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Quantum groups and ribbon G-categories

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Abstract

For a group G, the notion of a ribbon G-category was introduced in Turaev (Homotopy field theory in dimension 3 and crossed group-categories, preprint, math. GT/0005291) with a view towards constructing 3-dimensional homotopy quantum field theories (HQFTs) with target $K(G,1)$. We discuss here how to derive ribbon G-categories from a simple complex Lie algebra $\mathfrak{g}$ where G is the center of $\mathfrak{g}$. Our construction is based on a study of representations of the quantum group $U_q(\mathfrak{g})$ at a root of unity ε. Under certain assumptions on ε, the resulting G-categories give rise to numerical invariants of pairs (a closed oriented 3-manifold M, an element of $H^1(M;G)$) and to 3-dimensional HQFTs.

MSC: Primary: 57M27; secondary: 17B37; 18D10

0. Introduction

In order to construct 3-dimensional homotopy quantum field theories (HQFTs), the second author introduced for a group G the notions of a ribbon G-category and a modular (ribbon) G-category. The aim of this paper is to analyze the categories of representations of quantum groups (at roots of unity) from this prospective. The role of G will be played by the center of the corresponding Lie algebra.

We begin with a general theory of ribbon G-categories with abelian G. According to Turaev [18], any modular (ribbon) G-category $\mathscr{C}$ gives rise to a 3-dimensional HQFT

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Table 1

	A_ℓ	B_ℓ	B_ℓ	C_ℓ	D_ℓ	D_ℓ	E_6	E_7	E_8	F_4	G_2
		ℓ odd	ℓ even		ℓ odd	ℓ even					
d	1	2	2	2	1	1	1	1	1	2	3
D	$\ell+1$	2	1	1	4	2	3	2	1	1	1
G	$\mathbb{Z}_{\ell+1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	1	1	1
h	$\ell+1$	2ℓ	2ℓ	2ℓ	$2\ell-2$	$2\ell-2$	12	18	30	12	6
$h^\vee$	$\ell+1$	$2\ell-1$	$2\ell-1$	$\ell+1$	$2\ell-2$	$2\ell-2$	12	18	30	9	4

with target $K(G,1)$. Such an HQFT comprises several ingredients including a "homotopy modular functor" assigning modules to pairs (a surface Σ, an element of $H^1(\Sigma;G)$) and a numerical invariant of pairs (a closed oriented 3-manifold M, a cohomology class $\xi\in H^1(M;G)$). We introduce here a larger class of weakly non-degenerate premodular G-categories. Similarly to [18], each such category $\mathscr{C}$ gives rise to a numerical invariant $\tau_{\mathscr{C}}(M,\xi)$ of pairs (M,ξ) as above. If $\mathscr{C}$ satisfies an additional assumption of regularity and G is finite then the standard Witten–Reshetikhin–Turaev invariant $\tau_{\mathscr{C}}(M)$ is also defined and splits as

$$\tau_{\mathscr{C}}(M)=|G|^{-(b_1(M)+1)/2}\sum_{\xi\in H^1(M;G)}\tau_{\mathscr{C}}(M,\xi),$$

where $b_1(M)$ is the first Betti number of M. Note that the invariant $\tau_{\mathscr{C}}(M,\xi)$ extends to an HQFT provided $\mathscr{C}$ is a modular G-category.

Let $\mathfrak{g}$ be a simple complex Lie algebra and ε be a complex root of unity. We show that under certain assumptions on the order of ε, the pair $(\mathfrak{g},\varepsilon)$ gives rise to a premodular ribbon G-category $\mathscr{C}=\mathscr{C}(\mathfrak{g},\varepsilon)$ where G is the center of $\mathfrak{g}$. (For the center groups of simple Lie algebras, see Table 1 below.) The definition of $\mathscr{C}$ is based on a study of the representations of the quantum group $U_q(\mathfrak{g})$, cf. [1,10,11]. We specify conditions on ε which ensure that $\mathscr{C}$ is regular so that we have numerical invariants of 1-cohomology classes and a splitting of the standard WRT-invariant as above. Another set of conditions ensures that $\mathscr{C}$ is a modular G-category. The resulting 3-dimensional HQFT is however not very interesting since it splits as a product of a standard TQFT and a homological HQFT (cf. Remark 3.5).

The paper consists of three sections. In Section 1 we discuss the theory of premodular G-categories with abelian G. In Section 2 we recall the definition of $\tau_{\mathscr{C}}(M,\xi)$ and briefly discuss the homotopy modular functor. In Section 3 we consider the category $\mathscr{C}(\mathfrak{g},\varepsilon)$.

1. Premodular G-categories

Throughout this section, G denotes an abelian group and K a field of characteristic zero.

1.1. Preliminaries on monoidal categories

We shall use the standard notions of the theory of monoidal categories, see [12]. Recall that a left duality in a monoidal category $\mathscr{C}$ associates to any object $V\in\mathscr{C}$ an object $V^*\in\mathscr{C}$ and two morphisms $b_V:\mathbf{1}\to V\otimes V^*$ and $d_V:V^*\otimes V\to\mathbf{1}$ satisfying the identities

$$(\mathrm{id}_V\otimes d_V)(b_V\otimes \mathrm{id}_V)=\mathrm{id}_V,\quad (d_V\otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*}\otimes b_V)=\mathrm{id}_{V^*}.$$

Here $\mathbf{1}$ denotes the unit object of $\mathscr{C}$ and for simplicity we omit the associativity isomorphisms and the canonical isomorphisms $\mathbf{1}\otimes V\approx V\approx V\otimes\mathbf{1}$. An object of $\mathscr{C}$ isomorphic to $\mathbf{1}$ is said to be *trivial*.

A monoidal category $\mathscr{C}$ is *K-additive* if all the Hom's in $\mathscr{C}$ are K-modules and both the composition and the tensor product of morphisms are bilinear over K.

We say that a K-additive monoidal category $\mathscr{C}$ *splits as a disjoint union of subcategories* $\{\mathscr{C}_\alpha\}$ numerated by certain α if:

- each $\mathscr{C}_\alpha$ is a full subcategory of $\mathscr{C}$,
- each object of $\mathscr{C}$ belongs to $\mathscr{C}_\alpha$ for a unique α,
- if $V\in\mathscr{C}_\alpha$ and $W\in\mathscr{C}_\beta$ with $\alpha\neq\beta$ then $\mathrm{Hom}_{\mathscr{C}}(V,W)=0$.

1.2. Ribbon G-categories

A *monoidal G-category over K* is a K-additive monoidal category with left duality $\mathscr{C}$ which splits as a disjoint union of subcategories $\{\mathscr{C}_\alpha\}$ numerated by $\alpha\in G$ such that:

(i) if $V\in\mathscr{C}_\alpha$ and $W\in\mathscr{C}_\beta$ then $V\otimes W\in\mathscr{C}_{\alpha+\beta}$;
(ii) if $V\in\mathscr{C}_\alpha$ then $V^*\in\mathscr{C}_{-\alpha}$.

