THE LMO INVARIANT

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Abstract. We present the theory of the LMO invariant and related topics.

0. Introduction

In Section 1 we briefly present the *framed* Kontsevich integral for *framed oriented links and tangles* and describe how the Kontsevich integral behaves under the handle slide moves. Section 2 is devoted to various definitions of the LMO invariant and a proof of the invariance. In section 3 we recall some fundamental concepts in the theory of finite type invariants of homology 3-spheres, following Gusarov and Habiro. In section 4 we prove the universality of the LMO invariant and some other corollaries.

1. Chord diagrams and the Kontsevich integral

1.1. **Chord diagrams.** Our definition of chord diagrams is more general than that of [BN, LM1, LM2]: it includes all "Chinese character diagrams" of [BN], and even more. All vector spaces are over the field $\mathbb{Q}$ of rational numbers.

A *uni-trivalent graph* is a graph every vertex of which is either univalent or trivalent. A uni-trivalent graph is *vertex-oriented* if at each trivalent vertex a cyclic order of edges is fixed. A 3-valent (resp. 1-valent) vertex is called an *internal* (*external*) vertex.

Let X be a compact oriented 1-dimensional manifold. A *chord diagram* with support X is the manifold X together with a vertex-oriented uni-trivalent graph whose external vertices are on X; and the graph does not have any connected component homeomorphic to a circle. In figures components of X are depicted by solid lines, while the graph is depicted by dashed lines, with the convention that the orientation at every vertex is counterclockwise. *There may be connected components of the dashed graph which do not have univalent vertices*, and hence do not connect to any solid lines.

Let $\mathcal{A}(X)$ be the vector space spanned by chord diagrams with support X, subject to the AS, IHX and STU relations (see Figure 1).

The *degree* of a chord diagram is half the number of (external and internal) vertices of the *dashed graph*. We also use $\mathcal{A}(X)$ to denote the completion of $\mathcal{A}(X)$ with respect to the grading.

Of particular interest is the space $\mathcal{A}(\emptyset)$, i.e. when there are no solid lines nor circles. Then every dashed graph must be 3-valent. This vector space $\mathcal{A}(\emptyset)$ is a commutative algebra in which the product of 2 graphs is just the union of them. All $\mathcal{A}(X)$ can be regarded as graded $\mathcal{A}(\emptyset)$-modules, where the product of a 3-valent graph $\xi \in \mathcal{A}(\emptyset)$ and a chord diagram $\xi' \in \mathcal{A}(X)$ is their disjoint union.

1.2. **Some operations on chord diagrams.** *Antipode.* Suppose C is a component of X. Let X' be the same X with reverse orientation on C. Let $S_{(C)} : \mathcal{A}(X) \to \mathcal{A}(X')$ be the linear map which transfers every chord diagram ξ in $\mathcal{A}(X)$ to $S_{(C)}(\xi)$ obtained from

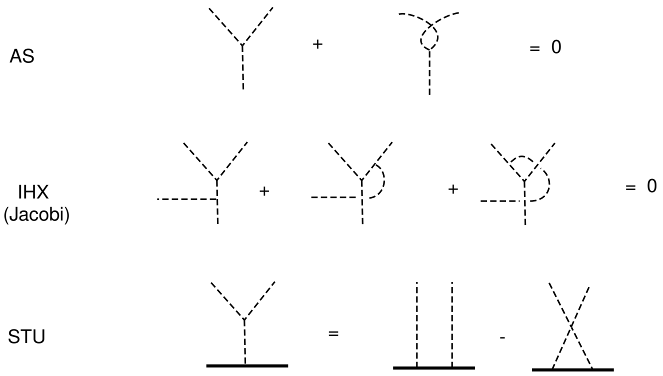


FIGURE 1

ξ by reversing the orientation of C and multiplying by $(-1)^m$, where m is the number of vertices of the dashed graph on the component C.

Doubling. Let $X^{(2,C)}$ be the space obtained from X by doubling the component C, using the framing. We define the linear operator $\Delta_{(C)} : \mathcal{A}(X) \to \mathcal{A}(X^{X^{(2,C)}})$ as in Figure 2.

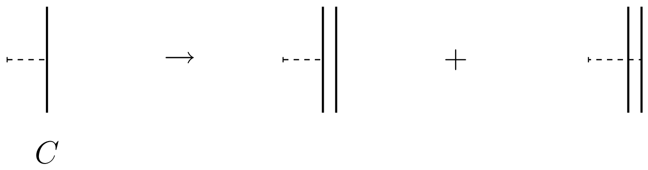


FIGURE 2.

Connected sum. When $X = S^1$, the connected sum defines an algebra structure on $\mathcal{A}(S^1)$ which is known to be commutative (see [BN]).

Co-product $\hat{\Delta} : \mathcal{A}(X) \to \mathcal{A}(X) \otimes \mathcal{A}(X)$. Let

$$\hat{\Delta}(\xi) = \sum \xi' \otimes \xi''.$$

Here the sum is over all chord sub-diagrams ξ' of ξ, and ξ'' is the complement of ξ'. This co-multiplication is co-commutative.

A similar co-product is defined for $\mathcal{A}(\emptyset)$, by the same formula, and $\mathcal{A}(\emptyset)$ becomes a commutative co-commutative Hopf algebra. Hence $\mathcal{A}(\emptyset)$ is the polynomial algebra on *primitive* elements, i.e. elements x such that $\hat{\Delta}(x) = 1 \otimes x + x \otimes 1$. It is easy to see that an element is primitive if and only if it is a linear combination of *connected* 3-valent graphs.

Algebra structure. In general, there is no natural algebra structure on $\mathcal{A}(X)$. However, when X is a set of n lines (imagine n vertical lines with orientation pointing downwards), then $\mathcal{P}_n := \mathcal{A}(X)$ has a natural algebra structure, where $\xi\xi'$ is obtained by stacking ξ on top of ξ'. Together with the co-product $\hat{\Delta}$, $\mathcal{P}_n$ is a Hopf algebra. It is known that $\mathcal{P}_1$ is isomorphic to $\mathcal{A}(S^1)$ (see [BN].

1.3. **The framed Kontsevich integral: a combinatorial definition.**

1.3.1. *q-tangles.* We fix an oriented 3-dimensional Euclidean space $\mathbb{R}^3$ with coordinates (x, y, t). *A tangle* is a smooth one-dimensional compact oriented manifold $L \subset \mathbb{R}^3$ lying between two horizontal planes $\{t = a\}, \{t = b\}, a < b$ such that all the boundary points are lying on two lines $\{t = a, y = 0\}, \{t = b, y = 0\}$, and at every boundary point L is orthogonal to these two planes.

A normal vector field on a tangle L is a smooth vector field on L which is nowhere tangent to L (and, in particular, is nowhere zero) and which is given by the vector $(0, -1, 0)$ at every boundary point. *A framed tangle* is a tangle enhanced with a normal vector field. Two framed tangles are isotopic if they can be deformed by a 1-parameter family of diffeomorphisms into one another within the class of framed tangles.

One assigns a symbol $+$ or $-$ to all the boundary points of a tangle according to whether the tangent vector at this point directs downwards or upwards. Then on the top boundary line of a tangle diagram we have a word w_t of symbols consisting of $+$ and $-$. Similarly on the bottom boundary line there is also a word w_b of symbols $+$ and $-$.

A non-associative word on $+, -$ is an element of the *free magma* generated by $+, -$ (see [Ser]). A *q-tangle* (or *non-associative* tangle) is a tangle whose top and bottom words w_t, w_b are equipped with some non-associative structure.

1.3.2. *Composition and tensor product.* If T_1, T_2 are tangles such that $w_b(T_1) = w_t(T_2)$ we can define the product $T = T_1 T_2$ by placing T_1 on top of T_2. In this case, if $\xi_1 \in \mathcal{A}(T_1), \xi_2 \in \mathcal{A}(T_2)$ are chord diagrams, then the *product* $\xi_1\xi_2$ is a chord diagram in $\mathcal{A}(T)$ obtained by placing ξ_1 on top of ξ_2.

For any two framed q-tangles T_1, T_2, we define their *tensor product* $T_1 \otimes T_2$ by putting T_2 on the right of T_1, with the obvious non-associative structure. Similarly, if $\xi_1 \in \mathcal{P}(T_1), \xi_2 \in \mathcal{P}(T_2)$ are chord diagrams, then one defines $\xi_1 \otimes \xi_2 \in \mathcal{P}(T_1 \otimes T_2)$ by the same way.

It is easy to see that every framed q-tangle T can be obtained from *elementary q-tangles*, using the composition and tensor product. Here the elementary q-tangles are listed in Figure 3 with blackboard framing, with all possible orientation. The q-tangle T_{w_1,w_2,w_3} has trivial underlying tangle, with $w_b = (w_1w_2)w_3$ and $w_t = w_1(w_2w_3)$.

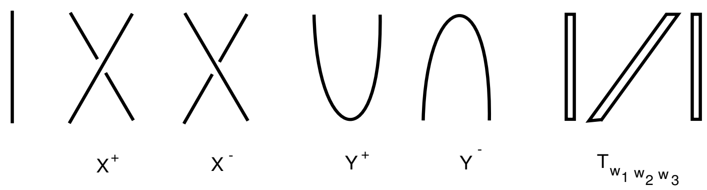


FIGURE 3. elementary q-tangles

1.3.3. *The framed Kontsevich integral.* The framed Kontsevich integral $Z(T) \in \mathcal{A}(T)$ is defined so that

$$Z(T_1T_2) = Z(T_1)Z(T_2), \qquad \text{and} \quad Z(T_1 \otimes T_2) = Z(T_1) \otimes Z(T_2)$$

With these requirements, one needs only to define Z of the elementary framed q-tangles. For $X^{\pm}$ with downward orientation, see Figure 4.

