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INTEGRALITY AND SYMMETRY OF QUANTUM LINK INVARIANTS

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0. Introduction. Quantum invariants of framed links whose components are colored by modules of a simple Lie algebra $\mathfrak{g}$ are Laurent polynomials in $v^{1/D}$ (with integer coefficients), where v is the quantum parameter and D an integer depending on $\mathfrak{g}$. We show that quantum invariants, with a suitable normalization, are Laurent polynomials in v^2.

We also establish two symmetry properties of quantum link invariants at roots of unity. The first asserts that quantum link invariants, at rth roots of unity, are invariant under the action of the affine Weyl group W_r, which acts on the weight lattice. A fundamental domain of W_r is the fundamental alcove $\bar{C}_r$, a simplex. Let G be the center of the corresponding simply connected complex Lie group. There is a natural action of G on $\bar{C}_r$. The second symmetry property, in its simplest form, asserts that quantum link invariants are invariant under the action of G if the link has zero linking matrix. The second symmetry property generalizes symmetry principles of Kirby and Melvin (the sl_2 case) and Kohno and Takata (the sl_n case) to arbitrary simple Lie algebra.

0.1. Quantum invariants. Suppose L is a framed link with m ordered components and $M_1, \dots, M_m$ are modules of a simple complex Lie algebra $\mathfrak{g}$. Then the quantum invariant $J_L(M_1, \dots, M_m)$ is a rational function in the variable $v^{1/D}$, where v is the *quantum parameter* and D is a number depending on $\mathfrak{g}$. (See [RT1], [Tu]; we recall the definition of quantum invariants in §1.) The Jones polynomial (see [Jo]) is the simplest in the family of quantum link invariants: When $\mathfrak{g} = \mathrm{sl}_2$ and the modules equal the fundamental representation, J_L is the Jones polynomial, with a suitable change of variable. The reader should be able to relate v to any other variable if it is known that the quantum integer $[n]$ is given by

$$[n] = \frac{v^n - v^{-n}}{v - v^{-1}}.$$

0.2. Integrality. A priori J_L is a rational function in $v^{1/D}$. Lusztig's result on the integrality of the R-matrix implies that J_L is a Laurent polynomial in $v^{1/D}$ with *integer coefficients* (see a detailed proof in §1.4.2 below). We study the integrality of the *exponents* of v. One of our main results shows that J_L is essentially a Laurent

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polynomial in v^2. More precisely, suppose all the modules $M_1, \dots, M_m$ are irreducible; then J_L belongs to $v^p\mathbb{Z}[v^{\pm 2}]$, where p is a rational number determined by the linking matrix of L (see the strong integrality theorem in §2). Thus one can get rid of fractional and odd powers of v by using a suitable normalization. For example, suppose that the normalization of the quantum invariant is chosen so that the value of the unknot is 1; then the value of any *unframed* knot is in $\mathbb{Z}[v^{\pm 2}]$.

The strong integrality theorem does not follow directly from the integrality result of Lusztig. To prove it, we have to use a geometric lemma about special presentation of links and a result of Andersen on quantum groups at roots of unity.

0.3. Symmetry I. To formulate the symmetry properties, it is more convenient to use another normalization of quantum invariants,

$$Q_L(M_1, \dots, M_m) := J_L(M_1, \dots, M_m) J_{U^{(m)}}(M_1, \dots, M_m),$$

where $U^{(m)}$ is the trivial link with m components and each has zero framing. This normalization is the one used in the definition of quantum 3-manifold invariants.

Since irreducible $\mathfrak{g}$-modules are parametrized by the set X_+ of *dominant weights*, both J_L and Q_L can be considered as functions from $(X_+)^m$ to $\mathbb{Z}[v^{\pm 1/D}]$. The set X_+ is the part of the weight lattice X which lies in a Euclidean space $\mathfrak{h}^*_{\mathbb{R}}$. For each positive integer r there is defined the *fundamental alcove* $\bar{C}_r$, which is a simplex in $\mathfrak{h}^*_{\mathbb{R}}$ (see §2). The reflections along the facets of $\bar{C}_r$ generate the affine Weyl group W_r, for which $\bar{C}_r$ is a fundamental domain. The affine Weyl group plays an important role in the theory of affine Lie algebras (see [Kac]).

We show that when v^2 is a primitive rth root of unity, quantum invariants have very nice symmetry properties expressed in the *first and the second symmetry principles*. The first asserts that Q_L is componentwise invariant under the action of the affine Weyl group. More precisely, when v^2 is an rth root of unity,

$$Q_L\big(\bar{\Lambda}_{w_1\cdot\mu_1}, \dots, \bar{\Lambda}_{w_m\cdot\mu_m}\big) = Q_L\big(\bar{\Lambda}_{\mu_1}, \dots, \bar{\Lambda}_{\mu_m}\big),$$

where $w_1, \dots, w_m \in W_r$ and $\bar{\Lambda}_\mu$ is the simple module of highest weight μ. Here the dot means the dot action, and all $\mu_1, \dots, \mu_m$, $w_1 \cdot \mu_1, \dots, w_m \cdot \mu_m$ are in X_+. For a stronger statement that describes the maximal group of symmetry, see §2.

Thus when considering quantum invariants at roots of unity, one could restrict the colors—that is, the modules assigned to components of links—to $\bar{C}_r$, a fundamental domain of the affine Weyl group. The simplex $\bar{C}_r$ contains only a finite number of elements in X_+. For example, the sum over all weights in X_+ could be replaced by the sum over all weights in $\bar{C}_r$. This happens in the theory of quantum 3-manifold invariants.