We shall write $\mathscr{C}=\amalg_\alpha\mathscr{C}_\alpha$ and call the subcategories $\{\mathscr{C}_\alpha\}$ of $\mathscr{C}$ the *components* of $\mathscr{C}$. The category $\mathscr{C}_0$ corresponding to the neutral element $0\in G$ is called the *neutral component* of $\mathscr{C}$. Conditions (i) and (ii) show that $\mathscr{C}_0$ is closed under tensor multiplication and taking the dual object. Condition (i) implies that $\mathbf{1}\in\mathscr{C}_0$. Thus, $\mathscr{C}_0$ is a monoidal category with left duality.

The standard notions of braidings and twists in monoidal categories apply in this setting without any changes. A braiding (resp. twist) in a monoidal G-category $\mathscr{C}$ is a system of invertible morphisms $\{c_{V,W}:V\otimes W\to W\otimes V\}_{V,W\in\mathscr{C}}$ (resp. $\{\theta_V:V\to V\}_{V\in\mathscr{C}}$) satisfying the usual conditions, see [6,16]. We say that a monoidal G-category is *ribbon* if it is ribbon in the sense of [16], i.e., if it has braiding and twist compatible with each other and with duality.

The standard theory of ribbon categories applies to any ribbon G-category $\mathscr{C}$. Suppose that L is a framed oriented m-component link in S^3 whose components are ordered, and $X_1,\dots,X_m$ are K-linear combinations of objects of $\mathscr{C}$. Then there is defined the quantum Reshetikhin–Turaev invariant $\langle L(X_1,\dots,X_m)\rangle\in\mathrm{End}_{\mathscr{C}}(\mathbf{1})$. In particular, for any object $X\in\mathscr{C}$, we have a dimension $\dim(X)=\langle U(X)\rangle\in\mathrm{End}_{\mathscr{C}}(\mathbf{1})$ where U is an oriented unknot with framing 0. For any endomorphism $f:X\to X$, we have a well-defined trace $\mathrm{tr}(f)\in\mathrm{End}_{\mathscr{C}}(\mathbf{1})$ so that $\mathrm{tr}(\mathrm{id}_X)=\dim(X)$.

1.3. Premodular G-categories

Let $\mathscr{C}$ be a ribbon G-category. An object V of $\mathscr{C}$ is *simple* if $\mathrm{End}_{\mathscr{C}}(V) = K\,\mathrm{id}_V$. It is clear that an object isomorphic or dual to a simple object is itself simple. The assumption that K is a field and [16, Lemma II.4.2.3] imply that any non-zero morphism between simple objects is an isomorphism.

We say that an object V of $\mathscr{C}$ is *dominated by simple objects* if there is a finite set of simple objects $\{V_r\}_r$ of $\mathscr{C}$ (possibly with repetitions) and morphisms $\{f_r : V_r \to V, g_r : V \to V_r\}_r$ such that $\mathrm{id}_V = \sum_r f_r g_r$. Clearly, if $V \in \mathscr{C}_\alpha$ then without loss of generality we can assume that $V_r \in \mathscr{C}_\alpha$ for all r.

We say that a ribbon G-category $\mathscr{C}$ is *premodular* if it satisfies the following three axioms:

(1.3.1) the unit object **1** is simple;

(1.3.2) for each $\alpha \in G$, the set I_α of the isomorphism classes of simple objects of $\mathscr{C}_\alpha$ is finite;

(1.3.3) for each $\alpha \in G$, any object of $\mathscr{C}_\alpha$ is dominated by simple objects of $\mathscr{C}_\alpha$.

According to [18, Remark 7.8] the set $G_{\mathscr{C}} = \{\alpha \in G \mid \mathscr{C}_\alpha \neq \emptyset\}$ is a subgroup of G. The theory certainly reduces to $G_{\mathscr{C}}$. For simplicity, we will assume from now on that $G = G_{\mathscr{C}}$.

Recall that U is an unknot with framing 0. Let $U^{\pm}$ be an unknot with framing ± 1. For each $\alpha \in G$, consider the formal linear combination

$$\omega_\alpha = \sum_{i \in I_\alpha} \dim(V_i) V_i,$$

where V_i is a simple object in $\mathscr{C}_\alpha$ representing $i \in I_\alpha$. Define

$$\Delta_\alpha = \langle U(\omega_\alpha) \rangle = \sum_{i \in I_\alpha} \dim(V_i)^2,$$

$$\Delta_\alpha^{\pm} = \langle U^{\pm}(\omega_\alpha) \rangle = \sum_{i \in I_\alpha} v_i^{\pm 1} \dim(V_i)^2,$$

where the twist θ acts on V_i as the scalar operator $v_i\,\mathrm{id}$. We recall here the following result [18, Lemma 6.6.1].

Lemma 1.1. *If $V \in \mathscr{C}_\beta$, then $V \otimes \omega_\alpha = \dim(V)\,\omega_{\alpha+\beta}$.*

The equality here is understood as an equality in the Verlinde algebra of $\mathscr{C}$. Applying to both sides the K-linear homomorphism dim from the Verlinde algebra to K, sending the class of any object to its dimension, we obtain that $\dim(V)\Delta_\alpha = \dim(V)\Delta_{\alpha+\beta}$. Taking as V a simple object in $\mathscr{C}_{-\alpha}$ and using the fact that $\dim(V) \neq 0$ (see [18, Lemma 6.5]) we obtain the following corollary.

Corollary 1.2. *For every $\alpha \in G$,*

$$\Delta_\alpha = \Delta_0.$$

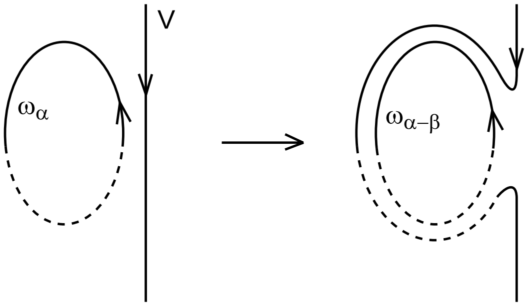

Fig. 1. Graded sliding property.

Corollary 1.3. *One has that* $\omega_\alpha \otimes \omega_\beta = \Delta_0 \omega_{\alpha+\beta}$.

This follows from Lemma 1.1 and Corollary 1.2.

An important property of any premodular category is the sliding property. Here is the version for premodular G-category.