For T_{w_1,w_2,w_3}, with downward orientation:

$$Z(T_{w_1w_2w_3}) = \Delta^{(|w_1|)} \otimes \Delta^{(|w_2|)} \otimes \Delta^{(|w_3|)}(\Phi).$$

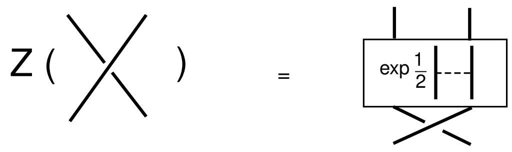


FIGURE 4. Values of the Kontsevich integral of a crossing

Here $|w|$ is the length of the word w and $\Delta^{(1)} = id$, $\Delta^{(2)} = \Delta$, $\Delta^{(n)} = \Delta_1 \circ \Delta_1 \circ \cdots \circ \Delta_1$ ($n-1$ times). The operation $\Delta^{(n)}$ replaces one string of the support by n strings. Here Φ is a special element in $\mathcal{P}_3$ called the *associator* (see §1.4). The right hand side of the above equation means that we apply $\Delta^{(|w_1|)}$ to the first string, $\Delta^{(|w_2|)}$ to the second, and $\Delta^{(|w_3|)}$ to the third string of the support of Φ.

For $Y^{\pm}$, we put $Z(Y^{\pm}) = \sqrt{\nu}$, where ν^{-1} is defined as in Figure 5. Here C_2 is the second string. Note that ν is Z of the unknot.

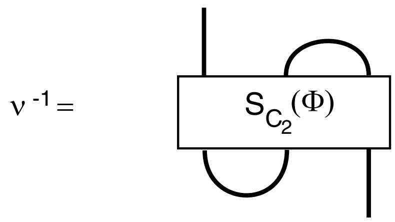


FIGURE 5. The value of the unknot ν^{-1}

If T' is obtained from T by reversing the orientation of a component C, then we put $Z(T') = S_C[Z(T)]$.

These requirements define $Z(T)$ uniquely. It is known (see [LM1, LM2]) that Z is well-defined and is an isotopy invariant of framed q-tangles. In fact, $Z(T)$ is a universal finite type invariant of framed q-tangles. For more properties of Z, see [LM2].

For a knot K, the natural projection from $\mathcal{A}(S^1)$ to $\mathcal{A}(S^1)/\approx$ takes $Z(K)$ to the original Kontsevich integral $\tilde{Z}(K)$ of K (see [LM1]). Here $\approx$ is the equivalence relation generated by: any chord diagram with an isolated dashed chord is equivalent to 0. The original Kontsevich integral is given by an explicit formula involving iterated integral (see [Ko1, BN]).

Exercise 1.1. Show that $Z(T)$ is always a group-like element in $\mathcal{A}(T)$.

Hint. It is sufficient to show this fact for elementary tangles.

1.4. **Even Associator.** Associator is an element in $\mathcal{P}_3$ satisfying some equations. See [LM2] for the list of equations. The most well-known associator is the KZ associator which is a special monodromy of the Knizhnik-Zamolodchikov equation. The KZ associator gives rise invariant of tangles which is a special limit of the original Kontsevich integral (see [LM1]). In [LM2] it was proved that any two associators are *gauge equivalent*, implying that the invariants they define are the same for links. For q-tangles, the invariants corresponding to different associators may not be the same, but they are always conjugate. For technical convenience, it is sometimes desirable to have an associator which makes Z have nice symmetry.

1.4.1. *Symmetry.* Let f be a rotation by 180^o about a horizontal or vertical line. The symmetry we want is:

$$Z(fT) = fZ(T), \tag{1.1}$$

for every q-tangle T.

For elementary q-tangles, except for T_{w_1,w_2,w_3} the above equation holds true trivially. The only remaining case actually is the case of the q-tangle T_a in Figure 6 whose value is the associator.

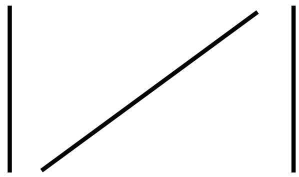


FIGURE 6. The tangle corresponding to the associator

If f is the rotation about a vertical line, then Equation (1.1) for T_a amounts to

$$\Phi^{321} = \Phi^{-1},$$

which always holds, since it is one of the axioms of the associator. Here Φ^{321} is obtained from Φ be permuting the strings of the support: $1 \to 3, 3 \to 1, 2 \to 2$.

The case when f is a rotation about a horizontal line is more complicated. Let us restrict ourselves only to the case when the associator is "horizontal", that is, $\Phi = \varphi(r_{12}, r_{23})$, where r_{12}, r_{23} are the two chord diagrams depicted in Figure 7 and $\varphi(A, B) \in \mathbb{C}\langle\langle A, B\rangle\rangle$

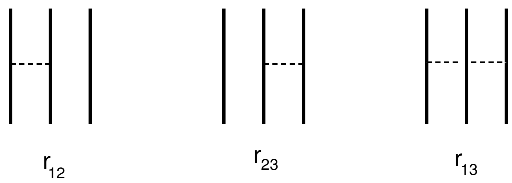


FIGURE 7. The three fundamental horizontal chord diagram

is a formal power series in two non-commuting variables A, B. For example, the KZ associator has this form. For an explicit formula of the KZ associator, see [LMO].

After the rotation about a horizontal line, Φ maps to $\varphi^{op}(r_{12}, r_{23})$, where $op : \mathbb{C}\langle\langle A, B\rangle\rangle \to \mathbb{C}\langle\langle A, B\rangle\rangle$ is the anti-automorphism mapping A to A and B to B. Thus to get identity (1.1) for the horizontal rotation one needs

$$\varphi^{op} = \varphi^{-1}. \tag{1.2}$$

Drinfeld showed that many associators φ could be found in the form $\varphi = \exp(\phi)$, where $\phi \in Lie(A, B)$ is a Lie series in 2 variables A, B. Equation (1.2) is equivalent to

$$\phi^{op} = -\phi.$$

On the vector space $Lie(A, B)$ the involution op has 2 eigenspaces, the $+1$ eigenspace consists of all elements of odd length, the -1 eigenspace – elements of even length. Hence, if φ is an even associator of the above form then Z enjoy all the symmetries described in Equation (1.1). Luckily, Drinfeld had proved the existence of such an even associator [Dri]. From now on we will use an such an even associator.

1.4.2. *Doubling formula.* For an even associator the doubling formula is valid for every q-tangle:

Proposition 1.1. *Suppose $T^{(2,C)}$ is obatined from T by doubling the component C, using the framing. Then*

$$Z(T^{(2,C)}) = \Delta_{(C)}(Z(T)).$$

Exercise 1.2. Prove this proposition. (Or see a proof in [LM3]).

Hint: For all elementary q-tangles, except for the one corresponding to the maximal points, the statement is trivial.

Note that if one uses a general associator, for example the KZ associator, then the proposition does not hold true.

1.5. **The first non-trivial term.** Suppose that T is a string link (or pure tangle) of l components. Denote the first non-trivial term of $Z(T)$ by $s(T)$, i.e.

$$Z(T) = 1 + s(T) + h.o.t.$$

Exercise 1.3. Show that $s(T)$ is a linear combination of *g-connected* chord diagrams – chord diagrams whose dashed graphs are connected.

Hint: Use the fact that $Z(T)$ is a group-like element in the co-commutative Hopf algebra $\mathcal{P}_m$.

Suppose that α, β are pure braid. The following proposition, simple but important, relates commutator in the pure braid group to commutators in the algebra of chord diagrams.

Proposition 1.2.

$$s(\alpha\beta\alpha^{-1}\beta^{-1}) = s(\alpha)s(\beta) - s(\beta)s(\alpha).$$

The proof is left as an exercise.

Let g_{13}, g_{23}, g_{123} be the pure braids depicted in Figure 8.

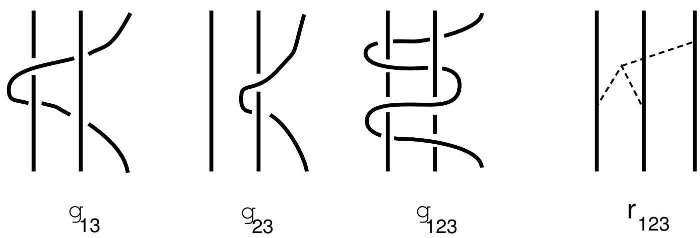


FIGURE 8

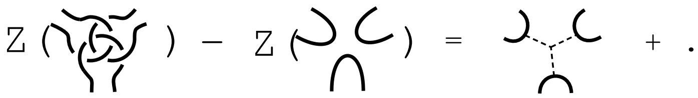
The following then follows.

Proposition 1.3. *One has that*

$$Z(\gamma_{123}) = 1 + r_{123} + (\textit{terms having} \geq 2 \textit{ internal vertices}).$$

Proof. Note that $g_{123} = g_{13}^{-1} g_{23} g_{13} g_{23}^{-1}$. It is easy to show that $s(g_{ij}) = r_{ij}$. Hence $s(g_{123}) = [r_{23}, r_{13}]$, by the previous proposition. It remains to notice that $r_{123} = [r_{23}, r_{13}]$. ☐

From this we obtain



where the dots stand for terms with ≥ 2 internal vertices. (Since the difference between the two tangles is the same as the difference between γ_{123} and the trivial 3-string pure braid.)

Further generalization explains the relation between the Kontsevich integral and Milnor invariants (see [HM]).

Conjecture . The first non-trivial term of $\tilde{Z}(K)$ has integer coefficients. Here $\tilde{Z}(K)$ is the unframed version (or the original version) of the Kontsevich integral.

If the conjecture is not true, then there exists a non-trivial FTI over a field of positive characteristic. Since this conjecture follows from the conjecture that the set of rational FTI is as powerful as the set of FTI invariants.