0.4. Symmetry II. Let G be the (necessarily finite abelian) center of the simply connected complex Lie group associated with $\mathfrak{g}$. The group G is also known as the fundamental group; it is isomorphic to the quotient of the weight lattice by the root lattice. There is a natural action of G on $\bar{C}_r$ (see §2). Suppose v^2 is a primitive rth root of unity. In its simplest form, the second symmetry principle says that Q_L is invariant under the action of G if the linking matrix of L is zero. In general, under the action of G, Q_L is multiplied by a *twisting factor* determined by the linking matrix of L. More precisely, suppose $\mu_1, \dots, \mu_m \in \bar{C}_r$, $g_1, \dots, g_m \in G$; then

$$Q_L\big(\overline{\Lambda}_{g_1\cdot\mu_1}, \dots, \overline{\Lambda}_{g_m\cdot\mu_m}\big) = v^{rt} Q_L\big(\overline{\Lambda}_{\mu_1}, \dots, \overline{\Lambda}_{\mu_m}\big),$$

where the dot action is, as usual, the one shifted by the half-sum of positive roots and where

$$t = (r-h) \sum_{1\le i,\, j\le m} l_{ij}\big(g_i \mid g_j\big) + 2 \sum_{1\le i,\, j\le m} l_{ij}\big(g_i \mid \mu_j\big),$$

with h being the Coxeter number (see Table 1). Here (l_{ij}) is the linking matrix, and $(\cdot \mid \cdot)$'s are scalar products naturally defined using the standard scalar product on $\mathfrak{g}$. For a stronger statement, see §2.

The action of G is induced from that of the *extended affine Weyl group*. The second symmetry principle, in fact, describes how quantum invariants behave under actions of the extended affine Weyl group.

For $\mathfrak{g} = \mathrm{sl}_2$, the second symmetry principle was discovered by Kirby and Melvin [KM] and for $\mathfrak{g} = \mathrm{sl}_n$ by Kohno and Takata [KT1]. Our contribution in these cases is the explicit relation between the twisting factor and the scalar product of $\mathfrak{g}$. This relation makes the second symmetry principle more understandable and easier to deal with. We also consider all primitive rth roots of unity, not only $e^{2\pi i/r}$. Our proof is different from those of [KM] and [KT1], though it borrows some ideas from [KM]. In order to handle all simple Lie algebras, we have to use deep results of Lusztig and Andersen on quantum groups.

0.5. In [KM] and [KT2], the second symmetry principle was used to define a finer version—the projective version—of quantum 3-manifold invariants. The values of the projective version, so far defined only for $\mathfrak{g} = \mathrm{sl}_n$, were proved to be algebraic integers (see [Mu], [MR], [TY]). Then Ohtsuki showed that, for $\mathfrak{g} = \mathrm{sl}_2$, the projective version has a *perturbative expansion*, which is a power series invariant of rational homology 3-spheres; see [Oh1] (see also [Le2] for the sl_n case). This result led Ohtsuki to the definition of finite-type invariants of 3-manifolds (see [Oh2]). In a forthcoming paper [Le3], we will generalize these results to arbitrary simple Lie algebras.

0.6. Various properties of quantum link invariants were proved by first establishing the properties for *fundamental modules* and then using *cablings* (see, e.g., [MW], [Yo]). This approach has been widely used for classical Lie algebras (series ABCD), since the invariants corresponding to fundamental representations are essentially the Homflypt and the Kauffman polynomials, which have simple skein relations. The case of exceptional Lie algebras has not been well studied. We do not use that approach in this paper. To uniformly handle all simple Lie algebras we extensively utilize results in quantum group theory.

TABLE 1

	A_ℓ	B_ℓ ℓ odd	B_ℓ ℓ even	C_ℓ	D_ℓ ℓ odd	D_ℓ ℓ even	E_6	E_7	E_8	F_4	G_2
d	1	2	2	2	1	1	1	1	1	2	3
D	$\ell+1$	2	1	1	4	2	3	2	1	1	1
G	$\mathbb{Z}_{\ell+1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	1	1	1
h	$\ell+1$	2ℓ	2ℓ	2ℓ	$2\ell-2$	$2\ell-2$	12	18	30	12	6
$h^\vee$	$\ell+1$	$2\ell-1$	$2\ell-1$	$\ell+1$	$2\ell-2$	$2\ell-2$	12	18	30	9	4

0.7. The paper is organized as follows. In §1 we recall necessary facts about quantum groups and the definition of quantum link invariants. The integrality theorem, the two symmetry principles, their refinements, and their corollaries are presented in §2. Finally, §3 contains proofs of main theorems.

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1. Quantum groups and quantum link invariants

1.1. Quantum groups. We recall here some facts from the theory of quantum groups, following [Lu2] (see also [Ka]). We do not use the h-adic version, so the R-matrix does not lie in the quantum group.

1.1.1. Cartan matrix and roots. Let $(a_{ij})_{1\le i,j\le\ell}$ be the Cartan matrix of a simple complex Lie algebra $\mathfrak{g}$. There are relatively prime integers $d_1,\dots,d_\ell$ in $\{1,2,3\}$ such that the matrix $(d_i a_{ij})$ is symmetric. Let d be the maximal of (d_i). The reader uncomfortable with Lie algebra theory might want to consider only the case $d=1$, that is, the simply laced case (series ADE), for which many formulas become much simpler. The values of d and other data for various Lie algebras are listed in Table 1.

We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and basis roots $\alpha_1,\dots,\alpha_\ell$ in the dual space $\mathfrak{h}^*$. Let $\mathfrak{h}^*_{\mathbb{R}}$ be the $\mathbb{R}$-vector space spanned by $\alpha_1,\dots,\alpha_\ell$. The root lattice Y is the $\mathbb{Z}$-lattice generated by α_i, $i=1,\dots,\ell$. Define the scalar product on $\mathfrak{h}^*_{\mathbb{R}}$ so that $(\alpha_i \mid \alpha_j)=d_i a_{ij}$. Then $(\alpha\mid\alpha)=2$ for every *short* root α.

Let $\mathbb{Z}_+$ be the set of all nonnegative integers. The weight lattice X (resp., the set of dominant weights X_+) is the set of all $\lambda\in\mathfrak{h}^*_{\mathbb{R}}$ such that $\langle\lambda,\alpha_i\rangle := (2(\lambda\mid\alpha_i))/(\alpha_i\mid\alpha_i)$ $\in\mathbb{Z}$ (resp., $\langle\lambda,\alpha_i\rangle\in\mathbb{Z}_+$) for $i=1,\dots,\ell$. Let $\lambda_1,\dots,\lambda_\ell$ be the fundamental weights; that is, the $\lambda_i\in\mathfrak{h}^*_{\mathbb{R}}$ are defined by $\langle\lambda_i,\alpha_j\rangle=\delta_{ij}$ or $(\lambda_i\mid\alpha_j)=d_i\delta_{ij}$. Then X is the

$\mathbb{Z}$-lattice generated by $\lambda_1, \dots, \lambda_\ell$. The root lattice Y is a subgroup of the weight lattice X, and the quotient $G = X/Y$ is called the *fundamental group*. If $\mu \in X$ and $\alpha \in Y$, then $(\mu \mid \alpha)$ is always an integer.

