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Kontsevich's integral for the Homfly polynomial and relations between values of multiple zeta functions

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Abstract

Kontsevich's integral for the Homfly polynomial is studied by using representations of the chord diagram algebras via classical r-matrices for sl_N and via a Kauffman type state model. We compute the actual value of the image of $W(\gamma)$ by these representations, where γ is the normalization factor to construct an invariant from the integral. This formula implies relations between values of multiple zeta functions.

Keywords: Kontsevich's integral; Homfly polynomial; Zagier's multiple zeta function

AMS (MOS) Subj. Class.: 57 M25, 11M41

0. Introduction

In [8] Kontsevich defined a knot invariant by using an iterated integral to obtain the monodromy of the Knizhnik–Zamolodchikov (KZ) equation. His description of the invariant in [8] is very brief, but the (partially expository) article [2] contains further details. He considers the KZ equation with values in an algebra $\mathscr{A}_{\mathscr{C}}$ which is the linear span of chord diagrams with relations corresponding to the flatness of the KZ equation. These relations are similar to the classical Yang–Baxter equation (CYBE) (without spectral parameters), which is also known as an infinitesimal pure braid relation in Kohno's work [7], first introduced as a relation of a holonomy Lie algebra in [6]. We construct a state model or a "representation" of the algebra $\mathscr{A}$ by using a classical r-matrix, a solution of the CYBE [2]. This state

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model defines a mapping from $\mathscr{A}$ to $\mathbb{C}$ called a weight, and applying it to the integral invariant, we get a $\mathbb{C}$-valued invariant.

In this paper, we extend the definition of Kontsevich's integral from knots to links, and then we give a precise correspondence of the Homfly polynomial of links and invariants coming from the integral related to the classical r-matrix associated with the vector representation of sl_N. This is a partial answer to Problem 4.9 of [2] in the case of the Homfly polynomial. In order to make the correspondence precise, we compute the normalization factor $W(\gamma)$ defined in Section 1.4 for Kontsevich's integral. This factor can be expressed in two ways: in terms of Zagier's multiple zeta functions [12],

$$\zeta(s_1, s_2, \ldots, s_k) = \sum_{0<m_1<m_2<\cdots<m_k\in\mathbb{Z}} \frac{1}{m_1^{s_1} m_2^{s_2} \cdots m_k^{s_k}}, \tag{1}$$

and using the correspondence set up in the above. These two formulas lead to relations between certain values of the multiple zeta function. Let I denote a sequence of positive integers with even number of elements $I=(p_1, q_1, \ldots, p_g, q_g)$. For such I, let $g(I)=g$, $p(\mathrm{I})=\Sigma_i p_i$, $q(I)=\Sigma_i q_i$, and $|I|=p(I)+q(I)$. Then, for nonnegative integers n and s with $n-s\geqslant 0$, we have the following relation:

$$\sum_{\substack{I:\, g(I)=n-s,\\ |I|=2n}} (-1)^{q(I)-n} \frac{\eta(I)}{\pi^{2n}} = \frac{1}{(2n+1)!} \sum_{r=0}^{s} \binom{2n+1}{2r} (2-2^{2r}) B_{2r}, \tag{2}$$

where

$$\eta(I) = \zeta\Big(\underbrace{1, \ldots, 1}_{p_1-1}, q_1+1, \underbrace{1, \ldots, 1}_{p_2-1}, q_2+1, \ldots, \underbrace{1, \ldots, 1}_{p_g-1}, q_g+1\Big), \tag{3}$$

and B_{2r} is the Bernoulli number. Since the Homfly polynomial is defined by the skein relation, (2) is a consequence of the skein relation. From (2), we can compute the values of $\zeta(2, \ldots, 2)$ and then reproduce the famous theorem of Euler which explains $\zeta(2n)$ in terms of a Bernoulli number. However, many of our relations seem to be quite new. It may be interesting to apply our method to other invariants to find more relations.

The Homfly polynomial is coming from the quantum R-matrix associated with the vector representation of sl_N as in [11]. If r is the classical limit of a quantum R-matrix, then it is clear more or less from Drinfeld's work [4] that Kontsevich's integral, via the weight of r, should be the same as the invariant coming from the R-matrix as in the Reshetikhin–Turaev approach [10]. The reason is the corresponding quasi-Hopf algebras are gauge equivalent. But since Drinfeld's work [4] does not treat knot invariant thoroughly, and since at the moment literature on Kontsevich's integral does not describe the quasi-Hopf origin of the invariant, here we present a direct proof that these invariants are the same. For braids, Kohno [7] investigated such iterated integral and found the skein relation in it. We apply his result for the iterated integral of links.

In Section 1 we review the material we need from [8,2]. (Readers who are familiar with [2] will be able to omit this section.) In Section 2 we discuss the construction of a weight system from both the classical r-matrix and quantum R-matrix. The key idea is that the skein relation which determines the Homfly polynomial also determines (in a related way) a system of weights on chord diagrams. See equations (9) and (10). The relations which we find between certain values of Zagier's multiple zeta function are given in Section 3.