1.6. **The behavior of the Kontsevich integral under the second Kirby move.** The handle slide move can be seen as first to make a double, then a band sum, see the figure 9.

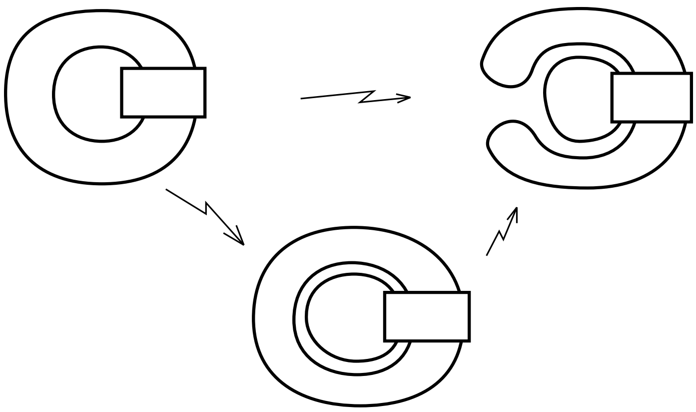
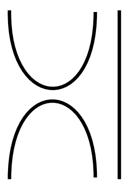
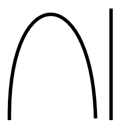


FIGURE 9. The Handle slide move

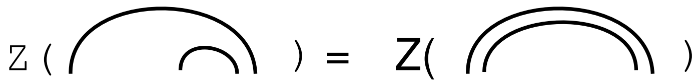
We expect the Kontsevich integral behaves in the same way: doubling followed by a band sum. For this purpose, as it is clear from the picture, we need to calculate the Kontsevich integral of the tangle a

a = 

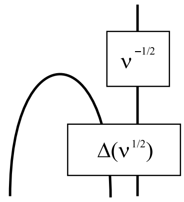
Because of the symmetry, we need only to calculate Z of

b = 

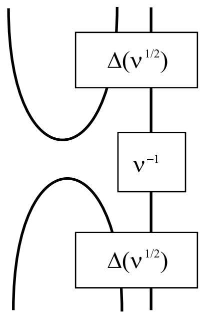
Exercise 1.4. Show that Z of the following two q-tangles are the same



From this exercise one sees at once that $Z(b)$ is



Hence $Z(a)$ is



Let us define another normalization $\check{Z}(L)$ of $Z(L)$ by

$$\check{Z}(L) = Z(L)\#\nu^{\otimes m},$$

where the right hand side means taking connected sum of ν with each component of L.

It follows that if L' is obtained from L a handle slide move, then

$$\check{Z}(L') = \text{band sum of (doubling of } \check{Z}(L)).$$

Warning. The band sum is not well-defined. The above statement just says that one of the values of the band sum is $\check{Z}(L')$. For details see [LMO].

2. Definitions of the LMO invariants

2.1. Removing solid loops: the maps ι_n.

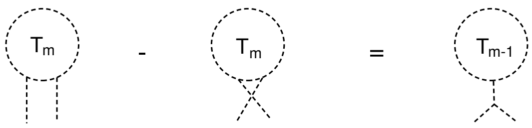
2.1.1. *A general way to remove solid loops.* Let $\mathcal{A}(m)$, for any positive integer m, be the vector space spanned by uni-trivalent vertex-oriented graphs with exactly m 1-valent vertices numbered $1, 2, \ldots, m$, subject to the AS and IHX relations.

Exercise 2.1. Show that $\mathcal{A}(1) = 0$.

We would like to replace solid circles with dashed graphs. A natural approach is the following. Suppose C is a solid circle (of some chord diagram) with m external vertices on it. Number the vertices, beginning at any vertex and following the orientation of C, by 1,2,...,m. Now remove the solid circle C, and glue the external vertices to the corresponding vertices of a *fixed* element T_m in $\mathcal{A}(m)$. Do this with all solid circles of the chord diagram; and we get a map j which transfers chord diagrams on l solid loops to chord diagrams without solid components. We always suppose that $T_0 = T_1 = 0$.

This map j is well-defined if and only if the elements $T_m, m = 2, 3, ...$ satisfy the following conditions (*) and (**).

$$T_m \text{ is invariant under cyclic permutation} \qquad (*)$$



(**)

The second condition makes the map j compatible with the STU relation. The compatibility with the AS and IHX relation is obvious.

Actually in the image of j there may be some chord diagrams containing dashed loops (without vertices). Let $\overset{o}{\mathcal{A}}(X)$ be the vector space spanned by chord diagrams with support X, subject to the STU, AS, and IHX relation, like in the case of $\mathcal{A}(X)$, but now the chord diagrams are allowed to contain dashed loops. So, in general, j is a linear map from $\mathcal{A}(\coprod^l S^1)$ to $\overset{o}{\mathcal{A}}(\phi)$.

2.1.2. *Solutions of (*) and (**).* There are many solutions of (*) and (**). But if we restrict ourselves to $\mathcal{A}(m)_{\text{tree}}$, then there is a unique solution. Here $\mathcal{A}(m)_{\text{tree}}$ is the subspace of $\mathcal{A}(m)$ generated by uni-trivalent graphs which are trees.

Proposition 2.1. [LMO] *Up to constants, there exists a unique sequence* $T_m \in \mathcal{A}(m)_{tree}, m = 0, 1, 2, 3...$ *satisfying (*) and (**). The solution is given by* $T_0 = T_1 = 0$ *and*

$$T_m = \sum_{\tau \in S_{m-2}} \frac{(-1)^{r(\tau)}}{(m-1)\binom{m-2}{r(\tau)}} T_\tau$$

for $m \geq 2$, *where* $r(\tau)$ *is the number of* $k \in \{1, 2, \ldots, m-2\}$ *satisfying* $\tau(k) > \tau(k+1)$ *and* T_τ *is given in Figure 10.*

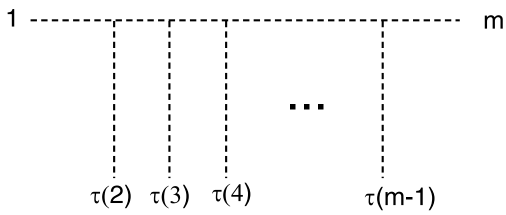


FIGURE 10

Exercise 2.2. Prove the uniqueness.

See Figure 11 for the first terms of T_m.

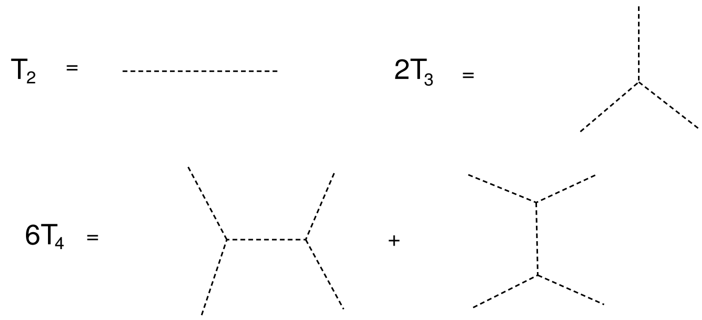


FIGURE 11. Simple cases of T_m

2.1.3. *New solutions from old ones.* There is a simple way to produce new solutions of (*) and (**) from known ones. Consider the space $\prod_{m=2}^{\infty} \mathcal{A}(m)$. In addition to the usual grading, it has another grading by the number of external vertices. We call this grading the e-grading. There is a shuffle product in $\prod_{m=2}^{\infty} \mathcal{A}(m)$ defined as follows. Suppose D is a graph in $\mathcal{A}(m)$, D' a graph in $\mathcal{A}(m')$. In the set of external vertices of D there is an order given by $1 < 2 < \cdots < m$. Consider the disjoint union of D and D', and a bijection from the external vertices of D and D' to the set $\{1, 2, \dots, m+m'\}$ which preserves the order of external vertices of D and D'. There are $\binom{m+m'}{m}$ such bijections, each gives a graph in $\mathcal{A}(m+m')$. Summing up all such possible graphs we get the shuffle product of D and D', denoted by $D \bullet D'$. The following is an exercise.

Proposition 2.2. *Suppose both T_m and $T'_m, m = 2, 3, \dots,$ satisfy (*) and (**). Let $T = \sum_{m=2}^{\infty} T_m, T' = \sum_{m=2}^{\infty} T'_m$. Then $(T \bullet T')_m, m = 2, 3, \dots,$ also satisfy (*) and (**), where $(T \bullet T')_m$ is the part of e-grading m of $T \bullet T'$.*

2.1.4. *The map j_n.* From now on let T_m be as in Proposition 2.1, and $T = \sum_{m=2}^{\infty} T_m$. Let $T^{\bullet n}$ be the n-th power of T in the shuffle product. Denote by T_m^n the part of $(T^{\bullet n}/n!)$ of e-grade m. Then for each positive integer n, the sequence $T_m^n, m = 2, 3, \dots,$ satisfies (*), (**), and hence defines a map $j_n : \mathcal{A}(\coprod^l S^1) \to \overset{o}{\mathcal{A}}(\phi)$ (see Figure 12).

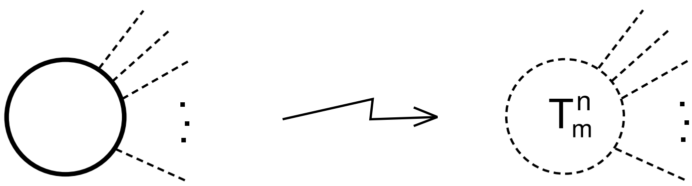


FIGURE 12. The definition of j_n

Note that if $m < 2n$, then, by definition, $T_m^n = 0$. The first non-trivial element $T_{2n}^n \in \mathcal{A}(2n)$ is the most important and is the following. Partition $2n$ points $\{1, 2, \dots, 2n\}$ into n pairs (there are $(2n-1)!!$ ways to do this), and then connect the two points of each pair by a dashed line, we get an element of $\mathcal{A}(2n)$. Summing up all such possible elements, we get T_{2n}^n. For T_4^2 see Figure 13.