Let ρ be the half-sum of all positive roots. Then $\rho = \lambda_1 + \cdots + \lambda_\ell \in X_+$. Finite-dimensional simple $\mathfrak{g}$-modules are parametrized by X_+: for every $\lambda \in X_+$, there corresponds a unique simple $\mathfrak{g}$-module $\bar{\Lambda}_\lambda$.

1.1.2. The Hopf algebra $\mathcal{U}$ and its integral form. Consider the algebra $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ and its fractional field $\mathbb{Q}(v)$, where v is an indeterminate. The Hopf algebra $\mathcal{U}$, known as the quantum group associated with $\mathfrak{g}$, is defined over $\mathbb{Q}(v)$ and is generated by E_i, F_i, K_α, with $i = 1, \dots, \ell$ and $\alpha \in Y$, subject to some relations. We refer the reader to [Lu2] for the set of relations and the definitions of the coproduct Δ and the antipode S; the precise formulas are not used in the sequel. Note that the coproduct in [Lu2] is the same as the one in [Tu], but opposite to the one in [Ka] and [KM]; correspondingly, our antipode is the inverse of that in [Ka] and [KM]. In [Lu2], the two lattices X, Y are in different spaces, dual to each other. Here we consider both X and Y as subsets of the same space $\mathfrak{h}^*_{\mathbb{R}}$ (using the scalar product).

One of the relations says that $K_{\alpha+\beta} = K_\alpha K_\beta = K_\beta K_\alpha$ and $K_0 = 1$. Hence $K_{-\alpha} = K_\alpha^{-1}$.

Lusztig introduced an integral version ${}_{\mathcal{A}}\mathcal{U}$ of $\mathcal{U}$, similar to the Kostant $\mathbb{Z}$-form of classical Lie algebras. For each positive integer p, let

$$E_i^{(p)} = \frac{E_i^p}{[p]_i!}, \qquad F_i^{(p)} = \frac{F_i^p}{[p]_i!}, \quad \text{where } [p]_i! = \prod_{n=1}^{p} \frac{v^{d_i n} - v^{-d_i n}}{v^{d_i} - v^{-d_i}}.$$

Then ${}_{\mathcal{A}}\mathcal{U}$ is the $\mathcal{A}$-subalgebra of $\mathcal{U}$ generated by $E_i^{(p)}, F_i^{(p)}, K_\alpha$, with $i = 1, \dots, n$, $p \in \mathbb{Z}_+$, and $\alpha \in Y$. It is known that ${}_{\mathcal{A}}\mathcal{U}$ inherits the Hopf algebra structure of $\mathcal{U}$.

1.2. Category of $\mathcal{U}$-modules

1.2.1. Finite-dimensional $\mathcal{U}$-modules of type 1. Suppose M is a $\mathcal{U}$-module. For every $\nu \in X$, let

$$M^\nu = \left\{x \in M \mid K_\alpha(x) = v^{\langle \nu, \alpha \rangle} x \text{ for every root } \alpha\right\}.$$

The subspace M^ν is called the subspace of weight ν; its elements are vectors of weight ν.

Let $\mathcal{C}$ be the category of *finite-dimensional* (over $\mathbb{Q}(v)$) $\mathcal{U}$-modules M such that

$$M = \bigoplus_{\nu \in X} M^\nu.$$

It is known that on every $M \in \mathcal{C}$, both $E_i^{(p)}$ and $F_i^{(p)}$ equal zero for sufficiently large p.

A morphism in $\mathcal{C}$ is just a $\mathcal{U}$-linear homomorphism. If M, N are in $\mathcal{C}$, then $M \otimes N$

is also in $\mathscr{C}$. Thus $\mathscr{C}$ is a *tensor category* (also known as a monoidal category; see, e.g., [Ka], [Tu]).

The category $\mathscr{C}$ is semisimple, and its simple objects are parametrized by the set of dominant weights X_+. For every $\lambda \in X_+$, there is a unique simple $\mathscr{U}$-module $\Lambda_\lambda \in \mathscr{C}$, with a vector x of weight λ such that $E_i^{(p)}(x) = 0$ for every $i = 1, \dots, \ell$ and $p \geq 1$. The module Λ_λ, called the *module of highest weight* λ, can be considered as the *deformation* of the corresponding module $\bar{\Lambda}_\lambda$ of the Lie algebra $\mathfrak{g}$. Every $\mathscr{U}$-module in $\mathscr{C}$ is the direct sum of simple modules of the form Λ_λ. The decomposition of the tensor product of two simple $\mathscr{U}$-modules in $\mathscr{C}$ is exactly the same as that of the tensor product of corresponding $\mathfrak{g}$-modules. Hence, the tensor category $\mathscr{C}$ is tensorly equivalent to the tensor category of finite-dimensional $\mathfrak{g}$-modules.

If ν is a weight of Λ_λ, then $\lambda - \nu$ is a sum of positive roots. In particular, $\lambda - \nu \in Y$.

1.2.2. Dual modules. As usual, using the antipode S, for every $M \in \mathscr{C}$ one can define the dual $\mathscr{U}$-module $M^* \in \mathscr{C}$. By definition, $M^* = \mathrm{Hom}_{\mathbb{Q}(v)}(M, \mathbb{Q}(v))$, and for every $a \in \mathscr{U}$, $f \in M^*$, $x \in M$, one has $(af)(x) = f(S(a)x)$. The dual of Λ_μ is Λ_ν, where $\nu = -w_0(\mu)$ and w_0 is the longest element of the Weyl group.

1.2.3. The element $\tilde{K}_{\pm 2\rho}$. For $\beta \in Y$, $\beta = \sum_{i=1}^{\ell} k_i \alpha_i$, let $\tilde{K}_\beta = \prod_{i=1}^{n} K_{k_i d_i \alpha_i}$. Replacing K by $\tilde{K}$ has the following effect: If x is a vector of weight ν, then $\tilde{K}_\beta(x) = v^{(\nu|\beta)}(x)$ (replacing the bracket $\langle \lambda, \beta \rangle$ by the scalar product $(\lambda \mid \beta)$).