Proposition 1.4 (Graded sliding property). *Let* $\mathscr{C}$ *be a premodular G-category. Let* L, L' *be framed oriented ordered links in* S^3 *such that* L' *is obtained from* L *by sliding the second component over the first one* (*see* Fig. 1). *Then for every* $V \in \mathscr{C}_\beta$, *one has*

$$\langle L(\omega_\alpha, V, \ldots)\rangle = \langle L'(\omega_{\alpha-\beta}, V, \ldots)\rangle. \tag{1.1}$$

The proof is the same as in the case of a premodular category, see for example [2]. One has to take into account the colors of the components of tensor product decomposition.

Remark 1.5. This proposition extends to tangles in the obvious way. One has to be a bit careful about colorings of a tangle. Any non-circle component must be colored with an object of $\mathscr{C}$, not a K-linear combination of objects as for circle components.

1.4. Transparent objects

Suppose that $\mathscr{S}$ is a set of objects of a ribbon G-category $\mathscr{C}$. An object $V \in \mathscr{C}$ is $\mathscr{S}$-*transparent* if for every $W \in \mathscr{S}$,

$$c_{W,V} c_{V,W} = \mathrm{id}.$$

This means that one can always move a string colored by V past a string colored by W, see Fig. 2. Let $\mathscr{M}_\mathscr{C}$ denote the set of all $\mathscr{C}_0$-transparent simple objects of $\mathscr{C}$.

Lemma 1.6. *Let* V *be a simple object of a premodular G-category* $\mathscr{C}$ *with* $\Delta_0 \neq 0$. *The operator of the tangle in* Fig. 3a *is non-zero for some* $\alpha \in G$ *if and only if* $V \in \mathscr{M}_\mathscr{C}$, *i.e.*, V *is* $\mathscr{C}_0$-*transparent.*

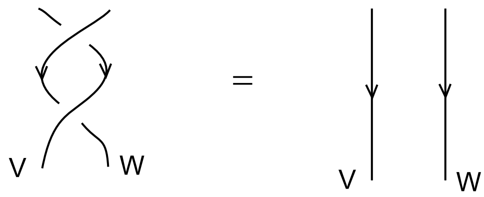

Fig. 2. Transparent object.

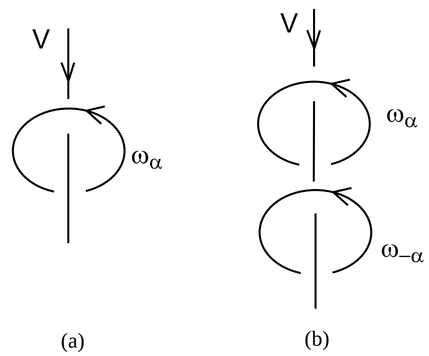

Fig. 3.

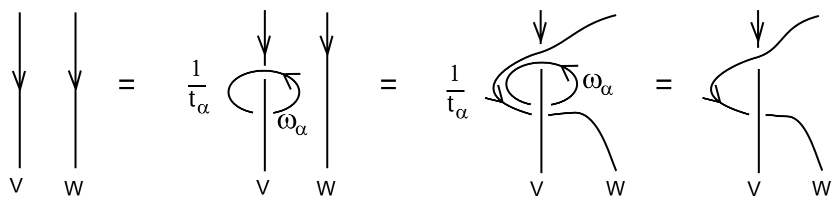

Fig. 4.

Proof. Since V is simple, the operator in Fig. 3 is equal to a scalar operator $t_\alpha\,\mathrm{id}$ with $t_\alpha \in K$.

Suppose $t_\alpha \neq 0$. Let $W \in \mathscr{C}_0$. Fig. 4 shows that $V \in \mathscr{M}_{\mathscr{C}}$; the second equality uses the graded sliding property. (This argument was first used by Blanchet and Beliakova in [2].)

Now suppose $V \in \mathscr{M}_{\mathscr{C}}$. It is clear that $t_0 = \Delta_0 \neq 0$. By Corollary 1.3, one has $\omega_\alpha \otimes \omega_{-\alpha} = \Delta_0 \omega_0$. The operator of the tangle in Fig. 3b is $t_\alpha t_{-\alpha}\,\mathrm{id}$. It is also equal to (by combining the two parallel components) $\Delta_0\, t_0\,\mathrm{id} = \Delta_0^2\,\mathrm{id}$. Thus $t_\alpha t_{-\alpha} = \Delta_0^2$, and $t_\alpha \neq 0$. □

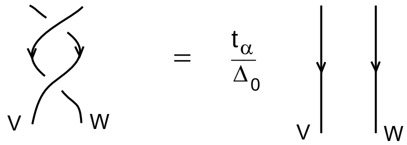

Fig. 5.

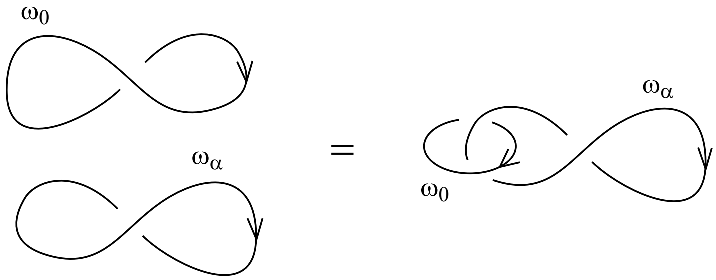

Fig. 6.

Remark 1.7. Under the assumption of Lemma 1.6, if V is $\mathscr{C}_0$-transparent, then V is "almost" transparent with respect to any object in $\mathscr{C}$. More precisely, the same argument as in the proof of Lemma 1.6 shows that for any $W \in \mathscr{C}_\alpha$, one has

1.5. Regular G-categories

A premodular G-category $\mathscr{C}$ is *weakly non-degenerate* if $\Delta_0^+ \Delta_0^- \neq 0$. It is known (see [4]) that weak non-degeneracy implies $\Delta_0 \neq 0$.

A premodular G-category $\mathscr{C}$ is *regular* if it is weakly non-degenerate and $\mathscr{M}_{\mathscr{C}} \subset \mathscr{C}_0$.

Lemma 1.8. *If $\mathscr{C}$ is a regular premodular G-category and $\alpha \neq 0$ then $\Delta_\alpha^\pm = 0$.*

Proof. Consider the equality in Figs. 5 and 6 which is obtained by sliding the top component of the left hand side over the bottom one. By the regularity of $\mathscr{C}$, there are no $\mathscr{C}_0$-transparent simple objects in $\mathscr{C}_\alpha$. Hence, by Lemma 1.6, the right-hand side on Fig. 6 is 0. It follows that $\Delta_\alpha^+ = 0$, since $\Delta_0^- \neq 0$. Similarly $\Delta_\alpha^- = 0$. □

1.6. Modular G-categories

Let $\mathscr{C}$ be a premodular G-category. For $i, j \in I = \bigcup_{\alpha \in G} I_\alpha$, choose simple objects $V_i, V_j \in \mathscr{C}$ representing i, j, respectively, and set

$$S_{i,j} = \langle H(V_i, V_j) \rangle = \mathrm{tr}(c_{V_j,V_i} \circ c_{V_i,V_j} : V_i \otimes V_j \to V_i \otimes V_j) \in \mathrm{End}_{\mathscr{C}}(\mathbf{1}) = K.$$

Here H is the standard Hopf link in S^3 with framing 0. It follows from the properties of the trace that $S_{i,j}$ does not depend on the choice of V_i, V_j in the isomorphism classes i, j.