Thus if a solid circle of D has exactly $2n$ vertices, then apply j_n to D means to remove the solid circle, then close the resulting 1-valent vertices (see §2.5.1).

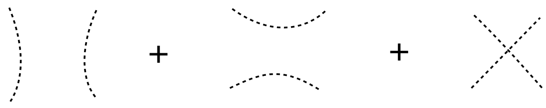


FIGURE 13. The element T_4^2

For $m > 2n$, T_m^n is more complicated and always contains internal vertices. We show the picture of T_5^2 in Figure 14. Actually, explicit formulas of T_m^n with $m > 2n$ is not important. It follows from the definition that j_n decreases the degree of a chord diagram

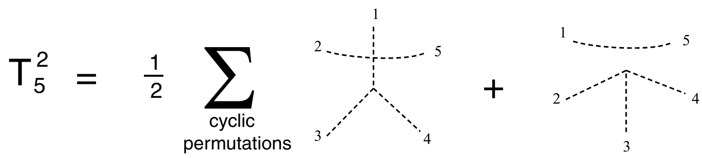


FIGURE 14. The picture of T_5^2

by ln, where l is the number of solid circles of the chord diagram.
Exercise. Suppose $D \in \mathcal{A}(S^1)$. Show that

$$j_n(D) = \frac{1}{n!} j_1[\Delta^{(n)}(D)].$$

Write down a similar formula for the case when the support of D contains many loops.

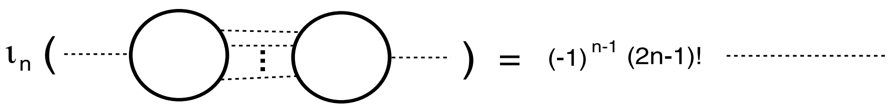
2.1.5. *The map ι_n.* For a fixed n, and $D \in \mathcal{A}(S^1 \sqcup S^1 \cdots \sqcup S^1)$ let $\iota_n(D)$ be obtained from $j_n(D)$ by keeping only the part of degree $\leq n$ and replacing every dashed circle by $-2n$. We have $\iota_n(D) \in \mathrm{Grad}_{\leq n}\mathcal{A}(\emptyset)$.

We will see we need only to calculate ι_n for the case when there are exactly $2n$ external vertices on each solid lines. That is, we need only T_{2n}^n.

Exercise 2.3. Suppose D is a chord diagram on one solid circle with exactly n dashed chord. Show that $\iota_n(D) = (-2)^n n!$. More generally, suppose D has $2n$ external vertices on its solid circle, and $2k$ of them are vertices of k dashed chords, with the remaining vertices of a uni-trivalent dashed graph ξ. Then $\iota_n(D) = (-2)(-4)\dots(-2k)\hat{\xi}$, where $\hat{\xi}$ is the closure of ξ.

A similar calculation gives

Lemma 2.3. *One has that*



(there are $2n-1$ in each bunch of parallel dashed lines).

Let U_+ (resp. U_-) be the unknot with framing $+1$ (resp. -1).

Exercise 2.4. Show that $\iota_n(\check{Z}(U_\pm)) = (\mp 1)^n +$ (elements of degree ≥ 1), hence their inverses exist.

2.2. **Definition of the invariant Z^{LMO}.** Suppose M is an oriented closed 3-manifold obtained from S^3 by surgery on a framed l-component unoriented link L. Providing L with an arbitrary orientation, we can define $Z(L)$ and $\check{Z}(L)$. We will construct an invariant $Z^{LMO}(M) \in \mathcal{A}(\emptyset)$, using $\check{Z}(L)$. The degree n part $\mathrm{Grad}_n(Z^{LMO})$ is constructed using $\mathrm{Grad}_{\le (l+1)n}[Z(L)]$.

Suppose the linking matrix of L has σ_+ positive eigenvalues and σ_- negative eigenvalues. Define ([LMO])

$$\Omega_n(L) = \frac{\iota_n(\check{Z}(L))}{(\iota_n(\check{Z}(U_+)))^{\sigma_+}(\iota_n(\check{Z}(U_-)))^{\sigma_-}} \in \mathrm{Grad}_{\le n}(\mathcal{A}(\emptyset)). \tag{2.1}$$

Theorem 2.4. ([LMO]) *$\Omega_n(L)$ does not depend on the orientation of L and does not change under the Kirby moves. Hence $\Omega_n(L)$ is an invariant of the 3-manifold M.*

Exercise 2.5. Show that $\Omega_n(L)$ does not depend on the orientation of L.

Recall that $\mathcal{A}(\emptyset)$ is a commutative co-commutative Hopf algebra. Let

$$Z^{LMO}(M) = 1 + \mathrm{Grad}_1(\Omega_1(M)) + \cdots + \mathrm{Grad}_n(\Omega_n(M)) + \cdots \in \mathcal{A}(\emptyset).$$

Proposition 2.5. ([LMO]) *$Z^{LMO}(M)$ is a group-like element, i.e.*

$$\hat{\Delta}(Z^{LMO}(M)) = Z^{LMO}(M) \otimes Z^{LMO}(M).$$

Hence $\ln(Z^{LMO}(M))$ *is a linear combination of* connected *3-valent vertex-oriented graphs.*

This follows from the fact that the Kontsevich integral of every string link is a group-like element (see exercise 1.1).

In general, $\Omega_n(M)$ is not equal to $\mathrm{Grad}_{\le n}(Z^{LMO}(M))$. Let $d(M)$ be the cardinality of $H_1(M,\mathbb{Z})$ if the first Betti number of M is 0, otherwise let $d(M) = 0$.

Proposition 2.6. ([LMO]) *We have that* $\mathrm{Grad}_{\le n}\Omega_{n+1}(M) = d(M)\,\Omega_n(M)$*. Hence if M is an integral homology 3-sphere, then*

$$\Omega_n(M) = \mathrm{Grad}_{\le n}(Z^{LMO}(M)).$$

The proof is given later.

It follows immediately from the definition that Ω_n behaves well under connected sum: $\Omega_n(M\#M') = \Omega_n(M) \times \Omega_n(M')$. Hence we have

Proposition 2.7. ([LMO]) *If M_1, M_2 are* integral homology 3-spheres*, then*

$$Z^{LMO}(M_1\#M_2) = Z^{LMO}(M_1) \times Z^{LMO}(M_2).$$

In general, if M_1, M_2 are *rational* homology 3-spheres, then Proposition 2.7 does not hold true. However, if we modify Z^{LMO}:

$$\hat{Z}^{LMO}(M) = 1 + \frac{\mathrm{Grad}_1(\Omega_1(M))}{d(M)} + \cdots + \frac{\mathrm{Grad}_n(\Omega_n(M))}{d(M)^n} + \ldots$$

Then $\hat{Z}^{LMO}(M_1\#M_2) = \hat{Z}^{LMO}(M_1) \times \hat{Z}^{LMO}(M_2)$, and $\hat{\Delta}(\hat{Z}^{LMO}(M)) = \hat{Z}^{LMO}(M) \otimes \hat{Z}^{LMO}(M)$.

2.3. **A definition of Z^{LMO} without using j_n, ι_n.** For simplicity we will assume that M is obtained from S^3 by surgery along a framed knot K. The case of links is similar (see [Le2]).

Then $\check{Z}(K)$ takes value in $\mathcal{A}(S^1)$ which is isomorphic to $\mathcal{B}$, the space of uni-trivalent graphs (subject to the IHX, AS conditions), see [BN]. Formally, $\mathcal{B} = \prod_{m=0}^{\infty} \mathcal{A}(m)/S_m$, where the symmetric group S_m acts on $\mathcal{A}(m)$ by permuting the external vertices.

Considered $D \in \mathcal{B}$. Keep only the part having exactly $2n$ external vertices, and close it. By this we mean to partition the $2n$ external vertices into n pairs and glue each pair together, then take the sum over all obtained trivalent graphs. Next we replace each dashed circle by $-2n$ and keep only the part of degree $\leq n$. The result is $\iota_n(D)$.

Exercise 2.6. Show that this is the same as the previous definiton.

Hint. If $m > 2n$, then each term of T^n_m contains an internal vertex adjacent to two external vertices.

2.4. **The BGRT definition.** In 1997 Bar-Natan, Garoufalidis, Rozansky and Thurston came up with the Aarhus integral – an invariant of rational homology 3-spheres, which is coincident with $\hat{Z}^{LMO}$. Let us describe the mathematical definition here.

The main point is a formal integration theory, which helps to explain the appearance of some formulas, and what results to expect. The definition for the case when one does surgery on a knot K of framed b is as follows. The link case is similar. For details see [BGRT].

Again consider $\check{Z}(K)$ as an element of $\mathcal{B}$, which is an algebra where the product is the disjoint union $\sqcup$. Since the framing is b, one has that (in $\mathcal{B}$)

$$\check{Z}(K) = \exp_{\sqcup}(bw_1/2) \sqcup Y,$$

where w_1 is the "dashed interval" (without internal vertices), and Y is an element in $\mathcal{B}$ every term of which must have at least one internal vertex.

For uni-trivalent graphs $C, D \in \mathcal{B}$ let

$$\langle C, D\rangle = \begin{cases} 0 & \text{if the numbers of external vertices of } C, D \text{ are different} \\ \text{sum of all ways to glue external vertices of } C \text{ and } D \text{ together} & \end{cases}$$

For example, closing a uni-trivalent graph D with $2n$ external vertices is

$$\mathrm{cl}(D) = \frac{1}{(2n)!}\langle T^n_{2n}, D\rangle.$$

Then one defines

$$\int^{FG} \check{Z}(K) = \langle \exp_{\sqcup}(-w_1/2b), Y\rangle.$$

Since we divide by b, the definition is good only for the case when $b \neq 0$ (for the link case, the definition is good only if the linking matrix is invertible, i.e. the resulting manifold is a rational homology 3-sphere).