Note that 2ρ, as the sum of all positive roots, is always in the root lattice Y. Hence, $(2\rho \mid \mu) \in \mathbb{Z}$ for every $\mu \in X$. The elements $\tilde{K}_{\pm 2\rho}$ play an important role.

1.2.4. The evaluation and coevaluation maps. The ground field $\mathbb{Q}(v)$ is the $\mathscr{U}$-module Λ_λ, with $\lambda = 0$. The algebra $\mathscr{U}$ acts on $\mathbb{Q}(v)$ via the co-unit. The module $\mathbb{Q}(v)$ is the unit of the tensor product in $\mathscr{C}$.

The left evaluation map $\mathrm{ev}_l : M^* \otimes M \to \mathbb{Q}(v)$, defined by $\mathrm{ev}_l(f \otimes x) = f(x)$, is $\mathscr{U}$-linear. But the map $M \otimes M^* \to \mathbb{Q}(v)$ defined by $(x \otimes f) \to f(x)$ is not $\mathscr{U}$-linear. However, the one twisted by $\tilde{K}_{-2\rho}$ is: The map $\mathrm{ev}_r : M \otimes M^* \to \mathbb{Q}(v)$, defined by $\mathrm{ev}_r(x \otimes f) = f(\tilde{K}_{-2\rho} x)$, is $\mathscr{U}$-linear.

Similarly, the coevaluation maps

$$\mathrm{coev}_l : \mathbb{Q}(v) \longrightarrow M \otimes M^*, \quad \text{defined by } \mathrm{coev}_l(1) = \sum_s x_s \otimes x_s^*,$$

$$\mathrm{coev}_r : \mathbb{Q}(v) \longrightarrow M^* \otimes M, \quad \text{defined by } \mathrm{coev}_r(1) = \sum_s x_s^* \otimes \tilde{K}_{2\rho}(x_s),$$

are $\mathscr{U}$-linear. Here $\{x_s\}$ is a basis of M and $\{x_s^*\}$ is the dual basis in M^*.

1.2.5. Canonical basis. Lusztig and Kashiwara introduced a *canonical basis* B_λ for the $\mathscr{U}$-module Λ_λ. The set B_λ is a $\mathbb{Q}(v)$-basis of the $\mathbb{Q}(v)$-vector space Λ_λ. Let ${}_{\mathscr{A}}\Lambda_\lambda$ be the $\mathscr{A}$-lattice in Λ_λ generated by B_λ. Then ${}_{\mathscr{A}}\mathscr{U}$ leaves the lattice ${}_{\mathscr{A}}\Lambda_\lambda$ invariant, ${}_{\mathscr{A}}\mathscr{U}({}_{\mathscr{A}}\Lambda_\lambda) \subset {}_{\mathscr{A}}\Lambda_\lambda$. The set B_λ consists of weight vectors; that is, the intersection $B_\lambda \cap (\Lambda_\lambda)^\nu$ is a $\mathbb{Q}(v)$-basis of the vector space $(\Lambda_\lambda)^\nu$ for every weight ν.

1.3. The braiding and the twist

1.3.1. The quasi-R-matrix. Let Θ be the quasi-R-matrix of [Lu2]; it is an *infinite* sum

$$\Theta = \sum_s a_s \otimes b_s, \tag{1.1}$$

where a_s, b_s are in $\mathcal{U}$. So Θ belongs to an appropriate completion of $\mathcal{U}\otimes\mathcal{U}$. On every $M \in \mathcal{C}$, all the a_s, b_s, except for a finite number of them, act as zero. Hence it makes sense to consider the operator $\Theta : M \otimes N \to M \otimes N$ for every M, N in $\mathcal{C}$.

In particular, there is $\Theta : \Lambda_\lambda \otimes \Lambda_\mu \to \Lambda_\lambda \otimes \Lambda_\mu$. Lusztig proved that Θ is invertible and that both Θ, Θ^{-1} leave ${}_{\mathcal{A}}\Lambda_\lambda \otimes_{\mathcal{A}} {}_{\mathcal{A}}\Lambda_\mu$ invariant. Let us take the set $x \otimes y$, with $x \in B_\lambda$ and $y \in B_\mu$ as a basis of $\Lambda_\lambda \otimes \Lambda_\mu$, and call it the *tensor product basis*. Then, in this basis, the matrices of Θ, Θ^{-1} have entries in $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$.

Moreover, Lusztig proved that Θ, Θ^{-1} can be defined over $\mathcal{A}$. This implies that the a_s, b_s in the formula (1.1) of Θ can be chosen in ${}_{\mathcal{A}}\mathcal{U}$.

1.3.2. Braiding. Let D be the least positive integer such that $D(\mu \mid \nu) \in \mathbb{Z}$ for every $\mu, \nu \in X$. Equivalently, D is the least positive integer such that $DX \subset Y$. The number D (see Table 1) is always a divisor of the determinant of the Cartan matrix.

Let $v^{1/D}$ be a new variable such that $(v^{1/D})^D = v$. To define the braiding, we extend the ground field to $\mathbb{Q}(v^{1/D})$ by taking tensor products of $\mathcal{U}$ and every module in $\mathcal{C}$ with $\mathbb{Q}(v^{1/D})$. By abuse of notation, we still use $\mathcal{U}$, $\mathcal{C}$ to denote the corresponding objects (after taking tensor products).

For two $\mathcal{U}$-modules M, N in $\mathcal{C}$, let $\Psi : M \otimes N \to M \otimes N$ be defined by

$$\Psi(x \otimes y) = v^{(\nu|\mu)} x \otimes y,$$

if $x \in M^\nu$ and $y \in N^\mu$. Note that $(\nu \mid \mu)$ is always in $(1/D)\mathbb{Z}$.

Let $\sigma : M \otimes N \to N \otimes M$ be the flip: $\sigma(x \otimes y) = y \otimes x$. Then the braiding operator $c = c(M, N) : M \otimes N \to N \otimes M$ is defined by

$$c(M, N) = \sigma \Psi \Theta^{-1} : M \otimes N \longrightarrow N \otimes M.$$

Then c commutes with the action of $\mathcal{U}$. Actually, c is equal to the inverse of the commutativity isomorphism in [Lu2, Chapter 32].