A premodular G-category $\mathscr{C}$ is *modular* if the following axiom is satisfied.

(*Non-degeneracy axiom*) The (finite) square matrix $[S_{i,j}]_{i,j \in I_0}$ is invertible over K.

It follows from this axiom that the neutral component $\mathscr{C}_0$ of $\mathscr{C}$ is a modular ribbon category in the usual, ungraded sense (see [16]). It is known that the non-degeneracy implies the weak non-degeneracy.

A modular ribbon G-category $\mathscr{C}$ may be non-modular in the ungraded sense because the set $I = \bigcup_{\alpha \in G} I_\alpha$ of the isomorphism classes of simple objects in $\mathscr{C}$ may be infinite, or because the full S-matrix $[S_{i,j}]_{i,j \in I}$ may be non-invertible.

Proposition 1.9. *Suppose that G is finite and a modular G-category $\mathscr{C}$ is regular. Then $\mathscr{C}$ is a modular category in the ungraded sense.*

Proof. According to Bruguieres [4] and Beliakova [2], a premodular category $\mathscr{C}$ is modular if and only if it has no non-trivial $\mathscr{C}$-transparent simple objects and $\sum_{i \in I} (\dim(V_i))^2 \neq 0$. If V is a $\mathscr{C}$-transparent simple object of $\mathscr{C}$ then $V \in \mathscr{M}_{\mathscr{C}} \subset \mathscr{C}_0$. But since $\mathscr{C}_0$ is modular, V is a trivial object. By Corollary 1.2, $\sum_{i \in I} (\dim(V_i))^2 = |G|\, \Delta_0 \neq 0$, since $\mathscr{C}_0$ is modular. $\square$

2. Invariants of 3-manifolds and HQFTs

2.1. Invariants of 3-dimensional G-manifolds

Fix an abelian group G. Let $\mathscr{C}$ be a weakly non-degenerate premodular ribbon G-category over a field of zero characteristic K. We explain here following [18] that $\mathscr{C}$ gives rise to a topological invariant of 1-dimensional cohomology classes of 3-manifolds with coefficients in G.

Fix an element $\mathscr{D} \in K$ such that $\mathscr{D}^2 = \Delta_0$. Let M be a closed connected oriented 3-dimensional manifold and $\xi \in H^1(M; G)$. Present M as the result of surgery on S^3 along a framed oriented link $L = L_1 \cup \cdots \cup L_m$. Recall that M is obtained by gluing m solid tori to the exterior of L in S^3. This allows us to consider for $n = 1, \ldots, m$, the value, $\alpha_n \in G$, of ξ on the meridian of L_n and provide L_n with the color $\omega_{\alpha_n} = \sum_{i \in I_{\alpha_n}} \dim(V_i) V_i$. Let σ_+ (resp. σ_-) be the number of positive (resp. negative) squares in the diagonal decomposition of the intersection form $H_2(W_L) \times H_2(W_L) \to \mathbb{Z}$ where W_L is the compact oriented 4-manifold bounded by M and obtained from the 4-ball B^4 by attaching 2-handles along tubular neighborhoods of the components of L in $S^3 = \partial B^4$. Set

$$\tau_{\mathscr{C},\mathscr{D}}(M, \xi) = \mathscr{D}^{-b_1(M)-1} (\Delta_0^-)^{-\sigma_-} (\Delta_0^+)^{-\sigma_+} \langle L(\omega_{\alpha_1}, \ldots, \omega_{\alpha_m}) \rangle \in K,$$

where $b_1(M) = m - \sigma_+ - \sigma_-$ is the first Betti number of M.

It follows from [18, Theorem 7.3] that $\tau_{\mathscr{C},\mathscr{D}}(M, \xi)$ is a homeomorphism invariant of the pair (M, ξ) (in [18] this is stated for modular G-categories, but the proof remains true for weakly non-degenerate premodular categories). The factor $\mathscr{D}^{-b_1(M)-1}$ appears here for normalization purposes.

As in the standard theory, the invariant $\tau_{\mathscr{C},\mathscr{D}}(M,\xi)$ generalizes to an invariant of triples (M,Ω,ξ) where M is as above, Ω is a colored ribbon graph in M and $\xi\in H^1(M\setminus\Omega;G)$. Here a coloring is understood as the usual coloring of Ω over $\mathscr{C}$ such that the color of every 1-stratum t of Ω is an object in $\mathscr{C}_{\alpha(t)}$ where $\alpha(t)\in G$ is the value of ξ on the meridian of t. This notion of a coloring applies in particular to framed oriented links in M so that we obtain an isotopy invariant of triples (M,L,ξ) where M is as above, L is a colored framed oriented link in M and $\xi\in H^1(M\setminus L;G)$.

Suppose now that G is finite and $\mathscr{C}$ is regular. Set $\omega=\sum_{\alpha\in G}\omega_\alpha$. By Lemma 1.8,

$$\langle U^{\pm}(\omega)\rangle=\langle U^{\pm}(\omega_0)\rangle=\Delta_0^{\pm}\neq 0.$$

We can therefore consider the standard Witten–Reshetikhin–Turaev invariant $\tau_{\mathscr{C},\tilde{\mathscr{D}}}(M)$ of M where $\tilde{\mathscr{D}}=|G|^{1/2}\mathscr{D}$.

Proposition 2.1. *If G is finite and $\mathscr{C}$ is a regular premodular G-category, then for any closed connected oriented 3-manifold M,*

$$\tau_{\mathscr{C},\tilde{\mathscr{D}}}(M)=|G|^{-(b_1(M)+1)/2}\sum_{\xi\in H^1(M;G)}\tau_{\mathscr{C},\mathscr{D}}(M,\xi).$$

Proof. The proof in [2, Section 5] can be applied to get the result. □

2.2. The homotopy modular functor

A modular G-category $\mathscr{C}$ (not necessarily regular) gives rise to a 3-dimensional HQFT, see [18]. In particular, the homotopy modular functor $\mathscr{T}_{\mathscr{C}}$ associated with $\mathscr{C}$ assigns K-modules to so-called extended G-surfaces and assigns K-linear isomorphisms of these modules to weak homeomorphisms of such surfaces. We recall here the definition of an extended G-surface and the corresponding K-module in our abelian case.

