n)$ to the element η of $C^n(\mathscr{F}^{\mathbb{Z}}(m,k))$ obtained from ξ by changing the colors in θ_i to i. It can be verified at once that this is an isomorphism between the two complexes. □

Lemma 7.5. *The cohomology groups of the complex $\Gamma^{\mathbb{Z}}(m,k) \otimes_{\mathfrak{S}_k} (C^*(E(k)), d)$ is annihilated by $k!$, except for the k-th cohomology group.*

Proof. First let us consider the tensor product over $\mathbb{Z}$: $\Gamma^{\mathbb{Z}}(m,k) \otimes_{\mathbb{Z}} C^*(E(k)), d)$. This complex has 0 cohomology, except for the cohomology of dimension k, by Proposition 7.3 and the universal coefficient formula for homology.

There is a natural projection

$$p_n : \Gamma^{\mathbb{Z}}(m,k) \otimes_{\mathbb{Z}} C^n(E(k)) \to \Gamma^{\mathbb{Z}}(m,k) \otimes_{\mathfrak{S}_k} C^n(E(k)) \ .$$

The kernel $\ker(p_n)$ is a finite $\mathbb{Z}$-module whose cardinality is a divisor of $|\mathfrak{S}_k| = k!$. Consider the short exact sequence of cochain complexes:

$$0 \to \ker(p_n) \to \Gamma^{\mathbb{Z}}(m,k) \otimes_{\mathbb{Z}} C^n(E(k)) \to \Gamma^{\mathbb{Z}}(m,k) \otimes_{\mathfrak{S}_k} C^n(E(k)) \to 0 \ .$$

By the above observation, the cohomology of the middle complex vanishes unless in dimension k, while the cohomology of the left one is annihilated by $k!$. From the long exact sequence derived from this short exact sequence, we get the lemma. □

Lemma 7.6. *If* $x \in C^4(\mathscr{F}^{\mathbb{Z}}(m,k))$ *is symmetric and* $dx = 0$, *then there is a symmetric* $y \in C^3(\mathscr{F}(m,k))$, *such that* $dy = x$. *Moreover,* y *has denominator* $2k!$.

Proof. By Lemma 7.4 we may suppose that $x \in \Gamma^{\mathbb{Z}}(m,k) \otimes_{\mathfrak{S}_k} C^4(E(k))$. Consider two cases: $k = 4$ and $k \neq 4$.

Suppose $k = 4$. Then x is equal to $\gamma \otimes e$, where $\gamma \in \Gamma^{\mathbb{Z}}(m,4)$ and $e = (\{1\},\{2\},\{3\},\{4\})$, a partition of $\{1,2,3,4\}$.

Let

$$y' = (12,3,4) - (2,13,4) + (2,3,14) - (2,34,1) + (24,3,1) - (4,23,1) \ ,$$

where, for example, $(2,13,4)$ means the element $\gamma \otimes (\{2\},\{1,3\},\{4\})$ in $\Gamma^{\mathbb{Z}}(m,4) \otimes_{\mathfrak{S}_4} C^3(E(4))$. One can readily check that

$$dy' = x - x^{4321} = 2x \ .$$

Let $y = (1/4)[y' - (y')^{321}]$; we see that y is symmetric and $dy = x$.

Now consider the case $k \neq 4$. By Lemma 7.5 there is $y' \in \Gamma^{\mathbb{Z}}(m,k) \otimes_{\mathfrak{S}_k} C^3(E(k))$ such that $dy' = (k!)x$. Now $y = [y' - (y')^{321}]/(2k!)$ is the element to find. □

7.5. Proof of Proposition 2.2. Suppose $\xi \in C^4_{\mathrm{sym}}(\mathscr{Q}^{\mathbb{Z}})$ is of degree m and $d\xi = 0$. Then $\xi' = \chi^{-1}(\xi)$ is symmetric and in $C^4(\mathscr{F}^{\mathbb{Z}}(m))$. In addition, $\chi(\xi)$ has denominator $m!(m-1)!$, since ξ, as an element of degree m, can have at most $m+1$ external vertices. (Here we use the fact that ξ is a linear combination of g-connected elements, and Proposition 7.2, part b)).

Using the decomposition (7.1), we can assume that ξ' is in $C^4(\mathscr{F}(m,k))$ with some $k \leq m+1$. Besides ξ' is symmetric, has denominator $m!(m-1)!$ and $d\xi' = 0$. From Lemma 7.6 it follows that there is $y \in C^3(\mathscr{F}(m,k))$, symmetric, with denominator $2m!(m-1)!k!$ such that $dy = \xi'$. Since $k \leq m+1$, the number $2m!(m-1)!k!$ is a divisor of $2m!(m-1)!(m+1)!$. Now let $\eta = \chi^{-1}y$. Then

$$d\eta = \chi^{-1}(\xi') = \xi \ ,$$

and η is symmetric. By part b) of Proposition 7.2, η has denominator $2[m!(m-1)!]^2(m+1)!$. This completes the proof of Proposition 2.2.

7.6. On the numbers d_n. We will write $p \lhd q$ if p is a divisor of q.

Lemma 7.7. a) *For any positive integers p, q one has $(p+q)!d_p d_q \lhd d_{p+q}$.*

b) *For any positive integers p, q one has $d_p d_q \lhd 2d_{p+q-1}$.*

c) *For any integers $p, q \geq 2$ one has $d_p d_q \lhd 96 d_{p+q-2}$.*

Proof. We will prove a). The others can be proved in a similar way.