1. Kontsevich's integral

In this section, we extend some notions in [8,2] to links.

1.1. Chord diagrams

Let $\mathscr{A}_{\mathscr{C}}$ be the algebra of chord diagrams on S^1, which is the quotient of the linear span of chord diagrams by the four-term relation and the framing independence relation. The multiplication is given by the connected sum of chord diagrams. The chord diagram without any chord is a unit of this algebra and we denote it by $\bigcirc$. Similarly, let $\mathscr{A}_{\mathscr{C}}^{(k)}$ denote the quotient of the linear span of chord diagrams on k copies of S^1 by the four-term relation and the framing independence relation. Note that $\mathscr{A}_{\mathscr{C}}^{(1)}$ is equal to $\mathscr{A}_{\mathscr{C}}$.

Let D_1 and D_2 be two chord diagrams in $\mathscr{A}_{\mathscr{C}}^{(k_1)}$ and $\mathscr{A}_{\mathscr{C}}^{(k_2)}$, each with a noted Wilson loop. Remove an arc on each noted Wilson loop which does not contain any vertex and then using two lines to combine the two strings into one single loop. We get a chord diagram in $\mathscr{A}_{\mathscr{C}}^{(k_1+k_2-1)}$ called the product (or connected sum) of D_1 and D_2 along the noted loops. As in [2] it can be proved that this operation does not depend on the location of the arcs removed. This is a corollary of the four-term relation.

1.2. Iterated integral for links

Let L be an embedding of k oriented circles into $\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$. A point of $\mathbb{R}^3$ has coordinates $(t, z) \in \mathbb{R} \times \mathbb{C}$. A plane parallel to $\mathbb{C}$ is called horizontal. We assume that L is in a general position. Then we can define the $\mathscr{A}_{\mathscr{C}}^{(k)}$-valued integral for L as in [2].

$$Z(L) = \sum_{n=0}^{\infty} \frac{1}{(2\pi \mathrm{i})^n} \int_{t_{\min}<t_1<\cdots<t_n<t_{\max}} \sum_{\substack{\text{applicable pairings} \\ P=\{(z_i, z_i')\}}} (-1)^{\# P_{\downarrow}} D_P \times \bigwedge_{i=1}^{n} \frac{\mathrm{d}z_i(t_i) - \mathrm{d}z_i'(t_i)}{z_i(t_i) - z_i'(t_i)} \in \mathscr{A}_{\mathscr{C}}^{(k)}. \quad (4)$$

If L' is obtained from L by horizontal deformation, then $Z(L) = Z(L')$.

1.3. Chord diagrams on tangles

Suppose X is a one-dimensional piecewise smooth compact manifold. Let $\mathscr{A}_{\mathscr{C}}(X)$ be the vector space over $\mathbb{C}$ spanned by chord diagrams with support X (instead of S^1, or a union of copies of S^1) subject to the four-term relation and the framing independence relation. If $f: X \to Y$ is an embedding of a one-dimensional manifold to another then there is an associate mapping from $\mathscr{A}_{\mathscr{C}}(X)$ to $\mathscr{A}_{\mathscr{C}}(Y)$. A tangle T is a one-dimensional piecewise smooth compact oriented manifold lying between two horizontal planes such that all the boundary points are in these planes and there are only a finite number of points of T at which the tangent vectors are parallel to a horizontal plane. A horizontal plane lying between these two planes cut T into two tangles, the upper T_1 and the lower T_2. We will say that $T = T_1 \times T_2$. If D_1 (respectively D_2) is a chord diagram with support T_1 (respectively T_2) then one can define the product $D_1 \times D_2$ belonging to $\mathscr{A}_{\mathscr{C}}(T)$.

Now we can define $Z(T)$ for a tangle T by using the same formula (4). This integral $Z(T)$ is an element of $\mathscr{A}_{\mathscr{C}}(T)$. If $T = T_1 \times T_2$ then $Z(T) = Z(T_1) \times Z(T_2)$.

1.4. Invariant

Let L be an embedding of k oriented circles into $\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$. We assume that the components of L are numbered. For $i = 1, \ldots, k$, let s_i be the number of maximal points of the ith component. Suppose U is the diagram which looks like Fig. 1.

Put $\gamma = Z(U)$. Then γ is an element of $\mathscr{A}_{\mathscr{C}}$ which begins with the unit and hence we can find $\gamma^{-1} \in \mathscr{A}_{\mathscr{C}}$. Let

$$\hat{Z}(L) = (\gamma^{-s_1} \otimes \cdots \otimes \gamma^{-s_k}) \cdot Z(L), \tag{5}$$

here the right-hand side is an element of $\mathscr{A}_{\mathscr{C}}^{(k)}$ obtained by acting γ^{-s_i} on the ith string. As in [2], we get

Theorem 1.4.1. *$\hat{Z}(L)$ is an isotopy invariant of oriented links.*

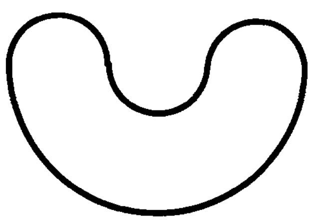


Fig. 1.