Exercise 2.7. Show that, when $b \neq 0$,

$$\int^{FG} \check{Z}(K) = \sum_{n=0}^{\infty} \frac{[\iota_n(\check{Z}(K))]_n}{(-b)^n},$$

where $[x]_n$ is the part of degree n of x.

The link case is also similar. Hence if one use $\int^{FG}$ to define invariants of rational homology 3-spheres, then the invariant is equal to $\hat{Z}^{LMO}$.

2.4.1. *Proof of Proposition 2.6.*

Proof. We give a proof for the case when $M = S^3_L$, where L is a knot of framing b. Suppose in $\mathcal{B}$

$$\check{Z}(L) = \exp_\sqcup(bw_1/2) \sqcup (Y_2 + Y_3 + \dots),$$

where Y_m has exactly m external vertices, and at least one internal vertices. We will use the definition of ι_n described in §2.3.

In order to apply ι_n, one needs $2n$ external vertices. Suppose $2k < 2n$. Then to get $2n$ external vertices we have to take the product with $(bw_1/2)^{n-k}/(n-k)!$ which comes from $\exp_\sqcup(bw_1/2)$. Thus the contribution of Y_{2k} in $\iota_n(\check{Z}(L))$ is

$$\iota_n(\frac{b^{n-k}}{2^{n-k}(n-k)!} Y_{2k} \sqcup w_1^{n-k})$$

which by the result of Exercise 2.3, is equal to $(-b)^{n-k}\mathrm{cl}(Y_{2k})$.

Similarly, the contribution of the same Y_{2k} in $\iota_{n+1}(\check{Z}(L))$ is $(-b)^{n+1-k}\mathrm{cl}(Y_{2k})$. Remembering that $\iota_n(U_\pm) = (\mp 1)^n + \dots$, we see that the contribution of Y_{2k} in Ω_{n+1} is $|b|$ times its contribution in Ω_n. □

2.5. Proof of invariance of Z^{LMO}.

2.5.1. *Closing a uni-trivalent graph.* We recall the closing operation. Suppose D consists of a uni-trivalent dashed graph, some of its univalent vertices are on a solid X, some are not. The closure of D, $\mathrm{cl}(D)$, is defined as follows. If the number of univalent vertices not on X is odd, let $\mathrm{cl}(D) = 0$. Otherwise partition these vertices into pairs and glue the two points of each pair together. Sum over all possible partition. The result is $\mathrm{cl}(D)$, which is an element of $\overset{o}{\mathcal{A}}(X)$.

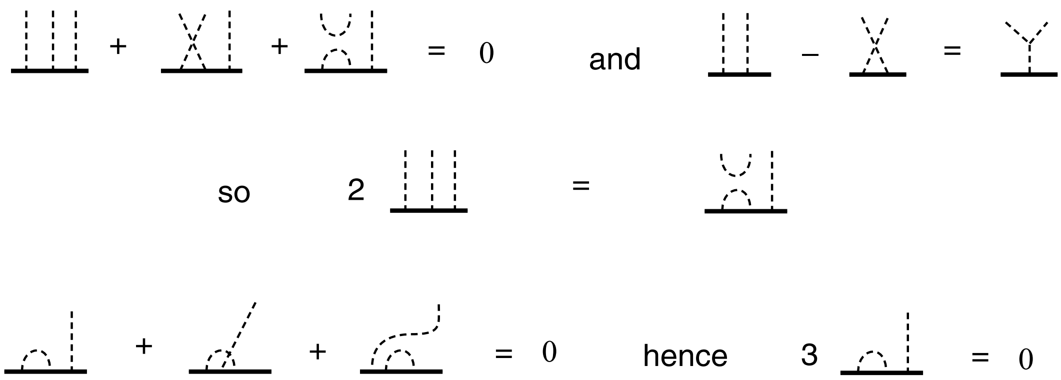
Let $\mathrm{cl}_n(D) = 0$ if D has other than $2n$ external vertices. Otherwise let $\mathrm{cl}_n(D) = \mathrm{cl}(D)$. As noted before,

$$\mathrm{cl}_n(D) = \frac{1}{(2n)!}\langle T^n_{2n}, D\rangle.$$

2.5.2. *Relation P_n.* The relation P_n (on the vector space $\overset{o}{\mathcal{A}}(X)$ says that $\mathrm{cl}_n(D) = 0$ for every D with support X. In other words, the relation P_n simply says an element containing T^n_{2n} is 0. The doubling operator is still well-defined if we impose P_n.

Proposition 2.8. *Suppose X consists of solid loops. In the space $\overset{o}{\mathcal{A}}(X)/P_{n+1}$, every chord diagram is equivalent to a linear combination of chord diagrams having $\leq 2n$ external vertices on each solid circle.*

Let us show the proof for $n = 2$, for general n one can generalize the proof, using induction.



Here = means equal up to an element with ≤ 2 external vertices.

2.5.3. *Invariance under 2-nd Kirby move in presence of P_{n+1}.* The invariance is easy to prove. Suppose D' is obtained from D by a chord-diagram-handleslide move, by a doubling followed by a band sum. Note that if D has less than $2n$ vertices on any solid component, then $j_n(D) = 0$.

By Proposition 2.8 we may assume that each solid circle of D has $\leq 2n$ vertices. If one of the circle has $< 2n$ vertices, then $j_n(D) = 0 = j_n(D')$. Suppose each solid circle of D has exactly $2n$ vertices. To get D' from D we do the doubling on a component C, getting 2 components C_1 and C_2, then band sum C_1 with another component C'. The total external vertices on C_1 and C_2 is $2n$. Hence the only non-trivial contribution to $j_n(D')$ is from the case when C_2 has $2n$ vertices and C_1 has 0. But in this case the contribution is equal to $j_n(D)$.

2.5.4. *Removing the relation P_{n+1}.* We have seen that $j_n(D) \in \overset{o}{\mathcal{A}}(\emptyset)/(P_{n+1})$ is invariant under the second Kirby move. The following shows that in degree $\leq n$, the relation P_{n+1} is redundant. Let O_n be the equivalent relation: a dashed loop is equal to $-2n$.

Proposition 2.9. *The naturally defined map*

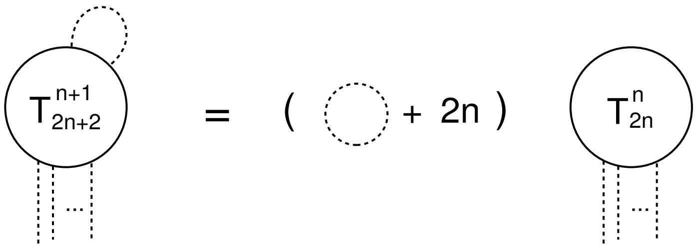
$$Grad_{\leq n}\mathcal{A}(\emptyset) \to Grad_{\leq n}\overset{o}{\mathcal{A}}(\emptyset)/(O_n, P_{n+1})$$

is an isomorphism.

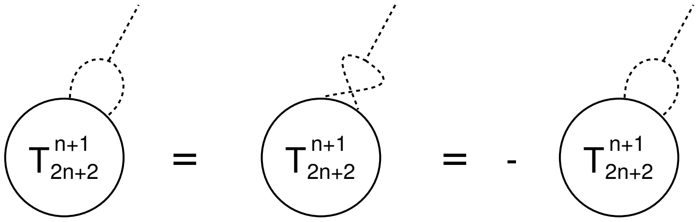
It follows that $\iota_n(D)$ is invariant under the 2-nd Kirby move. Hence Ω_n is an invariant of 3-manifolds.

Proof. We can remove the dashed loop in $\overset{o}{\mathcal{A}}$ using the O_n relation. It is enough to show that a non-trivial $x \in \overset{o}{\mathcal{A}}(\emptyset)$ containing T^{n+1}_{2n+2} must have degree $> n$, i.e. has more then $2n$ vertices.

Choose an external vertex v of T^{n+1}_{2n+2}. There must be a path in $x \setminus T^{n+1}_{2n+2}$ to another vertex v' of T^{n+1}_{2n+2} (why?). If there is no internal vertex on the path, then $x = 0$ by the O_n relation:



If there is only one internal vertex on the path, then $x = 0$ by the anti-symmetry relation:



If every such path has at least 2 vertices, then x has at least $2n+2$ vertices, hence degree of x is greater than n. □

2.6. **Reversing orientation.**

Proposition 2.10. *Suppose $\bar{M}$ is the manifold M with reverse orientation. Then*

$$Grad_n Z^{LMO}(\bar{M}) = (-1)^{n(b_1+1)} Grad_n Z^{LMO}(M),$$

where b_1 is the first Betti number.

Proof. we define a linear map $\hat{S} : \mathcal{A}(X) \to \mathcal{A}(X)$ by $\hat{S}(D) = (-1)^n D$ for a chord diagram D of degree n. Then for the Kontsevich integral one has $Z(\bar{L}) = \hat{S}(Z(L))$, where $\bar{L}$ is the mirror image of L.

Suppose L has l components. Then ι_n lowers the degree by ln, hence $\iota_n(\check{Z}(\bar{L})) = (-1)^{nl}\hat{S}[\iota_n(\check{Z}(L))]$. Similarly, since $\bar{U}_+ = U_-$, one has $\iota_n(\check{Z}(U_-)) = (-1)^n\hat{S}[\iota_n(\check{Z}(U_+))]$. Since $\bar{M}$ is obtained by surgery along $\bar{L}$, one has

$$\Omega_n(\bar{M}) = (-1)^{nl-n\sigma_+-n\sigma_-}\hat{S}\Omega_n(M).$$

It remains to remember that $l - \sigma_- - \sigma_+ = b_1$. □

2.7. **Extension to Links in 3-manifolds.**

The invariant can be easily extended to the case of a framed link L in a 3-manifold M. Suppose M is obtained by surgery along a link L' in S^3, and the link L is the image of another link in S^3, also denoted by L. Apply the removing-solid-circle map ι_n to components of L' only, from $\check{Z}'(L \cup L')$ we get $\iota'_n(L \cup L')$. Here $\check{Z}'(L \cup L')$ means the result of taking connected sum of $Z(L \cup L')$ with ν along each component of L'.