Let Υ be a closed oriented surface. A point $p\in\Upsilon$ is *marked* if is equipped with a sign $\varepsilon_p=\pm1$ and a tangent direction, i.e., a ray $\mathbb{R}_+v$ where v is a non-zero tangent vector at p. A *marking* of Υ is a finite (possibly void) set of distinct marked points $P\subset\Upsilon$. A *G-marking* of Υ is a marking $P\subset\Upsilon$ endowed with a cohomology class $\xi\in H^1(\Upsilon\setminus P;G)$. A G-marking $P\subset\Upsilon$ is *colored* if it is equipped with a function x which assigns to every point $p\in P$ an object $x_p\in\mathscr{C}_{\xi(\mu_p)}$ where μ_p is a small loop in $\Upsilon\setminus P$ encircling p in the direction induced by the orientation of Υ if $\varepsilon_p=+1$ and in the opposite direction otherwise.

An *extended G-surface* comprises a closed oriented surface Υ, a colored G-marking $P\subset\Upsilon$, and a Lagrangian space $\lambda\subset H_1(\Upsilon;\mathbb{R})$. The corresponding K-module is defined as follows. Assume that Υ is a connected surface of genus n and $P=\{p_1,\dots,p_m\}\subset\Upsilon$. For $r=1,\dots,m$, set $\varepsilon_r=\varepsilon_{p_r}=\pm1$, $\mu_r=\mu_{p_r}$ and $x_r=x_{p_r}\in\mathscr{C}$. The group $\pi_1(\Upsilon\setminus P)$ is generated by the homotopy classes of loops $\mu_1',\dots,\mu_m'$ homotopic to $\mu_1,\dots,\mu_m$, respectively, and by $2n$ elements $a_1,b_1,\dots,a_n,b_n$ subject to the only relation

$$(\mu_1')^{\varepsilon_1}\dots(\mu_m')^{\varepsilon_m}[a_1,b_1]\dots[a_n,b_n]=1,$$

where $[a,b]=aba^{-1}b^{-1}$. Then

$$\mathscr{T}_{\mathscr{C}}(\Upsilon,P,\xi,x,\lambda)$$
$$=\bigoplus_{i_1\in I_{\xi(a_1)},=2E..,i_n\in I_{\xi(a_n)}}\mathrm{Hom}_{\mathscr{C}}\left(\mathbf{1},(x_1)^{\varepsilon_1}\otimes\cdots\otimes(x_m)^{\varepsilon_m}\otimes\bigotimes_{s=1}^{n}(V_{i_s}\otimes V_{i_s}^*)\right),$$

where for an object $x\in\mathscr{C}$ we set $x^1=x$ and $x^{-1}=x^*$. Observe that the dimension of $\mathscr{T}(\Upsilon,P,\xi,x,\lambda)$ over K does not depend on the values of ξ on $b_1,\dots,b_n$. This implies that this dimension does not depend on the choice of ξ. For $\xi=0$, this dimension is computed by the Verlinde formula. In particular, if $P=\emptyset$ we obtain

$$\mathrm{Dim}\,\mathscr{T}_{\mathscr{C}}(\Upsilon,P,\xi,x,\lambda)=\mathrm{Dim}\,\mathscr{T}_{\mathscr{C}}(\Upsilon,P,0,x,\lambda)=\mathscr{D}^{2n-2}\sum_{i\in I_0}(\dim(V_i))^{2-2n}.$$

The same formula can be deduced from the generalised Verlinde formula for HQFTs given in [17, Section 4.11].

Remark 2.2. Since we restrict ourselves here to abelian G, the homotopy classes of maps from a manifold M to the Eilenberg–MacLane space $K(G,1)$ are classified by elements of $H^1(M;G)$. This allows one to formulate HQFTs with target $K(G,1)$ in terms of 1-dimensional cohomology classes. We call the HQFTs arising from modular G-categories with abelian G and trivial crossed structure (as everywhere in this paper) *abelian*. The abelian HQFTs are simpler than the general HQFTs. One of simplifications is that in the abelian case we can forget about base points (in the general case one has to work in the pointed category). Also, in the abelian case the canonical action of G on the spaces of conformal blocks $\mathscr{T}(\Upsilon,P,\xi,x,\lambda)$ defined in [18, Section 10.3] is trivial.

3. Premodular G-categories derived from quantum groups

3.1. Fusion category associated with quantum groups at roots of unity

We will briefly recall the fusion categories of quantum groups at roots of unity, mainly to fix notation. We refer the reader to [1,5,8] for details.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$. Fix a Cartan subalgebra and a system of basis roots $\alpha_1,\dots,\alpha_\ell$. Choose the inner product on $\mathfrak{h}^*=\mathbb{R}\text{-span}\langle\alpha_1,\dots,\alpha_\ell\rangle$ such that the square length of any *short* root is 2. Let $G=X/Y$ be the quotient of the weight lattice X by the root lattice Y. It is known that G is isomorphic to the center of the simply-connected Lie group associated with $\mathfrak{g}$, and $|G|$ is equal to the determinant of the Cartan matrix. Let d be the maximal absolute value of the non-diagonal entries of the Cartan matrix. Let h denote the Coxeter number of $\mathfrak{g}$ and $h^\vee$ the dual Coxeter number of $\mathfrak{g}$. By D we denote the smallest positive integer such that $D(x|y)\in\mathbb{Z}$ for all $x,y\in X$. The data associated with simple Lie algebras is given in Table 1.

The quantum group $U_v(\mathfrak{g})$, as defined in [1,11], is a Hopf algebra over $\mathbb{Q}(v)$, with v a formal variable. There is an integral version of $U_v(\mathfrak{g})$, defined over the ring

$\mathbb{Z}[v, v^{-1}]$, introduced by Lusztig. For $\varepsilon \in \mathbb{C}$, let $U_\varepsilon(\mathfrak{g})$ be the Hopf algebra over $\mathbb{C}$ obtained by tensoring the integral version of $U_v(\mathfrak{g})$ with $\mathbb{C}$, where $\mathbb{C}$ is considered as a $\mathbb{Z}[v, v^{-1}]$-module by $v \mapsto \varepsilon$ (see [1,11]).