The map Ψ is called the diagonal part of the braiding. It is because of the diagonal part that we need to extend the ground field to $\mathbb{Q}(v^{1/D})$. It is clear now that if we take the tensor product bases as the bases of $\Lambda_\lambda \otimes \Lambda_\mu$ and $\Lambda_\mu \otimes \Lambda_\lambda$, then the matrix of the braiding c has entries in $\mathbb{Z}[v^{\pm 1/D}]$. A slightly stronger result is given below.

Suppose $M = \Lambda_\mu$ and $N = \Lambda_\nu$. If μ', ν' are weights of M, N, respectively, then both $\mu - \mu'$ and $\nu - \nu'$ are in the root lattice Y. Since the scalar product of an element in Y and an element in X is always an integer, we see that

$$(\mu' \mid \nu') \equiv (\mu \mid \nu) \pmod{\mathbb{Z}}.$$

Hence, in any basis that is the tensor product of bases consisting of weight vectors, the matrix of Ψ has entries in $v^{(\mu|\nu)}\mathcal{A}$. Thus we have the following.

PROPOSITION 1.1. *In the product bases, the matrix of the braiding* $c : \Lambda_\mu \otimes \Lambda_\nu \to \Lambda_\nu \otimes \Lambda_\mu$ *has entries in* $v^{(\mu|\nu)}\mathcal{A} = v^{(\mu|\nu)}\mathbb{Z}[v^{\pm 1}]$, *and its inverse has entries in* $v^{-(\mu|\nu)}\mathcal{A}$.

1.3.3. The twist. Recall that $\Theta = \sum a_s \otimes b_s$, where the sum is infinite and $a_s, b_s \in \mathcal{U}$. Let (see [Lu2, Chapter 6])

$$\Omega = \sum_s S(a_s) b_s.$$

This sum should be considered as an element of some completion of $\mathcal{U}$. Since on $M \in \mathcal{C}$ only a finite number of terms in the sum survive, one can define $\Omega : M \to M$.

For every $M \in \mathcal{C}$, let $\theta = \theta(M) : M \to M$ be defined by

$$\theta(x) = v^{(\nu+2\rho|\nu)}\Omega(x) \quad \text{if } x \in M^\nu.$$

Then θ is invertible, commutes with $\mathcal{U}$-actions, and is known as a quantum Casimir element (see [Lu2, Chapter 6]). We call it the *twist*. Moreover, for any $M, N \in \mathcal{C}$, one has that

$$\theta(M \otimes N)\big[\theta(M) \otimes \theta(N)\big]^{-1} = c(N, M)c(M, N), \tag{1.2}$$

whose proof is similar to that of [Ka, Proposition VIII.4.5].

The twist θ can also be described as follows. First, note that $\theta(M \bigoplus N) = \theta(M) \bigoplus \theta(N)$. Every M in $\mathcal{C}$ can be uniquely expressed in the form

$$M = \bigoplus_{\lambda \in X_+} M_{(\lambda)},$$

where $M_{(\lambda)}$ is the direct sum of a finite number of copies of the simple module Λ_λ. The twist θ acts on $M_{(\lambda)}$ as the scalar $v^{(\lambda+2\rho|\lambda)}$ times the identity.

1.3.4. $\mathcal{C}$ is a ribbon category. It is known that $\mathcal{C}$, together with the braiding c and the twist θ, is a *ribbon category* (see [Ka], [Tu]). Actually, $\mathcal{C}$ is the same as the category $U_h(\mathfrak{g})\text{-Mod}_{fr}$ in [Ka, Chapter XVII] or the category $\mathcal{V}_q\mathfrak{g}$ in [Tu, Chapter XI]. Both [Ka] and [Tu] use the h-adic version of quantum group, which is not suitable for studying the roots of unity case.

1.3.5. The variable q. In knot theory, another variable $q = v^2$ is usually used. The reader should not confuse this q with the quantum parameter used in the definition of quantum groups by several authors. For example, our q is equal to q^2 in [Ka] and [Tu]. In the expression "quantum invariant at an rth root of unity," the rth root of unity is q (but not v).

1.4. Quantum link invariants. It is known that any ribbon category gives rise to operator invariants of *framed links*, whose components are colored by objects of the

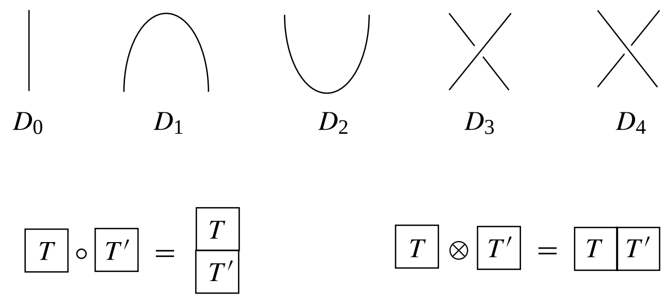


FIGURE 1. Elementary tangle diagrams, composition, and tensor product

category. We first review the definition, following [KM] and [Tu].

1.4.1. Framed tangles. A *tangle* T is an oriented 1-manifold properly embedded (up to isotopy) in $\mathbb{R}^2 \times [0,1]$, with $\partial T \subset 0 \times \mathbb{R} \times \partial[0,1]$. Define $\partial_- T = T \cap (\mathbb{R}^2 \times 0)$ and $\partial_+ T = T \cap (\mathbb{R}^2 \times 1)$, and call T a (k,l)-tangle if $|\partial_- T| = k$ and $|\partial_+ T| = l$. Thus a link is a $(0,0)$-tangle.

A *framed tangle* is a tangle T equipped with a normal vector field that is standard $(1,0,0)$ on ∂T. As usual, we consider framed tangles up to isotopy relative to the boundary. In $\mathbb{R}^3$, there is a natural way to identify framings of a component with integers.

A diagram of a tangle is its regular projection on $0 \times \mathbb{R}^2$, together with the information on over- or undercrossings. A diagram defines a blackboard framing in which the normal vector is always $(1,0,0)$. A diagram of T is *good* if the blackboard framing is coincident with the framing of T.