2. Weights for chord diagrams

2.1. Weights from the classical r-matrices

Let V be a finite-dimensional vector space, $N = \dim V$, and $\{e_1, e_2, \dots, e_N\}$ be a basis of V. Let $(R = R(q), \mu(q), \alpha(q), \beta(q))$ be an enhanced Yang–Baxter (EYB) operator in Turaev's sense [11, 2.3] for V. We assume that $\mu(1) = \mathrm{id}$. Let R_{ij}^{st} be the matrix element of R with $R(e_i \otimes e_j) = \Sigma_{st} R_{ij}^{st} e_s \otimes e_t$. Put $q = \exp(h)$, $R' = \alpha^{-1} R$ and $r = P(\mathrm{d}(R' - R'^{-1}/\mathrm{d}h)|_{h=0}$, where $P(u_1 \otimes u_2) = u_2 \otimes u_1$. Comparing the degree-two terms with respect to h of the braid relation for R, we get the following.

Lemma 2.1.1. *The matrix r satisfies the four-term relation (infinitesimal braid relation) $[r_{12}, r_{13} + r_{23}] = 0$, where $r_{ij} \in \mathrm{End}(V^{\otimes 3})$ acts on the ith and jth component of $V^{\otimes 3}$ by r.*

Comparing the degree-one terms with respect to h of the conditions of 2.3.1 in [11] for μ, we get the following.

Lemma 2.1.2. *For any i, $k \in \{1, 2, \dots, N\}$, $\Sigma_k r_{ik}^{kj} = 0$.*

Besides r is symmetric in the sense that $r_{kl}^{ij} = r_{lk}^{ji}$. As in [2,9] we construct a $\mathbb{C}[[h]]$-valued state model for chord diagrams, where h is a variable. Let D be a chord diagram on k strings. A mapping $f: \{\text{arcs of } D\} \to \{1, 2, \dots, N\}$ is called a *state* of D. For every state of D, we assign $r_{f(a_1)f(a_2)}^{f(a_3)f(a_4)} h$ for each chord c as in Fig. 2. Let $W_r(D)$ be a state sum on D defined by

$$W_r(D) = \sum_{f:\{\text{arcs}\} \to \{1,2,\dots,m\}} \prod_{\text{chord } c \text{ of } D} h r_{f(a_1)f(a_2)}^{f(a_3)f(a_4)}. \tag{6}$$

This model is called a *weight system* of chord diagrams associate with r. Due to Lemmas 2.1.1 and 2.1.2, W_r can be thought as a mapping from $\mathscr{A}_{\mathscr{G}}^{(k)}$ to $\mathbb{C}[[h]]$. In particular, W_r gives a mapping from $\mathscr{A}_{\mathscr{G}}$ to $\mathbb{C}[[h]]$.

Proposition 2.1.3. *For two chord diagrams D_1 and D_2, we have $W_r(D_1 \# D_2) = W_r(D_1)W_r(D_2)/N$, where $D_1 \# D_2$ is a connected sum of D_1 and D_2 along arbitrary components and $N = \dim V$.*

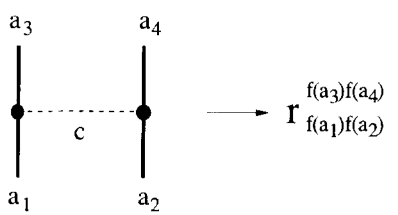


Fig. 2.

Proof. After removing a small arc from D_1 and D_2, we get chord diagrams D'_1 and D'_2, each having two endpoints. We extend the definition of W_r so that $W_r(D'_1)$ and $W_r(D'_2)$ are matrices in End(V), where rows and columns are corresponding to the state of two arcs containing the endpoints. We have $W_r(D_i) = \mathrm{Tr}\ W_r(D'_i)$ for $i = 1$, 2. So $W_r(D_1 \# D_2) = \mathrm{Tr}(W_r(D'_1)W_r(D'_2))$. Since the representation of the Lie algebra on U is irreducible and both $W_r(D'_1)$ and $W_r(D'_2)$ commute with this representation (see [2]), we have $W_r(D'_1) = \mathrm{const}_1 \cdot \mathrm{id}$ and $W_r(D'_2) = \mathrm{const}_2 \cdot \mathrm{id}$. It follows that

$$\mathrm{Tr}(W_r(D'_1)W_r(D'_2)) = (\mathrm{Tr}\ W_r(D'_1))(\mathrm{Tr}\ W_r(D'_2))/N. \qquad \square$$

The amount $W_r(\hat{Z}(L)) \in \mathbb{C}[[h]]$ is an isotopy invariant of links, but it does not take value 1 on the trivial knot. We use a normalization $\kappa_r(L) = N^{-2}W_r(\gamma)W_r(\hat{Z}(L))$, or by Proposition 2.1.3,

$$\kappa_r(L) = N^{s(L)-2}W_r(Z(L))W_r(\gamma)^{1-s(L)}, \tag{7}$$

where $s(L)$ is the number of maximal points of L. Since $Z(L)/\gamma^{s(L)-1}$ is an invariant of L, Proposition 2.1.3 implies the following.