Note that $d_{p+1} = d_p \times [(p+1)!]^3(p+2)!$. We use induction on q. The statement is true for $q = 1$. Suppose it has been true for q. Then,

$$(p+q+1)!d_p d_{q+1} = (p+q+1)!d_p d_q[(q+1)!]^3(q+2)!$$

By the induction hypothesis, the latter is a divisor of $(p+q+1)d_{p+q}[(q+1)!]^3(q+2)!$

Since $(p+q+1)(q+1)! \lhd (p+q+1)!$, the number $(p+q+1) d_{p+q}[(q+1)!]^3(q+2)!$ is a divisor of $d_{p+q}[(p+q+1)!]^3(p+q+2)!$ which is d_{p+q+1}. This completes the proof of a). □

Corollary 7.8. *For every positive integers $n_1, n_2, \ldots n_k$, the number $k!d_{n_1} \cdots d_{n_k}$ is a divisor of $d_{n_1+\cdots+n_k}$.*

Proof. From part a) of the previous lemma, one has that $(n_1 + \cdots + n_k)!d_{n_1} \cdots d_{n_k}$ is a divisor of $d_{n_1+\cdots n_k}$. It remains to notice that $k \leq (n_1 + \cdots + n_k)$, since each n_i is a positive integer. □

We recall here the Campbell-Hausdorff formula. Let $B = \mathbb{Q}\langle\langle x^{(1)}, x^{(2)}, \ldots, x^{(l)}\rangle\rangle$ be the algebra of formal power series in l non-commuting variables. This algebra is graded by the degree of monomials in $x^{(j)}, j = 1, 2, \ldots, l$. The free Lie algebra $\mathscr{L}$ over $\mathbb{Z}$ (the set of integers) generated by $x^{(j)}, j = 1, 2, ..., l$ is a subset of B, and an element in B is called a *Lie polynomial* if it is a $\mathbb{Q}$-linear combination of elements in $\mathscr{L}$.

The Campbell-Hausdorff formula says that

$$\exp(x^{(1)}) \cdots \exp(x^{(l)}) = \exp\left[\sum_{k=1}^{\infty} f_k(x^{(1)}, \ldots, x^{(l)})\right],$$

where f_k is a homogeneous Lie polynomial of total degree k. Moreover from the Dynkin's form of the Campbell-Hausdorff formula (see, for example [Ser]), one can easily see that each f_k is a $\mathbb{Q}$-linear combination of elements in $\mathscr{L}$ whose coefficients have denominators $(k!)^2$.

7.7. Proof of Lemmas 4.2 and 4.4. We have that (modulo part of degree $\geq 2m+3$)

$$1+\psi = \exp(\phi_{2m}^{312})\exp\left(\frac{r^{13}}{2}\right)\exp(-\phi_{2m}^{132})\exp\left(\frac{r^{23}}{2}\right) \times \exp(\phi_{2m})\exp\left(\frac{-r^{13}-r^{23}}{2}\right) .$$

Hence

$$\psi = \sum_k f_k\left(\phi_{2m}^{312}, \frac{r^{13}}{2}, -\phi_{2m}^{132}, \frac{r^{23}}{2}, \phi_{2m}, \frac{-r^{13}-r^{23}}{2}\right) . \tag{7.2}$$

Recall that ϕ_{2m} is a sum of even degree parts, the part of degree $2k$ has denominator d_{2k}. The element $r^{ij}/2$ has degree 1, and has denominator $d_1 = 2$. Replacing ϕ_{2m} by the sum of its degree $2k$ parts, $k = 1, \ldots, m$, then expanding the right hand side, we see that ψ is a sum of elements having denominator $(k!)^2 d_{n_1} \cdots d_{n_k}$, where $n_1 + \cdots + n_k = 2m+2$. So we need to show that $(k!)^2 d_{n_1} \cdots d_{n_k}$ is a divisor of $[(2m+2)!]^2 d_{2m}$ if $n_1 + \cdots + n_k = 2m+2$. Since all $n_i \leq 2m$, the number k is greater than 1. We assume that $n_1 \leq n_2 \leq \cdots \leq n_k$. Consider several cases.

The case $k = 2$. Then each of n_1, n_2 must be greater than 1. By Lemma 7.7, part c),

$$(k!)^2 d_{n_1} d_{n_2} = 4 d_{n_1} d_{n_2}$$

is a divisor of $4 \times 96 d_{2m}$, which, in turn, is a divisor of $[(2m+2)!]^2 d_{2m}$ if $m \geq 2$.

The case $2m+1 \geq k \geq 3$. Applying Lemma 7.7, part b), repeatedly, we have that

$$d_{n_1} d_{n_2} \cdots d_{n_k} \lhd 2^{k-1} d_{n_1+n_2+\cdots+n_k-k+1} = 2^{k-1} d_{2m+3-k} .$$

If $k = 3$, then $(k!)^3 d_{n_1} d_{n_2} d_{n_3} \lhd (3!)^2 \times 2^2 \times d_{2m}$, which is a divisor of $[(2m+2)!]^2 d_{2m}$.

If $k \geq 4$, then $2m+3-k \leq 2m-1$. Noting that $k \leq 2m+1$, one has

$$(k!)^2 d_{n_1} \cdots d_{n_k} \lhd 2^{k-1}(k!)^2 d_{2m-1} \lhd 2^{2m}[(2m+1)!]^2 d_{2m-1} .$$

The latter is a divisor of $[(2m+2)!]^2 d_{2m}$.

The case $k = 2m + 2$. Then each $n_i, i = 1, 2, \ldots, 2m + 2$ is 1. One has

$$(k!)^2 d_{n_1} \cdots d_{n_{2m+2}} = 2^{2m+2} [(2m + 2)!]^2$$

is a divisor of $[(2m + 2)!]^2 d_{2m}$. This completes the proof of Lemma 4.2. The proof of Lemma 4.4 is similar, even easier.

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