Let ε be a root of unity. The fusion category $\tilde{\mathscr{C}}(\mathfrak{g}, \varepsilon)$ is the quotient of the category of all tilting $U_\varepsilon(\mathfrak{g})$-modules by negligible modules and negligible morphisms (see [1], there $\tilde{\mathscr{C}}(\mathfrak{g}, \varepsilon)$ is denoted by $\mathscr{C}^-$). The category $\tilde{\mathscr{C}}(\mathfrak{g}, \varepsilon)$ is a semisimple monoidal $\mathbb{C}$-abelian category with duality. The isomorphism classes of simple objects in $\tilde{\mathscr{C}}(\mathfrak{g}, \varepsilon)$ are parametrized by the dominant weights μ such that $\mu + \rho \in C_\varepsilon$ where ρ is the half-sum of all positive roots and C_ε is defined as follows. There are two cases depending on the order r of the root of unity ε^2.

Case 1: The numbers r and d are co-prime and $r > h$. Then

$$C_\varepsilon = \{x \in C \mid (x|\alpha_0) < r\},$$

where C is the Weyl chamber and α_0 is the short highest root, i.e., the only root in C with square length 2.

Case 2: r is divisible by d and $r/d > h^\vee$. Then

$$C_\varepsilon = \{x \in C \mid (x|\beta_0) < r\},$$

where β_0 is the long highest root, i.e., the only root in C with square length $2d$. Note that when $d = 1$, the two cases overlap and the definitions of C_ε agree. In [9], C_ε is denoted by C'_r. For a dominant weight μ such that $\mu + \rho \in C_\varepsilon$ there is a simple $U_\varepsilon(\mathfrak{g})$-module $\Lambda_\mu \in \tilde{\mathscr{C}}(\mathfrak{g}, \varepsilon)$, known as the Weyl module, which is a deformation of the corresponding classical $\mathfrak{g}$-module. It represents the isomorphism class numerated by μ. The decomposition of a tensor product $\Lambda_\mu \otimes \Lambda_\nu$ is described in [1] (the tensor product is denoted there by $\underline{\otimes}$, but we will use the usual notation). For our purposes, we notice that if Λ_γ appears as a summand in $\Lambda_\mu \otimes \Lambda_\nu$, then $\mu + \nu - w \cdot \gamma \in Y$ for some $w \in W_\varepsilon$ where W_ε is the group of affine transformations of $\mathfrak{h}^*$ generated by the reflections in the walls of the simplex $\bar{C}_\varepsilon$, the topological closure of C_ε, and the dot action is the one shifted by ρ so that $w \cdot x = w(x + \rho) - \rho$. Note that $\bar{C}_\varepsilon$ is a fundamental domain of W_ε. For further use, we state here a simple lemma.

Lemma 3.1. *For every $x \in X$ and $w \in W_\varepsilon$, one has $x - w \cdot x \in Y$.*

Proof. It is known that $W_\varepsilon = W \ltimes Q$, where W is the usual Weyl group, and Q is the $\mathbb{Z}$-lattice spanned by the long roots. (One considers Q as a group of translations). Obviously, $Q \subset Y$. If $w \in Q$, the statement is trivial. If $w \in W$, the statement is well-known in the theory of simple Lie algebras. □

3.2. Braiding and twist in $\tilde{\mathscr{C}}(\mathfrak{g}, \varepsilon)$

There are a braiding and a twist in $\tilde{\mathscr{C}}(\mathfrak{g}, \varepsilon)$ which make $\tilde{\mathscr{C}}(\mathfrak{g}, \varepsilon)$ a ribbon category. They depend on a choice of a complex root of unity ζ such that $\zeta^D = \varepsilon$, where D is as in Table 1. Let us denote the resulting ribbon category by $\tilde{\mathscr{C}}(\mathfrak{g}, \varepsilon; \zeta)$. The tensor structure in $\tilde{\mathscr{C}}(\mathfrak{g}, \varepsilon; \zeta)$ and the corresponding Verlinde algebra do not depend on the choice of ζ. The ribbon category $\tilde{\mathscr{C}}(\mathfrak{g}, \varepsilon; \zeta)$ is premodular, since it has only a finite number of isomorphism classes of simple objects. It is also Hermitian, see [8], so that $\dim(V) \in \mathbb{R}$ for every $V \in \tilde{\mathscr{C}}(\mathfrak{g}, \varepsilon; \zeta)$ (This can also be deduced from the explicit formulas for the quantum dimensions of the Weyl modules.). Thus for every simple object x, one has $(\dim x)^2$ is a positive real number (since $\dim(x) \neq 0$). In Case 2 if $\varepsilon = e^{\pi i/r}$, then this Hermitian structure is known to be unitary, see [19]. Then $\dim(V) > 0$ for every $V \in \tilde{\mathscr{C}}(\mathfrak{g}, \varepsilon; \zeta)$. Again this can also be deduced from the explicit formulas for the quantum dimensions of the Weyl modules. It is clear that $\zeta^{2Dr} = 1$ where r is the order of ε^2. When r is divisible by d, $r/d > h^\vee$, and ζ has order $2Dr$, the category $\tilde{\mathscr{C}}(\mathfrak{g}, \varepsilon; \zeta)$ is modular. These are the well-known, and so far the main, examples of modular category.

3.3. $\tilde{\mathscr{C}}(\mathfrak{g}, \varepsilon; \zeta$ *as a premodular* G*-category)*

Let $\pi : X \to G = X/Y$ be the projection. For $\alpha \in G$, let I_α be the set of dominant weights μ such that $\mu + \rho \in C_\varepsilon$ and $\pi(\mu) = \alpha$. Let $\mathscr{C}_\alpha = \mathscr{C}_\alpha(\mathfrak{g}, \varepsilon; \zeta)$ be the set of all objects in $\tilde{\mathscr{C}}(\mathfrak{g}, \varepsilon; \zeta$ which are direct sums of Λ_μ with $\mu \in I_\alpha$. Set $\mathscr{C} = \bigsqcup_{\alpha \in G} \mathscr{C}_\alpha$. We consider $\mathscr{C}$ as a full subcategory of $\tilde{\mathscr{C}}(\mathfrak{g}, \varepsilon; \zeta)$.

Proposition 3.2. *The category* $\mathscr{C} = \bigsqcup_{\alpha \in G} \mathscr{C}_\alpha$ *is a premodular* G*-category.*

Proof. One only needs to check (i) and (ii) of Section 1.2. (i) Let $\mu \in I_\alpha$, $\nu \in I_\beta$, and Λ_λ be a direct summand of $\Lambda_\mu \otimes \Lambda_\nu$. Then $\mu + \nu - w \cdot \lambda \in Y$ for some $w \in W_\varepsilon$. Lemma 3.1 shows that $\pi(\lambda) = \pi(\mu) + \pi(\nu)$, or, in other words, $\lambda \in I_{\alpha+\beta}$. (ii) The dual of Λ_μ is $\Lambda_{-w_0(\mu)}$, where w_0 is the longest element of the Weyl group. Again Lemma 3.1 shows that $-w_0(\mu) \in I_{-\alpha}$. □

If all the colors of a framed link L are simple $\mathfrak{g}$-modules with highest weights in the root lattice, then the quantum invariant of L is in $\mathbb{Z}[v^2, v^{-2}]$, according to the integrality, see [9]. Hence the fact that $\mathscr{C}$ is a modular, or weakly non-degenerate, G-category does not depend on the choice of the Dth root ζ of ε; it totally depends on the order r of ε^2. Note that the fact $\tilde{\mathscr{C}}(\mathfrak{g}, \varepsilon; \zeta)$ is a (non-graded) modular category does depend on the choice of ζ.