Theorem 2.1.4. *For a link L, $\kappa_r(L)$ is an ambient isotopy invariant of L. It satisfies $\kappa_r(\bigcirc) = 1$ for the trivial knot $\bigcirc$, $\kappa_r(L_1 \# L_2) = \kappa_r(L_1)\kappa_r(L_2)$ for a connected sum of two links L_1, L_2 along arbitrary components, and $\kappa_r(L_1 \cup L_2) = \kappa_r(L_1)\kappa_r(L_2)N^2/W_r(\gamma)$ for the disjoint union of L_1, L_2.*

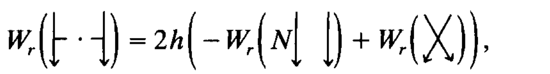
2.2. sl_N case

Let $V = \mathbb{C}^N$ be the fundamental representation of sl_N and (R, μ, α, β) be the EYB operator for V in [11, §4.2]. Put $q = \exp(h)$. Then $R = -q\Sigma_i E_{i,i} \otimes E_{i,i} - \Sigma_{i \neq j} E_{i,j} \otimes E_{j,i} + (q^{-1} - q)\Sigma_{i<j} E_{i,i} \otimes E_{j,j}$, $\mu = \mathrm{diag}(\mu_1, \dots, \mu_N)$ where $\mu_i = q^{2i-N-1}$, $\alpha = -q^N$ and $\beta = 1$. Let

$$r = P\left.\frac{\mathrm{d}(R' - R'^{-1})}{\mathrm{d}h}\right|_{h=0}, \tag{8}$$

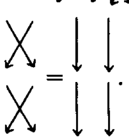
where $R' = \alpha^{-1}R$. Then r is given by $r = 2(P - N\ \mathrm{id})$. Another construction for r is the following. Let $\mathrm{Tr}(AB)$ be the usual scalar product on sl_N, and I_v is an orthonormal basis of sl_N. Then $r = 2(\Sigma_v I_v \otimes I_v)$. By Theorem 2.1.4, we get an isotopy invariant $\kappa_r(L) \in \mathbb{C}[[h]]$.

Next we will present a graphical algorithm to compute W_r for a chord diagram. Let r the matrix given by (8). Then $r = 2(P - N\ \mathrm{id})$ and is graphically presented by



(9)

where P is the permutation matrix given by $P_{ij}^{kl} = \delta_{il}, \delta_{jk}$, which is expressed graphically by $\times$. In graphical expression, the relation $P^2 = \mathrm{id}$ is expressed as



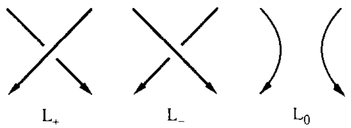


Fig. 3.

W_r also satisfies

$$W_r(D \cup \bigcirc) = NW_r(D),$$
$$W_r(\bigcirc) = N. \tag{10}$$

This interpretation resembles Kauffman's state model for the Jones polynomial. For a chord diagram D, we can compute easily $W_r(D)$ from (9) and (10).

2.3. *Equivalence of invariants from the integral and the Homfly polynomial*

For the sl_N case, Kohno already showed in Theorem 4.1 of [7] that the representation of the braid groups coming from the iterated integral satisfies the skein relation. We show that the link invariant κ_r is equal to the Homfly polynomial P with two variables m and l defined by

$$l^{-1}P_{L_+}(m, l) - lP_{L_-}(m, l) = mP_{L_0}(m, l),$$
$$P_{\bigcirc}(m, l) = 1, \tag{11}$$

where L_+, L_-, L_0 are identical except within a ball as in Fig. 3.

Theorem 2.3.1. *The invariant κ_r is equal to the Homfly polynomial. More precisely, $\kappa_r(L) = P_L(\exp(h) - \exp(-h), \exp(-Nh))$.*

Proof. We start with the following lemma.

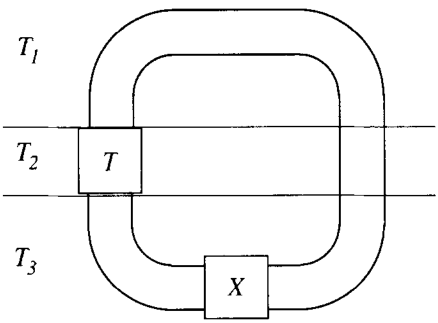


Fig. 4.