It is well known that every tangle diagram can be factored into the *elementary diagrams* D_0–D_4 depicted in Figure 1 using the *composition* $\circ$ (when defined) and the *tensor product* $\otimes$ of diagrams.

1.4.2. Operator invariants of colored framed tangles. A coloring of a tangle T is an assignment of an object in the category $\mathscr{C}$ to each component of T. This induces a coloring of ∂T as follows: If C is an arc of color M, then assign M to each endpoint of C where C is oriented down and assign the dual object M^* to each endpoint where C is oriented up. Tensoring from left to right, this gives the *boundary objects* $T_\pm$ assigned to $\partial_\pm T$. By convention, the empty product is the unit in $\mathscr{C}$ (the ground ring $\mathbb{Q}(v^{1/D})$).

There exists a unique $\mathscr{U}$-linear operator $J_T : T_- \to T_+$, assigned to each colored framed tangle T, that satisfies $J_{T\circ T'} = J_T \circ J_{T'}$, $J_{T\otimes T'} = J_T \otimes J_{T'}$, and, for the tangles given by the elementary diagrams with blackboard framing,

$$J_{D_0} = \mathrm{id}, \qquad J_{D_1} = \mathrm{ev}_l, \qquad J_{D_1'} = \mathrm{ev}_r,$$
$$J_{D_2} = \mathrm{coev}_l, \qquad J_{D_2'} = \mathrm{coev}_r, \qquad J_{D_3} = c, \qquad J_{D_4} = c^{-1}.$$

Here we assume that D_1, D_2 have orientation pointing from left to right, while D_1', D_2' are the same tangles D_1, D_2 but with reverse orientation. In particular, if L is a framed link with m ordered components, then $J_L(M_1, \ldots, M_m)$, for $M_1, \ldots, M_m \in \mathscr{C}$, is in $\mathbb{Q}(v^{1/D})$.

We see that only the braiding c and $\tilde{K}_{\pm 2\rho}$ take part in the construction of J_T. Many problems are thus reduced to questions about c and $\tilde{K}_{\pm 2\rho}$. Since in the product bases the matrices of c and $\tilde{K}_{\pm 2\rho}$ have entries in $\mathbb{Z}[v^{\pm 1/D}]$, we see that $J_L(\Lambda_{\mu_1}, \ldots, \Lambda_{\mu_m})$ is always a Laurent polynomial in $v^{\pm 1/D}$, that is, $J_L \in \mathbb{Z}[v^{\pm 1/D}]$. Masbaum and Wenzl in [MW] proved this fact for $\mathfrak{g} = \mathrm{sl}_n$ using idempotent decompositions.

Later we see that every link can be decomposed into *pure braids* and tangle diagrams without crossing points. We then prove that J_T, when T is a pure braid, can be expressed through the twist θ only (no need to use the braiding). Thus one needs to use only the twist and $\tilde{K}_{\pm 2\rho}$. Both are simple, since their actions on highest-weight modules are easily described.

1.4.3. Relation to the Kontsevich integral. Quantum invariants of links can also be defined through the Kontsevich integral; see [LM] and [Ka, Chapter XX]. Roughly speaking, one first takes the (framed version) Kontsevich integral of a link L, then plugs in the *weight system* coming from the modules $\bar{\Lambda}_{\mu_1}, \ldots, \bar{\Lambda}_{\mu_m}$ of the Lie algebra $\mathfrak{g}$. The result, after a change of variable, is $J_L(\Lambda_{\mu_1}, \ldots, \Lambda_{\mu_m})$.

This approach avoids the theory of quantum groups (although the Kontsevich integral has its origin in quantum group theory). Some properties of quantum link invariants can be easily seen from this point of view. Some other properties, such as the ones proved in this paper, are easier to prove using quantum group theory. Actually, we do not know how to prove the results of this paper by using the Kontsevich integral theory.

1.4.4. The trivial knot. Suppose U is the trivial knot. Then $J_U(M)$ is called the *quantum dimension* of M; its value is well known:

$$J_U(\Lambda_\mu) = \prod_{\text{positive roots } \alpha} \frac{v^{(\mu+\rho|\alpha)} - v^{-(\mu+\rho|\alpha)}}{v^{(\rho|\alpha)} - v^{-(\rho|\alpha)}}$$

$$= v^{-(\mu|2\rho)} \prod_{\text{positive roots } \alpha} \frac{v^{2(\mu+\rho|\alpha)} - 1}{v^{2(\rho|\alpha)} - 1} \tag{1.3}$$

$$= \frac{\sum_{w \in W} \mathrm{sn}(w) v^{2(\mu+\rho|w(\rho))}}{\sum_{w \in W} \mathrm{sn}(w) v^{2(\rho|w(\rho))}}, \tag{1.4}$$

where W is the Weyl group and $\mathrm{sn}(w)$ is the sign of the linear transformation w. Noting that $(\mu \mid 2\rho)$ is always an integer, we get the following.

COROLLARY 1.2. *One has that $J_U(\Lambda_\mu)$ is either in $\mathbb{Z}[v^2, v^{-2}]$ or in $v\mathbb{Z}[v^2, v^{-2}]$. More precisely, $J_U(\Lambda_\mu) \in v^{(\mu|2\rho)}\mathbb{Z}[v^2, v^{-2}]$.*

1.4.5. Sum, tensor product, and framing. The following facts are well known; see [Ka], [Tu]. One has the *sum formula*

$$J_L(M \oplus M', \dots) = J_L(M, \dots) + J_L(M', \dots), \tag{1.5}$$

where the dots denote the colors of the components other than the first.