3.4. *Modular* G*-categories*

The following proposition shows that in Case 1 of Section 3.1, the category $\mathscr{C}$ constructed above is a modular G-category, at least under the assumption that r is co-prime with $|G|$.

Proposition 3.3. *Suppose that the order* r *of* ε^2 *is co-prime with* $d|G|$ *and* $r > h$. *Then* $\mathscr{C}$ *is a modular* G*-category.*

Proof. We need only to prove that the S-matrix of the neutral component $\mathscr{C}_0$ is invertible. This was established in [10, Theorem 3.3]. □

The following proposition shows that $\mathscr{C}$ can be G-modular even when r is not co-prime with $d|G|$.

Proposition 3.4. *Suppose $\mathfrak{g}$ is a Lie algebra of series C_ℓ with odd ℓ. Assume that the order r of ε^2 is even but not divisible by 4 and $r > dh^\vee$. Then $\mathscr{C}$ is a modular G-category* (*Note that for C_ℓ, one has $d = 2$, $D = 1$, and $h^\vee = \ell + 1$.*).

A proof will be given later.

Remark 3.5. In the cases of Propositions 3.3 and 3.4, it can be shown that the category $\mathscr{C}$ is the product of its neutral component and a modular category associated with the center group G (see the definition in [14], see also [10]). The corresponding invariant of a 1-cohomology class on a 3-manifold M is then the product of the invariant of the cohomology class 0 with an invariant depending only on $H_1(M;\mathbb{Z})$ and the linking form. The theory in this case is rather trivial.

3.5. Regularity of $\mathscr{C}$

In Case 2 of Section 3.1, the premodular G-category $\mathscr{C}$ constructed above turns out to be regular, at least under a few further assumptions on ζ and r.

Proposition 3.6. *Suppose that r is divisible by d, $r/d > h^\vee$, and ζ is a root of unity of order $2Dr$. Assume that the number $k := r/d - h^\vee$ satisfies*: (*) *$k\ell\vdots 2|G|$ for Lie series A, C; $k\vdots|G|$ for Lie series B, E, F; and $k\vdots 2, k\ell\vdots 2|G|$ for Lie series* D, *where ℓ is the rank of $\mathfrak{g}$, and $a\vdots b$ means a is divisible by b. Then the premodular G-category $\mathscr{C}$ is regular.*

When $\mathfrak{g} = sl_n$, the condition on r can be rephrased as $r > n$ and $(r-n)(n-1)$ is divisible by $2n$. Thus, we recover the sl_2-case of [7,15] and the sl_n-case of [13,2]. Proposition 3.6 will be proven below.

Remark 3.7. One may ask when the G-category $\mathscr{C}$ is weakly non-degenerate. The answer is only when the order r of ε^2 is as one described in Propositions 3.3, 3.4, and 3.6, or one in the following additional cases: for the Lie algebra D_ℓ with odd ℓ the number r must satisfies $r \equiv 2 \pmod 4$; for the Lie algebra A_ℓ, one must have $k\ell(\ell+1)\vdots 2s^2$, where $s = (k, \ell+1)$.

3.6. Transparent objects

Our aim here is to study the set $\mathscr{M}_{\mathscr{C}}$ of $\mathscr{C}_0$-transparent objects of $\mathscr{C}$. To cover both Cases 1 and 2 of Section 3.1 we introduce a group G' equal to G in Case 1 and equal to $Hom(G, \mathbb{Q}/\mathbb{Z})$ in Case 2. More precisely, set $G' = X'/Y'$ where $X' = X$, $Y' = Y$ in Case 1 and $X' = Y^*, Y' = X^*$ are the $\mathbb{Z}$-dual lattices in Case 2. In both cases one has $X' \subset Y^*$. The group G' acts on $\bar{C}_\varepsilon$ as follows. Let $g \in G'$ with a lift $\tilde{g} \in X'$ and let $\mu \in \bar{C}_\varepsilon$. The element $r\tilde{g} + \mu$ may not lie in the simplex $\bar{C}_\varepsilon$ anymore, but there is $w \in W_\varepsilon$ which maps $r\tilde{g} + \mu$ into $\bar{C}_\varepsilon$. Set $g(\mu) = w(r\tilde{g} + \mu)$. Let the dot version of this action be the one shifted by ρ so that $g\mu = g(\mu + \rho) - \rho$.

Proposition 3.8. (i) *For every $g \in G'$, the module $\Lambda_{g\cdot 0}$ is $\mathscr{C}_0$-transparent.* (ii) *If r is divisible by d, $r/d > h^\vee$, and ζ is a root of unity of order $2Dr$ then every $\mathscr{C}_0$-transparent simple object in $\mathscr{C}$ is isomorphic to $\Lambda_{g\cdot 0}$ with $g \in G'$.*

Note that in the expression $g \cdot 0$ the zero stands for the weight $= 0$.

To prove Proposition 3.8 we first recall the so-called second symmetry principle for quantum link invariants (see [9]). It describes how these invariants change under the action of G' on the colors of link components. We record two corollaries of this principle. First

$$\dim(\Lambda_v)^2 = \dim(\Lambda_{g\cdot v})^2, \quad \forall g \in G'. \tag{3.1}$$

Second, for the Hopf link H

$$\langle H(\Lambda_\mu, \Lambda_{g\cdot v})\rangle \dim(\Lambda_\mu)\dim(\Lambda_{g\cdot v}) = \varepsilon^{2r(\tilde{g}|\mu)} \langle H(\Lambda_\mu, \Lambda_v)\rangle \dim(\Lambda_\mu)\dim(\Lambda_v), \tag{3.2}$$

where $\tilde{g} \in X'$ is a lift of g.