Lemma 2.3.2. *Suppose T_+, T_-, T_0 are the three tangles listed in Fig.* 4, *then*

$$\exp(Nh)W_r(Z(T_+)) - \exp(-Nh)W_r(Z(T_-))$$
$$= (\exp(h) - \exp(-h))W_r(Z(T_0)).$$

The above lemma is a corollary of Kohno's theorem [7], applied to the braid group B_2 on two strings. We can also obtain exact values for the two parameters of P from $W(Z(T_+)) = P\exp(rh/2)$, $W(Z(T_-)) = P\exp(-rh/2)$, $W(Z(T_0)) = \mathrm{id}$, and an explicit calculation of $\exp(rh/2)$, $\exp(-rh/2)$.

Proof of Theorem 2.3.1 (*continued*). By isotopy, we can push the local part containing the difference of the three links far away as in Fig. 4. In this figure the different parts of the three links are in the box denoted T. The complement parts are the same and are denoted by X. We suppose that the endpoints of X are (0, 0), (0, 1), (1, 0), (1, 1). In Fig. 4, L is decomposed into three tangles, the top is denoted by T_1, the middle by T_2, the bottom by T_3. The middle contains T and two extra lines parallel to the straight line $\mathbb{R}$. We suppose the upper endpoints of these two lines are $(l, 1)$, $(l+1, 1)$. We will consider the limit when $l \to \infty$, and write $T_1(l)$, $T_2(l)$, and $T_3(l)$. Let $Z(T_2(l)) = A + B(l)$ where $B(l)$ is the part containing all the chord diagrams with at least one "long" chord connecting a string of the left part of T_2 and a string of the right part of T_2, $A = Z(T)$ is the remaining. Of course A does not depend on l. The coefficient of a diagram of $B(l)$ tends to zero when l tends to infinity at least as fast as $\log(1 + 1/l)$. This also follows easily from the integral formula.

For all chord diagrams with less than k chords of $Z(T_1(l))$ or $Z(T_3(l))$, the coefficients tend to infinity when l tends to infinity, but at most as fast as $(\log l)^k$. This also follows easily from the integral formula. Using $\lim_{l\to\infty}\log(1 + 1/l)(\log l)^k = 0$, we see that

$$Z(L) = \lim_{l\to\infty} Z(T_1(l)) \times Z(T) \times Z(T_3(l)).$$

Now let T be respectively the diagrams of Fig. 3. Then, by applying Lemma 2.3.2 we get the first relation in (11). Hence κ_r is equal to the Homfly polynomial. □

Remark. An analogous proof yields the following result: For a weight W_r, if $P\exp(rh)$ satisfies some polynomial equation $f(t) = 0$ then this polynomial annihilates the invariants $W(\hat{Z})$ in the sense of Turaev [11].

3. Relations between values of multiple zeta functions

3.1. Computation of $\gamma = Z(U)$

Our method to compute $Z(U)$ is suggested by [1]. We place the diagram U as in Fig. 5 by using horizontal deformation for strings and vertical deformation for

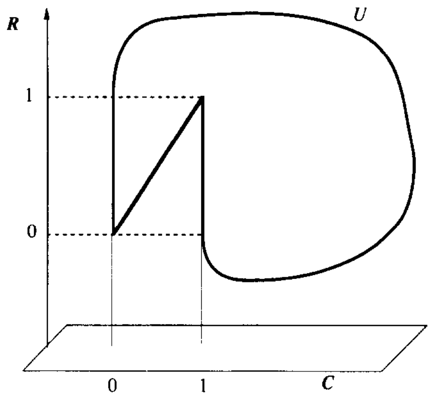


Fig. 5.

maximal and minimal points of U. The value of $\gamma = Z(U)$ in $\mathscr{A}_{\mathscr{C}}$ is a sum of iterated integrals for all the applicable pairings of U in Fig. 5. By using the technique to prove Lemma 2.3.2, we know the following.

Lemma 3.1.1. *If a chord diagram on a diagram U in Fig. 5 has a chord with an endpoint on the outside of the shape N, then the integral corresponding to this configuration goes to* 0 *when we move the outside string to far away.*