Let $T^{(2)}$ be the link obtained from T by replacing the first component by two of its parallel push-offs (using the framing). Then one has the *tensor product formula*

$$J_T(M \otimes N, \dots) = J_{T^{(2)}}(M, N, \dots). \tag{1.6}$$

Suppose L' is obtained from L by increasing the framing of the first component by 1. Then one has the *framing formula*

$$J_{L'}(\Lambda_\mu, \dots) = v^{(\mu+2\rho|\mu)} J_L(\Lambda_\mu, \dots). \tag{1.7}$$

1.4.6. $(1,1)$*-tangles.* Suppose that T is a $(1,1)$-tangle and that the open component of T is the first component whose color is M. Then J_T is an operator from M to M, commuting with the action of $\mathcal{U}$. When M is a simple $\mathcal{U}$-module, J_T is a scalar operator, and thus there is a scalar invariant $\tilde{J}_T(M, \dots) \in \mathbb{Z}[v^{\pm 1/D}]$ such that

$$J_T(M, \dots) = \tilde{J}_T(M, \dots) \times \mathrm{id}.$$

If we close the $(1,1)$-tangle T to get a framed link L, then

$$J_L(M, \dots) = \tilde{J}_T(M, \dots) \times J_U(M). \tag{1.8}$$

2. Integrality and symmetry

2.1. Integrality. By integrality we mean the integrality of the coefficients and the exponents of v in J_L. Recall that D is the least natural number such that $(\mu \mid \mu') \in (1/D)\mathbb{Z}$ for every μ, μ' in the weight lattice X.

2.1.1. Weak integrality. We have seen that $J_L(M_1, \dots, M_m)$ is always in $\mathbb{Z}[v^{\pm 1/D}]$. A little stronger statement is the following.

PROPOSITION 2.1 (Weak integrality). *The quantum invariant $J_L(\Lambda_{\mu_1}, \dots, \Lambda_{\mu_m})$ lies in $v^f\mathbb{Z}[v^{\pm 1}] = q^{f/2}\mathbb{Z}[q^{\pm 1/2}]$, where f is a (generally fractional) number determined by the linking matrix $(l_{ij})_{1\le i,j\le m}$ of L: $f = \sum_{1\le i,\, j\le m} l_{ij}(\mu_i \mid \mu_j)$.*

Proof. If $x \in M^\nu$, then $\tilde{K}_{\pm 2\rho}(x) = v^{(\pm 2\rho|\nu)}x$. Note that $(2\rho \mid \nu)$ is an integer, since 2ρ, as the sum of all positive roots, is always in the root lattice Y. It follows that in any basis consisting of weight vectors, $\tilde{K}_{\pm 2\rho}$ has entries in $\mathbb{Z}[v^{\pm 1}]$.

The braiding $c : \Lambda_\mu \otimes \Lambda_\nu \to \Lambda_\nu \otimes \Lambda_\mu$ has entries in $v^{(\mu|\nu)}\mathcal{A}$ (see Proposition 1.1) in the product bases; its inverse has entries in $v^{-(\mu|\nu)}\mathcal{A}$. Counting the positive and negative crossing points of a good diagram of L gives the desired result. □

2.1.2. Strong integrality. The weak integrality says that the quantum invariant is essentially a Laurent polynomial in v. The fractional power can be eliminated by a suitable normalization. We have here a stronger statement, which says that the quantum invariant is essentially a Laurent polynomial in v^2.

THEOREM 2.2 (Strong integrality). *The invariant $J_L(\Lambda_{\mu_1},\dots,\Lambda_{\mu_m})$ is in $v^p\mathbb{Z}[v^{\pm 2}] = q^{p/2}\mathbb{Z}[q^{\pm 1}]$, where p is a (generally fractional) number determined by the linking matrix l_{ij} of L:*

$$p = \sum_{1\le i,\, j\le m} l_{ij}\big(\mu_i \mid \mu_j\big) + \sum_{1\le i\le m} (l_{ii}+1)(2\rho \mid \mu_i) \in \frac{1}{D}\mathbb{Z}.$$

The proof, which is presented in §3.6, is much more difficult than that of the weak integrality. We have to use a geometric lemma about special presentation of links together with a result of Andersen on quantum groups at roots of unity. The use of quantum groups at roots of unity seems unnatural, since the strong integrality does not have anything to do with roots of unity (see also the remark in §3.6.7). Andersen constructed an algebra homomorphism from the quantum group at $v=-\varepsilon$ to the quantum group at $v=\varepsilon$, where ε is some root of unity. Heuristically, this implies some kind of symmetry between $-v$ and v, which leads to the fact that J_L depends essentially only on v^2.

Remarks. (a) The factor $q^{p/2}$ could be understood as the contribution of the diagonal part of the R-matrix.

(b) If the link L is replaced by a tangle T, then one cannot get such nice results about the exponents as in the strong integrality theorem.

(c) If we use the normalization

$$\hat{J}_L\big(\Lambda_{\mu_1},\dots,\Lambda_{\mu_m}\big) := v^{-p} J_L\big(\Lambda_{\mu_1},\dots,\Lambda_{\mu_m}\big),$$

then $\hat{J}_L$ is a link invariant with values in $\mathbb{Z}[v^2, v^{-2}]$. Note that we can define $\hat{J}_L$ only for *simple* modules in $\mathscr{C}$. The normalization $\hat{J}_L$ does not behave well under the action of the Weyl group (see below), and we do not use it in the sequel.

COROLLARY 2.3. *Consider the knot case. Let $J'_L(\Lambda_\mu)$ be the nonframed version of the quantum invariant of knots, normalized so that the unknot takes value* 1*, that is,*

$$J'_L(\Lambda_\mu) := \frac{J_{L^0}(\Lambda_\mu)}{J_U(\Lambda_\mu)},$$

where L^0 is the framed knot with framing zero and of knot type L. Then $J'_L(\Lambda_\mu) \in \mathbb{Z}[v^{\pm 2}]$.

Remark. When the link L has more than one component, then in general, the quotient $J_L/J_{U^{(m)}}$ is not a Laurent polynomial, but rather a rational function in $v^{1/D}$.

The following corollary is useful in the theory of quantum invariants of 3-manifolds.

(See [Le3]; for the case $\mathfrak{g} = \mathrm{sl}_n$, the corollary was proved in [Le2], using cabling.)

COROLLARY 2.4. *If all the μ_j's are in the root lattice, then $J_L(\Lambda_{\mu_1}, \dots, \Lambda_{\mu_m})$ is in $\mathbb{Z}[v^{\pm 2}] = \mathbb{Z}[q^{\pm 1}]$.*

Proof. The second term in the expression of the exponent p is in $2\mathbb{Z}$, since $(\rho \mid \mu_j)$ is in $\mathbb{Z}$. The first term is

$$\sum_{ij} l_{ij}\big(\mu_i \mid \mu_j\big) = \sum_i l_{ii}(\mu_i \mid \mu_i) + 2\sum_{i>j} l_{ij}\big(\mu_i \mid \mu_j\big).$$

Since $(\alpha \mid \alpha)$ is even for every α in the root lattice, we see that the first term is in $2\mathbb{Z}$, too. Hence, the exponent p is an even number. □

2.2. The first symmetry principle. Recall that $q = v^2$. We show that if q is an rth root of unity, then J_L has nice symmetry.