Let

$$\Omega_n(M, L) = \frac{\iota'_n(L \cup L')}{\iota_n(U_+)^{\sigma_+}\iota_n(U_-)^{\sigma_-}}$$

Then the same proof shows that $\Omega_n(M, L)$ is an invariant of pair (M, L). We again can combine these invariant into one by

$$Z(M, L) := 1 + \sum_{n=1}^{\infty} \mathrm{Grad}_n(\Omega_n(M, L)).$$

When M is a rational homology 3-sphere, $Z(M, L)$ is a strong invariant. There is a surgery formula which allows to calculate $Z^{LMO}(M_L)$ directly from $Z(M, L)$, without referring to links in S^3, see [LMO]. There is also a TQFT based on $Z^L MO$.

3. Review of the theory of Finite type invariants for 3-manifolds

The theory of finite type invariants (FTI) of 3-manifolds was initiated by Ohtsuki [Oh1]. In [Oh2], Ohtsuki found a perturbative expansion of the quantum $SO(3)$-invariants of homology 3-spheres, which led him to the definition of FTI. The theory has been developed rapidly by many authors. Later Gusarov and Habiro independently introduced clasper calculus, or Y-surgery, which is a powerful geometric technique in the theory. Gusarov-Habiro theory can be naturally generalized to other classes of 3-manifolds (with or without homology, spin structures...). We will review here the Gusarov-Habiro approach, then mention the original definition of Ohtsuki, and also the definitions given by Garoufalidis and Levine. Cochran and Melvin developed another theory of finite type invariants of 3-manifolds with homology.

3.1. **Generality, the knot case.**

3.1.1. *Elementary move and decreasing filtration.* In a theory of FTI, one considers a class of objects, and a "good" decreasing filtration on the vector space spanned by these objects. An invariant of the objects with values in the ground field is of order less than or equal to n if its restriction to the $n+1$-st term of the filtration is 0. Good here means at least the space of FTI of each order is finite-dimensional, and it is desirable to have an algorithm of polynomial time to calculate every FTI. Also one certainly wants the set of FTI invariants separates the objects (completeness).

Usually the filtrations are defined using an *elementary move*. For the class of knots the elementary move is given in Figure 15.

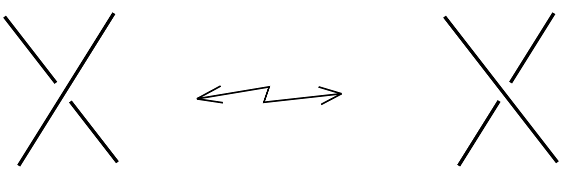


Figure 15

Any two knots can be connected by a finite sequence of such moves. The filtration $\mathcal{F}_n$ on the vector space spanned by knots is then defined so that if $K, K' \in \mathcal{F}_n$, then $K - K' \in \mathcal{F}_{n+1}$. Formally one proceeds as follows. Suppose S is a set of double points of a knot diagram D. Let

$$[D, S] = \sum_{S' \subset S} (-1)^{|S'|} D_{S'},$$

where $D_{S'}$ is the knot obtained by changing the crossing at every point in S'. Then $\mathcal{F}_n$ is the vector space spanned by all elements of the form $[D, S]$ with $|S| = n$.

3.1.2. *Another elementary move– the Δ-move.* Let us consider another elementary move for knots, see Figure 16.

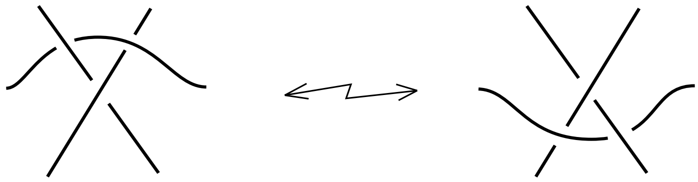


FIGURE 16

It is known as the Δ-move (H. Murakami-Nakashini [MN]), or the Borromeo move (Matveev, [Mat]). It is the same as the one in Figure 17,

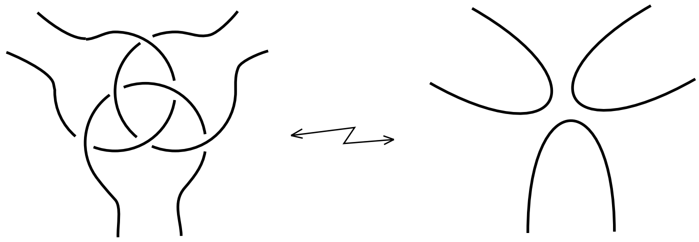


FIGURE 17

or, in surgery picture, see Figure 18, where each component of the surgery link has

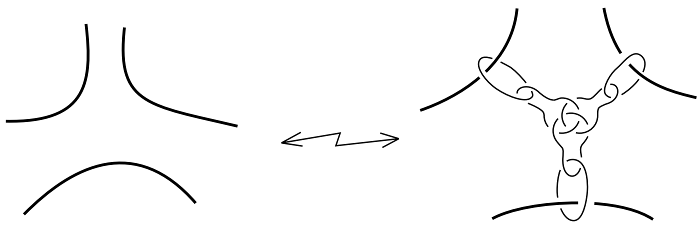


FIGURE 18

framing 0.

Exercise 3.1. Prove the equivalence between these 3 moves.

Murakami and Nakanishi showed that any 2 knots are related by a finite number of Δ-moves. Let us use this move to define a decreasing filtration $\mathcal{F}_n^{\Delta}, n = 0, 1, 2, ...$ on the vector space of knots. It turns out that the new theory of FTI is equivalent to the original one (with a shift of index). One has that $\mathcal{F}_n^{\Delta} = \mathcal{F}_{n+1}$. (Prove this!).

The new definition, seemingly more complicated than the previous one, provides a powerful geometric tool in studying FTI of knots. Many important results are obtained using this tool (See [Gu1, Hab]). The Δ-move naturally gives rise to 3-valent graphs. It corresponds to the commutator in group theory.

Exercise 3.2. a) Suppose D is a chord diagram on a solid circle, with connected dashed graph (g-connected). Let K be the knot whose surgery diagram is shown on the right hand side of Figure 19. Show that for every FTI v of order $< n$, one has $v(K) = v(unknot)$,

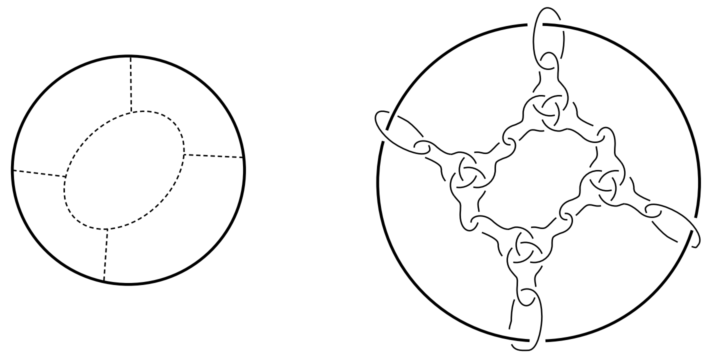


FIGURE 19

and for any FTI w of order n, $w(K) = w(D)$ (this is true for FTI with values in arbitrary abelian groups). This implies that the first non-trivial term in the Kontsevich integral of K is D.

b) Find a knot K such that the first non-trivial term of the Kontsevich integral is $-D$.

3.2. **The Gusarov-Habiro filtration for 3-manifolds.** We will consider only closed oriented manifolds, although the theory can be extended to manifolds with boundary or manifolds with spin structures. Consider the following as the elementary move

$$S^3_L \leftrightarrow S^3_{L'}, \qquad \text{(move 1)}$$

where S^3_L means the manifold obtained from S^3 by surgery along L, and L' is obtained from L by a Δ-move. To fix the place where the Δ-move happens, one can embed a framed Y-shape graph in $S^3 \setminus L$, with external vertices on L, as in Figure 20.

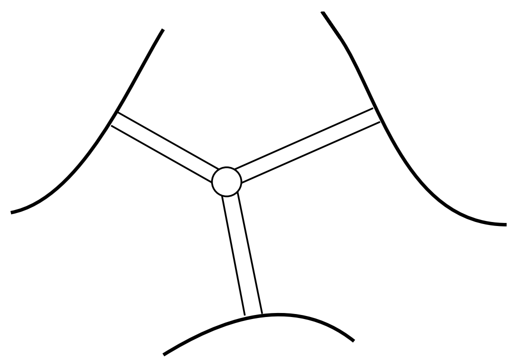


FIGURE 20

Another elementary move is the following move 2. Consider the *standard Y-graph-with-leaves* in $\mathbb{R}^3$, see Figure 21. A Y-graph-with-leaves C in M is the image of an embedding of a small neighborhood of the standard Y-graph-with-leaves into M. Let L be the six-component link in a small neighborhood of the standard Y-graph-with-leaves as shown in the figure, each component has framing 0. The surgery of M along the image of the

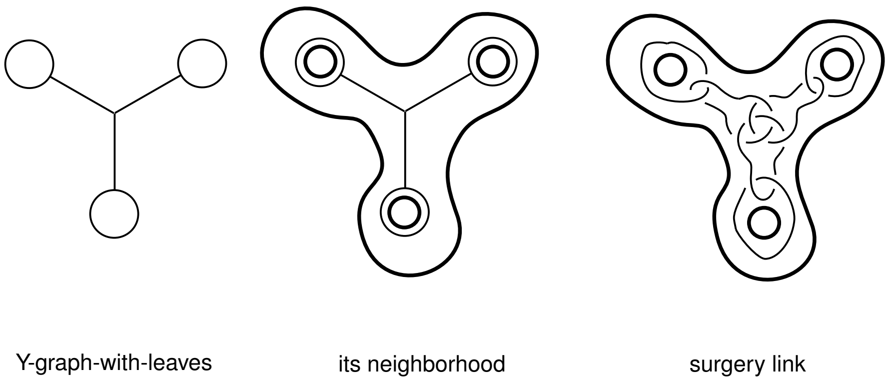


FIGURE 21

six-component link is called a Y-surgery along C, denoted by M_C. The change $M \leftrightarrow M_C$ is move 2.