For a sequence of positive integers $I = (p_1, q_1, p_2, q_2, \ldots, p_g, q_g)$, let $P(I)$ be the configuration as in Fig. 6. It is regarded as an element of $\mathscr{A}_{\mathscr{C}}$. Then, by Lemma 3.1.1, $Z(U)$ is given by iterated integrals for the applicable pairings $P(I)$, i.e.,

$$Z(U) = \bigcirc + \sum_{n=1}^{\infty} \sum_{g=1}^{[n/2]} \frac{1}{(2\pi\mathrm{i})^n} \sum_{g=1}^{[n/2]} \sum_{\substack{I:\, g(I)=g,\\ |I|=n}} D_{P(I)}$$

$$\times (-1)^{q(I)} \int_0^1 \int_0^{t_n} \cdots \int_0^{t_3} \int_0^{t_2} \underbrace{\frac{\mathrm{d}t_n}{t_n} \cdots \frac{\mathrm{d}t_{n-q_g+1}}{t_{n-q_g+1}}}_{q_g} \underbrace{\frac{\mathrm{d}t_{n-q_g}}{1-t_{n-q_g}} \cdots \frac{\mathrm{d}t_{n-q_g-p_g+1}}{1-t_{n-q_g-p_g+1}}}_{p_g}$$

$$\times \cdots \underbrace{\frac{\mathrm{d}t_{p_1+q_1}}{t_{p_1+q_1}} \cdots \frac{\mathrm{d}t_{p_1+1}}{t_{p_1+1}}}_{q_1} \underbrace{\frac{\mathrm{d}t_{p_1}}{1-t_{p_1}} \cdots \frac{\mathrm{d}t_1}{1-t_1}}_{p_1}, \tag{12}$$

where $\bigcirc$ is the unit of $\mathscr{A}_{\mathscr{C}}$, which is the chord diagram on S^1 without any chord. Note that the chord diagram $D_{P(I)}$ is zero if $p_1 = 0$ or $q_g = 0$ and so we omit these cases. To compute the above iterated integral for the cases with $p_1 \neq 0$ and $q_g \neq 0$, we introduce a function $F(s_1, s_2, \ldots, s_k;\, x)$ for positive integers $s_1, s_2, \ldots, s_k$ and $x \in (0, 1)$ by

$$F(s_1, \ldots, s_{k-1}, s_k;\, x) = \sum_{0<m_1<m_2<\cdots<m_k \in \mathbb{Z}} \frac{x^{m_k}}{m_1^{s_1} \cdots m_k^{s_k}}.$$

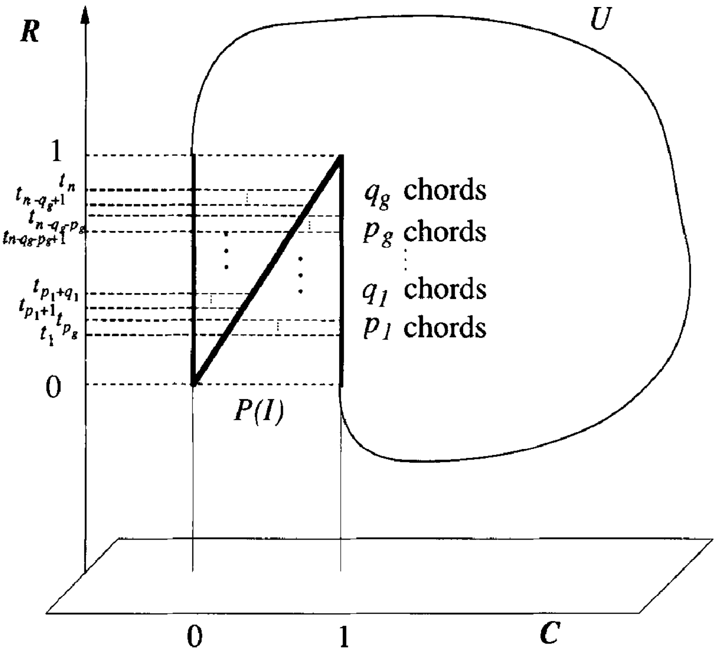


Fig. 6.

These functions satisfy:

$$\begin{aligned} F(1;\, x) &= \int_0^x \frac{1}{1-y}\, \mathrm{d}y = -\ln(1-x) = \sum_{m_1 \in \mathbb{N}} \frac{x^{m_1}}{m_1}, \\ F(s_1,\ldots,s_{k-1},\, s_k,\, 1;\, x) &= \int_0^x \frac{F(s_1,\ldots,s_{k-1},\, s_k;\, y)}{1-y}\, \mathrm{d}y, \\ F(s_1,\ldots,s_{k-1},\, s_k;\, x) &= \int_0^x \frac{F(s_1,\ldots,s_{k-1},\, s_k-1;\, y)}{y}\, \mathrm{d}y \quad \text{for } s_k \geqslant 2. \end{aligned} \tag{13}$$

Remark 3.1.2. The function F is an extension of dilogarithmic functions. Especially, $F(s_1, s_2,\ldots, s_k;\, 1) = \zeta(s_1, s_2,\ldots, s_k)$ for $s_k \geqslant 2$.

By using F and $\eta(I)$ in (3), we get the following from (12).