3.7. *Proof of Proposition 3.8*

Part (i): Set $V = \Lambda_{g\cdot 0}$. Let $v = 0$ in both (3.1) and (3.2), we see that

$$\langle H(\Lambda_\mu, V)\rangle = \varepsilon^{2r(\tilde{g}|\mu)} \dim(\Lambda_\mu)\dim(V).$$

Suppose $\mu \in I_0$, then $\mu \in Y$. Hence $(\tilde{g}|\mu) \in \mathbb{Z}$, since $\tilde{g} \in X' \subset Y^*$. This implies $\varepsilon^{2r(\tilde{g}|\mu)} = 1$, and hence

$$\langle H(\Lambda_\mu, V)\rangle = \dim(\Lambda_\mu)\dim(V).$$

This means for the Hopf link colored by V and any element in $\mathscr{C}_0$, we can unlink the Hopf link not altering the value of the quantum invariant. Thus the operator of Fig. 3a with $\alpha = 0$ is equal to $\Delta_0\,\mathrm{id}$. Now $\Delta_0 = \sum_{\mu \in I_0} \dim(\Lambda_\mu)^2 \neq 0$ since each $\dim(\Lambda_\mu)^2 > 0$. By Lemma 1.6, V is $\mathscr{C}_0$-transparent.

Part (ii): Consider the colored Hopf link H depicted in Fig. 7, where $\omega = \sum_{\alpha \in G} \omega_\alpha$. Applying Lemma 1.6 (for strings piercing the unknot with color ω_0), we obtain that

$$\langle H(\omega, \omega_0)\rangle = \Delta_0 \sum_{x \in T} (\dim x)^2,$$

where T is the set of isomorphism classes of objects in $\mathscr{M}_{\mathscr{C}}$. On the other hand, $\mathscr{C}$ is known to be modular in the ungraded sense. Hence a string colored by a simple object piercing through an unknot with color ω is non-zero only when the color of the string is a trivial object. Hence $\langle H(\omega, \omega_0)\rangle = \langle U(\omega)\rangle$ which, by Corollary 1.2, is $|G|\Delta_0$.

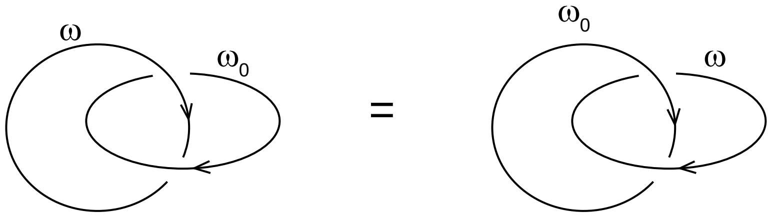

Fig. 7.

Thus

$$\sum_{x \in T} (\dim x)^2 = |G|.$$

From Part (i) we know that $\Lambda_{g \cdot 0} \in \mathscr{M}_{\mathscr{C}}$ for every $g \in G'$, and $\dim(\Lambda_{g \cdot 0})^2 = 1$ by (3.1). Since $r/d > h^\vee$, we can have $g \cdot 0 = 0$ only when $g = 0$. Hence we have at least $|G'| = |G|$ pairwise non-isomorphic simple objects in $\mathscr{M}_{\mathscr{C}}$, each has the square of the quantum dimension equal 1. Since $(\dim x)^2$ is a positive real number for any $x \in T$, we obtain that T consists only of the isomorphism classes of $\Lambda_{g \cdot 0}$ with $g \in G'$.

Remark 3.9. It can be shown that for every $g \in G'$ one has $(\Lambda_{g \cdot 0})^* = \Lambda_{(-g) \cdot 0}$ and $\Lambda_{g \cdot 0} \otimes \Lambda_\mu = \Lambda_{g \cdot \mu}$.

3.8. Proof of Proposition 3.6

We need to show that $\mathscr{M}_{\mathscr{C}} \subset \mathscr{C}_0$ and that $\Delta_0^\pm \neq 0$. Using the explicit description of the weight and root lattices, one can show the condition $(^*)$ on the number k ensures that (a) $r\tilde{g} \in Y$ for every $\tilde{g} \in X' = Y^*$, and (b) $\mathrm{d}k(\tilde{g}|\tilde{g}) \in 2\mathbb{Z}$ for every $\tilde{g} \in X'$.

Now (a) implies that $g \cdot 0 \in Y$ for every $g \in G'$. Proposition 3.8 (ii) shows that $\mathscr{M}_{\mathscr{C}} \subset \mathscr{C}_0$. Now we prove that $\Delta_0^\pm \neq 0$. By the second symmetry principle in [9], the twist θ acts on each $\Lambda_{g \cdot 0}$ as multiplication by $\varepsilon^{r\,\mathrm{d}k(\tilde{g}|\tilde{g})}$ where $\tilde{g} \in X'$ is a lift of $g \in G'$. Now (b) implies that the twist acts as the identity operator on all $\mathscr{C}_0$-transparent objects. From (3.1) we see that for $g \in G'$ one has $(\dim \Lambda_{g \cdot 0})^2 = 1$. If $\zeta = \exp(\pi i/Dr)$ (in that case $\varepsilon = \exp(\pi i/r)$) then $\dim \Lambda_{g \cdot 0}$ is positive, hence $\dim \Lambda_{g \cdot 0} = 1$. When ζ is an arbitrary root of unity of order $2Dr$, by considering a Galois action, we also have $\dim \Lambda_{g \cdot 0} = 1$.

Note also that $\Delta_0 = \sum \dim(x)^2 \neq 0$. It follows from the Bruguieres modularisation criteria [4] (simplified in [3]) that $\mathscr{C}_0$ is modularisable. This implies that $\Delta_0^\pm \neq 0$.

3.9. Proof of Proposition 3.4

We have $G' = \mathbb{Z}/2$. Let g be the non-trivial element of G'. Explicit calculation shows that $\Lambda_{g \cdot 0}$ is not in $\mathscr{C}_0$. It follows from Proposition 3.8 part (ii) that $\mathscr{C}_0$ does not have any non-trivial transparent objects. Hence by a criterion of Bruguieres (simplified in [3, Lemma 4.3]), $\mathscr{C}_0$ is modular.

Remark 3.10. 1. In the *BCD*-case, there is a decomposition of quantum invariants considered by Blanchet and Beliakova, using idempotents in the Birman–Wenzl–Murakami category. Their construction is probably different from ours because they do not consider spin representations. See also [14]. 2. For a quotient group $\tilde{G}$ of G, one can consider any G-category as a $\tilde{G}$-category. By choosing $\tilde{G}$ appropriately, we can make the proof of Proposition 3.6 valid even when the order r of ε^2 does not satisfy the conditions there. In this way, one can get a cohomology decomposition for sl_n in the case $1 < (r,n) < n$, as in [2]. Similar result holds true for the Lie series D. 3. Suppose $|G| > 1$ (otherwise the theory is reduced to the ungraded case). If the order r of ε^2 does not satisfy the conditions of Propositions 3.4 or 3.3, then it can be shown that $\mathscr{C}_0$ contains a non-trivial $\mathscr{C}_0$-transparent element of the form $\Lambda_{g\cdot 0}$, $g \in G'$, hence $\mathscr{C}$ cannot be modular (in the G-category sense).

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