Exercise 3.3. Show that moves 1 and 2 are equivalent.

Matveev [Mat] proved the following interesting result.

Theorem 3.1. *M and M' are related by a finite sequence of Y-surgery if and only if there is an isomorphism from $H_1(M,\mathbb{Z})$ onto $H_1(M',\mathbb{Z})$ preserving the linking form on the torsion group.*

Exercise 3.4. Prove the easy part of the theorem: If M is obtained from M' by an elementary move, then M and M' have the same H_1 and linking form.

Thus if we want to consider the theory of FTI based on this elementary move, we have to partition the set of 3-manifolds into smaller classes of the same H_1 and the linking form.

For a 3-manifold M let $\mathcal{M}(M)$ be the vector space spanned by all 3-manifolds with the same H_1 and linking form. The formal definition of the filtration is as follows. For a set S of a finite number of Y-graphs-with-leaves in a 3-manifold N let

$$[N,S] = \sum_{S'\subset S} (-1)^{|S'|} N_{S'}.$$

Define $\mathcal{F}_n$ to be the vector space spanned by all $[N,S]$ such that N is in $\mathcal{M}(M)$ and $|S| = n$.

3.2.1. *Other equivalent moves.* Let T_0 denote the 2-dimensional torus with two open discs removed. We call a framed link $L = \cup_{i=1}^{2n} L_i$ (in M) T_0-bounding if there are pairwise disjoint surface $F_1,\dots,F_n$, each homeomorphic to T_0, such that $\partial F_k = L_{2k-1} \cup L_{2k}$, and the framings of L_{2k-1} and L_{2k} are -1 and 1, assuming that the framing determined by the normal vector field of the surface F_k is 0.

Exercise 3.5. Show that the Y-move is equivalent to the move $M \leftrightarrow M_L$, where L is T_0 bounding.

This move was introduced by Matveev, and rediscovered by Garoufalidis and Levine under the name *blink* surgery.

Here is yet another elementary move, equivalent to the above one. Let $T^3 = V_1 \cup V_2$ be the Heegard splitting of the 3-torus, where V_1 is the standard handlebody of genus 3, and the characteristic link is shown in Figure 22. In this figure only one component of

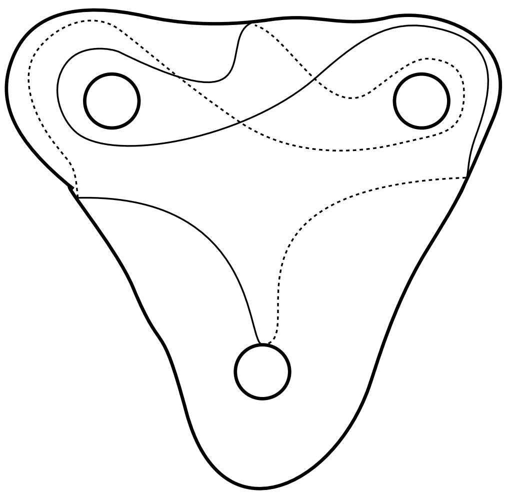


FIGURE 22

the characteristic link is shown, the other two are obtained from this one by rotation by $2\pi k/3$.

A Borromeo surgery (Matveev) of M is obtained by embedding V_1 into M, then constructing new 3-manifold M' by $M' = (M \setminus V_1) \cup V_2$.

Exercise 3.6. Show that the Borromeo surgery is equivalent to any of the above elementary moves.

3.2.2. *Main properties.* From now on we concentrate on the case of $\mathcal{M} = \mathcal{M}(S^3)$, i.e. $\mathcal{M}$ is the vector space spanned by integer homology 3-spheres.

Gusarov and Habiro showed that $\mathcal{F}_{2n-1} = \mathcal{F}_{2n}$. This means the order of a FTI must be even.

Invariants of order n is the dual space of $\mathcal{F}_n/\mathcal{F}_{n+1}$. There is a surjective map

$$W : \mathrm{Grad}_n \mathcal{A}(\emptyset) \to \mathcal{F}_{2n}/\mathcal{F}_{2n+1},$$

called the weight map, where Grad_n means the part of degree n. We will give the definition later.

Suppose v is an invariant of order $\leq 2n$. Then v restricted to a well-defined linear map from $\mathcal{F}_{2n}/\mathcal{F}_{2n+1}$ to the ground field. The composition of v and W is a functional on $\mathrm{Grad}_n \mathcal{A}(\emptyset)$ called the weight system of v. The LMO invariant will help to prove that infact W is an isomorphism.

3.2.3. *The weight map W.* We give here the definition of $W : Grad_n \mathcal{A}(\emptyset) \to \mathcal{F}_{2n}/\mathcal{F}_{2n+1}$. Suppose D is a trivalent graph of degree n. Then D has $2n$ vertices. Embed D into S^3 arbitrarily. Then from the image of D construct a set of Y-graphs-with-leaves as in Figure 23.

Then let $W(D) = [S^3, G]$. It is clear that $W(D) \in \mathcal{F}_{2n}$. It requires some geometric (not easy) arguments to show that W is a well-defined $W : Grad_n \mathcal{A}(\emptyset) \to \mathcal{F}_{2n}/\mathcal{F}_{2n+1}$. The most difficult is the IHX(or Jacobi) relation.

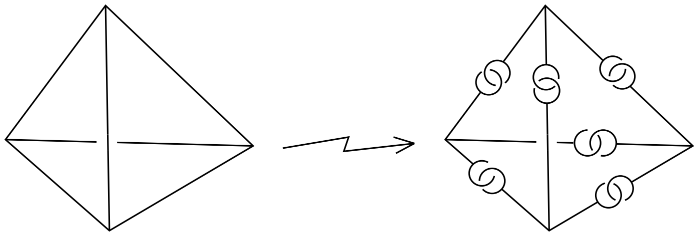


FIGURE 23

3.2.4. *Ohtsuki's and Garoufalidis' definitions.* We briefly recall here the original definition of Ohtsuki which is good for integral homology 3-spheres only. Here an elementary move is the move $M \leftrightarrow M_L$, where L is a *knot* with framing ± 1 in an integral homology 3-sphere M. Formally the definition is as follows.

A framed link L in M is *unit-framed* if the framing of each component is ± 1; L is *algebraically split* if the linking number of every two components is 0.

Define

$$[M, L] = \sum_{L' \subset L} (-1)^{|L'|} M_{L'}, \tag{3.1}$$

Consider the following decreasing filtration in $\mathcal{M}$. Let $\mathcal{F}_n^{Oht}(\mathcal{M})$ be the vector space generated by $[M, L]$, where M is an arbitrary homology 3-sphere and L is a unit-framed and algebraically split n-component link in M. Garoufalidis-Levine-Ohtsuki (see [GL1, GO1]) showed that $\mathcal{F}_{3n+1}(\mathcal{M}) = \mathcal{F}_{3n+2}(\mathcal{M}) = \mathcal{F}_{3n+3}(\mathcal{M})$. Hence the order of a FTI in this theory must be divisible by 3.

Garoufalidis introduced a similar filtration, replacing unit-framed algebraically split links by *boundary* link. The resulting filtration is denoted by $\mathcal{F}_n^{Gar}$. It had been conjectured that $\mathcal{F}_n^{Gar} = \mathcal{F}_{3n}^{Oht}$. This was later proved by Gusarov, by showing that each is equal to $\mathcal{F}_{2n}$:

Theorem 3.2. [Gu3] *One has that (over $\mathbb{Q}$):*

$$\mathcal{F}_n^{Gar} = \mathcal{F}_{2n} = \mathcal{F}_{3n}^{Oht}.$$

4. UNIVERSALITY OF THE LMO INVARIANT

4.1. **Universality.**

4.1.1. *Main theorem.* We consider only integral homology 3-spheres, i.e. the vector space $\mathcal{M}$.

Theorem 4.1. (Main Theorem)

a) For homology 3-spheres, Z_n^{LMO} is an invariant of order $2n$.

b) The map W is an isomorphism, i.e. every linear form from $\mathrm{Grad}_n(\mathcal{A}(\emptyset))$ *to $\mathbb{Q}$ is a weight system of some finite invariant of order $2n$.*

c) $Z_{\leq n}^{LMO}$ is a universal invariant of order $2n$, i.e. if $Z_{\leq n}^{LMO}(M) = Z_{\leq n}^{LMO}(M')$ then $I(M) = I(M')$ for every invariant I of order $\leq 2n$. Every rational invariant of order $\leq 2n$ factors thru $Z_{\leq n}^{LMP}$.

Remark . The universality holds true if we replace $\mathcal{M}$ by $\mathcal{M}(M)$, where M is any *rational* homology 3-sphere (Habiro). The proof is similar, the only new is the existence of the weight map.

4.1.2. *A reformulation.* Let the *product* of two homology 3-spheres be their connected sum. The unit is S^3. Define a co-product: $\Delta(M) = M \otimes M$, and co-unit: $\varepsilon(M) = 1$, for every homology 3-sphere M. Then $\mathcal{M}$ becomes a commutative co-commutative Hopf algebra.