Theorem 3.1.3.

$$\gamma = \bigcirc + \sum_{n=1}^{\infty} \frac{1}{(2\pi \mathrm{i})^n} \sum_{g=1}^{[n/2]} \sum_{\substack{I:\, g(I)=g, \\ |I|=n}} (-1)^{q(I)} \eta(I) D_{P(I)}.$$

By computing the integral from the top to the bottom, we get

Lemma 3.1.4 (inversion formula for ζ). *$\eta(I^*) = \eta(I)$, where $I^* = (q_g,\, p_g,\ldots, q_2,\, p_2,\, q_1,\, p_1)$.*

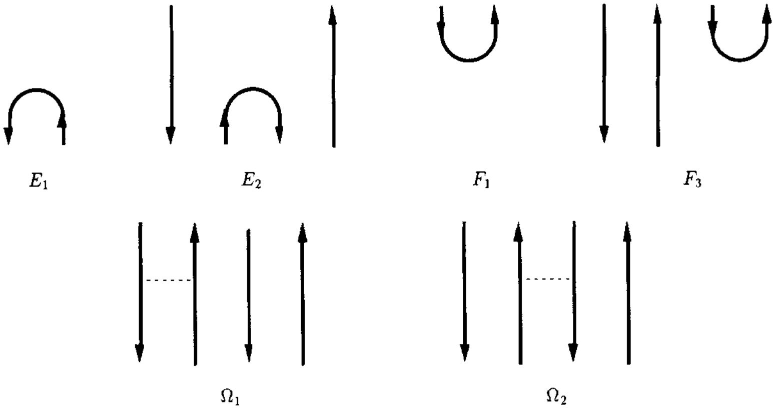


Fig. 7.

3.2. Computation of $W_r(Z(U))$

To apply the weight to the configuration $D_{P(I)}$, we denote $D_{P(I)}$ as a product $E_1E_2\Omega_1^{p_1}\Omega_2^{q_1}\cdots\Omega_1^{p_g}\Omega_2^{q_g}F_3F_1$, where E_1, E_2, F_1, F_3 are generators of tangles and Ω_1, Ω_2 are chord diagrams as in Fig. 7.

Lemma 3.2.1.

$$W_r(D_{P(I)}) = \frac{(-2hN)^{|I|}}{N^{2g(I)-1}}(1-N^2).$$

Proof. (By induction on $|I|$.) Let P_1, P_2 and id be the configurations as in Fig. 8. Then the state model replaces Ω_i by $2h(P_i - N\text{ id})$ for $i = 1, 2$. The definition of the state model implies the following: $W_r(E_1E_2\Omega_2\cdots) = W_r(\cdots\Omega_1F_3F_1) = 0$, $W_r(\cdots P_2\Omega_2\cdots) = W_r(\cdots P_1\Omega_1\cdots) = 0$, $W_r(E_1E_2P_1P_2F_3F_1) = N$, $W_r(E_1E_2P_1F_3F_1) = W_r(E_1E_2P_2F_3F_1) = N^2$. With these relations, we get the desired formula by induction. □

From the above lemma, we have

$$W_r(\gamma) = N + \sum_I \frac{(-1)^{q(I)}}{(2\pi\mathrm{i})^{|I|}} W_r(D_{P(I)})\eta(I)$$

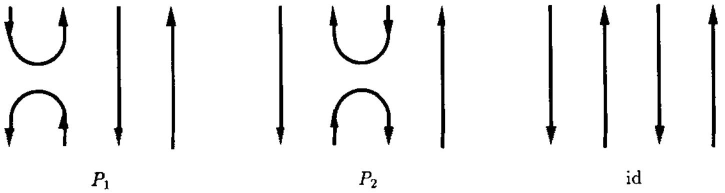


Fig. 8.

$$= N + \sum_I \frac{(-1)^{q(I)}}{(2\pi i)^{|I|}} \frac{(-2hN)^{|I|}}{N^{2g(I)-1}} (1-N^2)\eta(I)$$

$$= N\left[1 + \sum_{n=1}^{\infty} \sum_{g=1}^{[n/2]} \sum_{\substack{I:\, g(I)=g,\\ |I|=n}} (-h)^n \frac{(-1)^{q(I)}}{(\pi i)^n} (1-N^2) N^{n-2g} \eta(I)\right].$$

The inversion formula for ζ implies that the coefficient of h^n in the above formula is equal to 0 if n is odd because $\eta(I) = \eta(I^*)$, $W(D_{P(I)}) = W(\Omega(I^*))$, and $(-1)^{q(I)} = -(-1)^{q(I^*)}$. Hence we have

$$W_r(\gamma)$$

$$= N\left[1 + \sum_{n=1}^{\infty} \sum_{g=1}^{n} \sum_{\substack{I:\, g(I)=g,\\ |I|=2n}} (-1)^{q(I)-n} h^{2n} N^{2n-2g} (1-N^2) \frac{\eta(I)}{\pi^{2n}}\right]. \tag{14}$$

3.3. Another formula for $W_r(\gamma)$

Since the invariant κ_r satisfies the skein relation in Theorem 2.3.1, we have $\kappa_r(\bigcirc \cup \bigcirc) = \sinh Nh/\sinh h$. On the other hand, from Theorem 2.1.4, $\kappa_r((\bigcirc \cup \bigcirc)) = N^2/W(\gamma)$. Combining these relations, we get the following.

Theorem 3.3.1. $W_r(\gamma) = N^2 \sinh h/\sinh Nh$.

3.4. Relations for values of multiple zeta functions

Comparing (14) and Theorem 3.3.1 for $W_r(\gamma)$, we get relations for values of multiple zeta functions. To get these relations, we expand $N^2 \sinh h/\sinh Nh$. Since $t \exp(xt)/(\exp(t)-1) = \sum_{n=0}^{\infty} B_n(x) t^n/n!$ where $B_n(x)$ is the Bernoulli polynomial, we have

$$N\frac{\sinh h}{\sinh Nh} = 1 + \sum_{n=1}^{\infty} B_{2n+1}\left(\frac{N+1}{2N}\right) \frac{(2N)^{2n+1}}{(2n+1)!} h^{2n}. \tag{15}$$

We also know that

$$B_{2n+1}\left(\frac{N+1}{2N}\right) = -\sum_{s=0}^{n} \binom{2n+1}{2s} (1-2^{1-2s})(2N)^{-2n-1+2s} B_{2s}, \tag{16}$$

because $B_n(x+h) = \sum_{s=0}^{n} \binom{n}{s} B_s(x) h^{n-s}$, $B_n(1/2) = -(1-2^{1-n})B_n$ and $B_{2l+1} = 0$

for any positive integer l. Here $B_n = B_n(0)$ are the Bernoulli numbers. Hence comparing the coefficients of h^{2n} of (14) and Theorem 3.3.1, we have

$$1+\sum_{n=1}^{\infty}\frac{1}{(2n+1)!}\sum_{s=0}^{n}\binom{2n+1}{2s}(2-2^{2s})(N)^{2s}B_{2s}h^{2n}$$
$$=1+\sum_{n=1}^{\infty}\sum_{g=1}^{n}\sum_{\substack{I:\, g(I)=g,\\ |I|=2n}}(-1)^{q(I)-n}h^{2n}N^{2n-2g}(1-N^2)\frac{\eta(I)}{\pi^{2n}}. \tag{17}$$

Comparing the coefficient of $h^{2n}N^{2s}$, we get

$$\frac{1}{(2n+1)!}\binom{2n+1}{2s}(2-2^{2s})B_{2s}$$
$$=\sum_{\substack{I:\, g(I)=n-s,\\ |I|=2n}}(-1)^{q(I)-n}\frac{\eta(I)}{\pi^{2n}}-\sum_{\substack{I:\, g(I)=n-s+1,\\ |I|=2n}}(-1)^{q(I)-n}\frac{\eta(I)}{\pi^{2n}}. \tag{18}$$

This relation implies (2).

Examples. If $p=0$ then

$$\underbrace{\zeta(2,2,\ldots,2)}_{k}/\pi^{2k}=1/(2k+1)!. \tag{19}$$

If $p=n$ then

$$\frac{2-2^{2n}}{(2n)!}B_{2n}=-\sum_{\substack{p_1,q_1\geq 1\\ p_1+q_1=2n}}(-1)^{p_1-n}\frac{\zeta(\overbrace{1,\ldots,1}^{p_1-1},q_1+1)}{\pi^{2n}}$$
$$=(-1)^n\frac{1}{\pi^{2n}}[\zeta(2n)-\zeta(1,2n-1)+\cdots+\zeta(1,\ldots,1,2)].$$

By using $B_{2n}=2(2n)!(-1)^{n-1}(2\pi)^{-2n}\zeta(2n)$, we get

$$\zeta(2n)-\zeta(1,2n-1)+\cdots+\zeta(1,\ldots,1,2)=2\left(1-\frac{1}{2^{2n-1}}\right)\zeta(2n).$$

Hence

$$\left(\frac{1}{2^{2n-2}}-1\right)\zeta(2n)-\zeta(1,2n-1)+\cdots+\zeta(1,\ldots,1,2)=0.$$

For example, if $n=2$, $-\frac{3}{4}\zeta(4)-\zeta(1,3)+\zeta(1,1,2)=\frac{1}{4}\zeta(4)-\zeta(1,3)=0$ since $\zeta(1,1,2)=\zeta(4)$, and so $\zeta(1,3)=\frac{1}{4}\zeta(4)=\frac{1}{360}\pi^4$.

Remark 3.4.1. The Euler relation $B_{2n}=2(2n)!(-1)^{n-1}(2\pi)^{-2n}\zeta(2n)$ can be obtained from (19) and relations like $\zeta(4)=\zeta(2)^2-2\zeta(2,2)$.

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