It is easy to see that the product is compatible with the filtration

$$\mathcal{M} = \mathcal{F}_0\mathcal{M} \supset \mathcal{F}_2\mathcal{M} \supset \cdots \supset \mathcal{F}_{2n}(\mathcal{M}) \dots$$

This means, if $N_1 \in \mathcal{F}_{n_1}(\mathcal{M})$ and $N_2 \in \mathcal{F}_{n_2}(\mathcal{M})$, then their product is in $\mathcal{F}_{n_1+n_2}(\mathcal{M})$.

Hence the algebra structure of $\mathcal{M}$ induces an algebra structure on the associated (complete) graded vector space

$$\mathcal{GM} = \prod_{i=0}^{\infty} \mathcal{F}_{2i}(\mathcal{M})/\mathcal{F}_{2i+2}(\mathcal{M}).$$

A nice interpretation of the main theorem is the following.

Theorem 4.2. *a) The mapping* $Z^{LMO} : \mathcal{M} \to \mathcal{A}(\emptyset)$ *is a* Hopf algebra *homomorphism which maps* $\mathcal{F}_{2n}(\mathcal{M})$ *to* $\mathrm{Grad}_{\geq n}\mathcal{A}(\emptyset)$.

b) Z^{LMO} *induces an* algebra *isomorphism between* $\mathcal{GM}$ *and* $\mathcal{A}(\emptyset)$.

This theorem follows immediately from the previous one and Propositions 2.5, 2.7. Part *b*) shows that $\mathcal{GM}$ has a structure of a commutative co-commutative Hopf algebra.

4.1.3. *Some corollaries.* X. S. Lin proved (see [Lin])

Proposition 4.3. *The space of finite type invariants of homology 3-spheres with rational values is a commutative co-commutative Hopf algebra.*

Here this fact is a consequence of Theorem 4.2, *b*), as observed in [GO1], Remark 1.10.

Suppose I is an invariant of integral homology 3-spheres, K is a knot in S^3. Let us define $\lambda_I(K) = I(S^3_K)$, where we provide K with framing 1.

Proposition 4.4. *If* I *is a homology 3-sphere invariant of order* $2n$*, then* λ_I *is a knot invariant of order* $2n$.

This had been conjectured by Garoufalidis, and proved by Habegger, see [Hab]. Here this fact follows easily from the main theorem: for a knot K, the universal invariant $\Omega_n(S^3_K)$ is computed using $\mathrm{Grad}_{\leq 2n}(Z(K))$, hence $I_{\Omega_n}(K)$ is derived, by a linear map, from $\mathrm{Grad}_{\leq 2n}Z(K)$ which is a knot invariant of order $\leq 2n$.

The result of Gusarov and Habiro shows that this proposition holds true for FTI with values in arbitrary abelian group.

4.1.4. *3-manifold with pre-given first non-trivial term.*

Theorem 4.5. *For every* connected *3-valent vertex-oriented graph* D *of degree* n*, there are homology 3-spheres* $M^{\pm}$ *such that*

$$Z^{LMO}(M^{\pm}) = 1 + \pm D + (\text{elements of degree} > n).$$

This follows easily from Gusarov-Habiro theory: Consider the set G of Y-graphs-with-leaves constructed in §3.2.3. Since D is connected, $S^3_{G'}$, with G' a proper subset of G, is S^3. Thus $[S^3, G] = S^3 - S^3_G$. This means $S^3 - S^3_G \in \mathcal{F}_{2n}$. To get $-D$ one needs only to change tyhe orientation at one vertex.

Some authors (Habegger, Orr, and myself) had given another, more complicated proof.

4.2. **Proof of the universality.**

4.2.1. *Ω_n is of order $\le 2n$.* We have to show that $\Omega_n([M, G]) = 0$, where G is a set of $2n+1$ Y-graphs-with-leaves in a homology 3-sphere M. (Actually the proof holds true for every 3-manifolds).

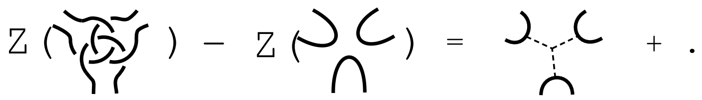
It suffices to prove the following.

Lemma 4.6. *Suppose L is an arbitrary framed link in S^3. Suppose there is a set Γ of $2n+1$ (framed) dashed Y-graphs, each has external vertices attached to components of L. Then $\iota_n(\delta_\Gamma(L)) = 0$, where $\delta_\Gamma(L)$ is the alternating sum obtained by doing Δ-moves along subset of Γ:*

$$\delta_\Gamma(L) = \sum_{\Gamma' \subset \Gamma} (-1)^{|\Gamma'|} L_{\Gamma'}.$$

Here $L_{\Gamma'}$ is the link obtained by doing Δ-moves at Γ'.

Proof. Each difference



gives us an *internal* vertex in $\check{Z}(\delta(L))$. Hence $\check{Z}(\delta(L))$ contains at least $2n+1$ internal vertices. Since the map ι_n does not touch the internal vertices, $\iota_n(\delta(L))$ has at least $2n+1$ vertices, and hence has degree $\ge n+1$. This means $\iota_n(\delta(L)) = 0$. □

4.2.2. *Proof that Ω_n gives the inverse of W.* We have just shown that Ω_n has degree $\le n$. Suppose D is a 3-valent graph of degree n, and G is the set of Y-graphs-with-leaves constructed in §3.2.3. Then $W(D) = [S^3, G]$ in $\mathcal{F}_{2n}$. We will show that $\Omega_n([S^3, G]) = \pm D + h.o.t.$ This will prove the universality of Ω_n.

Replace G by the $6n$-component link L, as in Figure 24, where in each disk the strands

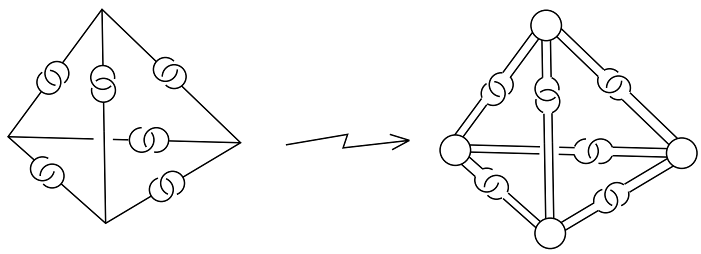


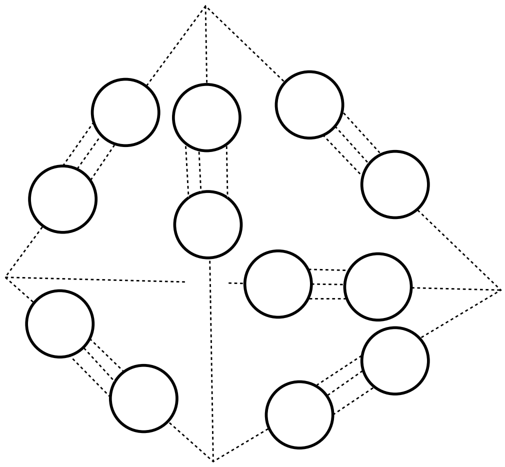
FIGURE 24

link like in a Borromeo link. Let $\gamma(L)$ be the following alternating sum

$$\gamma(L) = \sum (-1)^{|G'|} L',$$

where the sum is over all subset G' of G, and L' is the surgery link corresponding to G'.

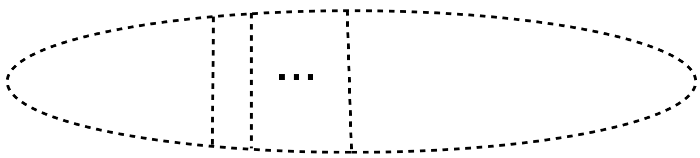
The counting argument of the previous subsection shows that the first non-trivial term of $\iota_n(\delta(\Gamma))$ has $2n$ vertices, and each trivalent vertex of G gives a trivalent vertex of this first non-trivial term. No other internal vertices are allowed. In $\check{Z}(\delta(\Gamma))$ the part which does not contains internal vertices comes only from the clasper (at the middle of edges of G). Among the T^n_m only T^n_{2n} does not have internal vertices. The contribution of a clasper is $\exp(r)$, where r is the dashed chord connecting the two components of the clasper. Since ι_n lower the degree by n per solid circle, the only non-trivial contribution in degree n of $\iota_n(\delta(\Gamma))$ is ι_n of $(\frac{1}{(2n-1)!})^{3n}$ times



(where each bunch has $2n-1$ parallel dashed lines).

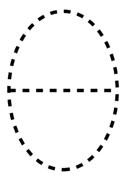
Applying Lemma 2.3, we see that the result is D, the original 3-valent graph.

4.2.3. *Proof that Ω_n has order exactly $2n$.* To prove that Ω_n has order exactly $2n$, one needs only to show that there exists a non-trivial element of order n in $\mathcal{A}(\emptyset)$. Using weight systems coming from a simple Lie algebra $\mathfrak{g}$ one can easily show that



is non-trivial. In fact, each isolated chord gives us a factor c – the eigenvalue of the Casimir element in the adjoint representation. Hence the value of the above 3-valent graph is $c^n \times dim\mathfrak{g}$.

4.3. **The first coefficient of Z^{LMO}.** Note that in degree 1, $\mathcal{A}(\emptyset)$ has dimension 1, and is spanned by the Θ graph



Thus for every 3-manifold M

$$Z^{LMO} = 1 + c_1\Theta + h.o.t.$$

Proposition 4.7. *The invariant c_1, up to a normalization, is the Casson-Walker-Lescop invariant λ:*

$$c_1(M) = (-1)^{b_1}\lambda(M)/2,$$

where b_1 is the Betti number, and λ is normalized so that it takes value 1 on the Poincare homology 3-sphere.

Exercise 4.1. Prove this proposition for the case when M can be obtained from S^3 by surgery along a framed knot.

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