2.2.1. The Weyl group and the affine Weyl group. Let C be the *fundamental chamber*:

$$C = \big\{x \in \mathfrak{h}^*_{\mathbb{R}} \mid 0 \le (x \mid \alpha_i),\ i = 1, \dots, \ell\big\}.$$

Then $X_+ = X \cap C$. The Weyl group W, by definition, is generated by reflections along the facets of C. It is a finite subgroup of the orthogonal group of $\mathfrak{h}^*_{\mathbb{R}}$, and C is a fundamental domain of it. Let W_r be the group of affine transformation of $\mathfrak{h}^*_{\mathbb{R}}$ generated by W and the translation group rY. Since the root lattice is invariant under the action of W, one has $W_r = W \ltimes rY$.

Let α_0 be the highest short root. When $d = 1$, that is, when all the roots have the same length, α_0 is simply the highest root. The *fundamental alcove* is defined by

$$C_r = \big\{x \in C \mid (x \mid \alpha_0) < r\big\} = \big\{x \in \mathfrak{h}^*_{\mathbb{R}} \mid 0 \le \langle x, \alpha\rangle < r \text{ for every positive root } \alpha\big\}.$$

Its topological closure $\bar{C}_r$ is an ℓ-simplex and is a fundamental domain of the affine Weyl group W_r. (See, e.g., [Kac, Chapter 6]; one has to apply the theory in [Kac] to the dual root lattice.) Moreover, W_r is generated by the reflections along the facets of $\bar{C}_r$.

2.2.2. J_L as a function on the weight lattice. Let us define, for $\mu_1, \dots, \mu_m \in \rho + X_+$,

$$J_L(\mu_1, \dots, \mu_m) := J_L\big(\Lambda_{\mu_1 - \rho}, \dots, \Lambda_{\mu_m - \rho}\big) \in \mathbb{Z}\big[v^{\pm 1/D}\big].$$

The shift by ρ is more convenient for us. The formula is good only when all the $\mu_1, \dots, \mu_m$ are in $\rho + X_+ = X \cap C^\circ$, where C° is the interior of the fundamental chamber C. We extend the definition to every point in X as follows.

If one of the μ_j is on the boundary of the chamber C, then let $J_L(\mu_1, \dots, \mu_m) = 0$. For every $\mu \in X$, there exists $w \in W$ such that $w(\mu) \in C$; moreover, if $w(\mu)$ is in the interior of C, then such a w is unique. For arbitrary $\mu_1, \dots, \mu_m \in X$, choose

$w_1,\dots,w_m \in W$ such that $w_j(\mu_j)\in X_+$. Then define (recall that $\mathrm{sn}(w)$ is the sign of w)

$$J_L(\mu_1,\dots,\mu_m)=\mathrm{sn}(w_1)\cdots\mathrm{sn}(w_m)\,J_L\big(w_1(\mu_1),\dots,w_m(\mu_m)\big).$$

Using formulas (1.3), (1.4) for the unknot, we see that the formula

$$J_U(\mu)=v^{-(\mu-\rho|\rho)}\prod_{\text{positive roots }\alpha}\frac{v^{2(\mu|\alpha)}-1}{v^{-2(\rho|\alpha)}-1} \tag{2.1}$$

is valid for every μ, not only in $\rho+X_+$, but also in X.

2.2.3. Another normalization. Let $U^{(m)}$ be the zero-framing trivial link of m components. Recall that

$$Q_L(\mu_1,\dots,\mu_m):=J_L(\mu_1,\dots,\mu_m)\times J_{U^{(m)}}(\mu_1,\dots,\mu_m).$$

This normalization is more suitable for the study of quantum 3-manifold invariants and helps us to get rid of the $\pm$ sign in many formulas. Then Q_L is componentwise invariant under the action of the Weyl group: For every $w_1,\dots,w_m\in W$,

$$Q_L\big(w_1(\mu_1),\dots,w_m(\mu_m)\big)=Q_L(\mu_1,\dots,\mu_m).$$

2.2.4. First symmetry principle. Recall that $q=v^2$. Suppose f,g belong to the same $q^a\mathbb{Z}[q^{\pm1}]$, where $a\in(1/2D)\mathbb{Z}$. We say that $f=g$ *at primitive rth roots of unity* and write

$$f\overset{(r)}{=}g$$

if, for every primitive rth root of unity ξ, one has

$$q^{-a}f|_{q=\xi}=q^{-a}g|_{q=\xi}.$$

There is no need to fix a $2D$th root of ξ. When writing $f\overset{(r)}{=}g$, we always assume that f and g belong to the same $q^a\mathbb{Z}[q^{\pm1}]$.

THEOREM 2.5 (First symmetry principle). *At primitive rth roots of unity, the quantum invariant Q_L is componentwise invariant under the action of the affine Weyl group W_r. This means, for every $w_1,\dots,w_m\in W_r$,*

$$Q_L\big(w_1(\mu_1),\dots,w_m(\mu_m)\big)\overset{(r)}{=}Q_L(\mu_1,\dots,\mu_m). \tag{2.2}$$

If one of the $\mu_1,\dots,\mu_m$ is on the boundary of $\bar{C}_r$, then $J_L(\mu_1,\dots,\mu_m)\overset{(r)}{=}0$.

Note that by the strong integrality, the left-hand side and the right-hand side of (2.2) belong to the same $q^a\mathbb{Z}[q^{\pm1}]$. We also show that J_L is componentwise *skew-invariant* under the affine Weyl group: For every $w_1,\dots,w_m\in W_r$,

$$J_L\big(w_1(\mu_1),\dots,w_m(\mu_m)\big)\overset{(r)}{=}\mathrm{sn}(w_1)\cdots\mathrm{sn}(w_m)\,J_L(\mu_1,\dots,\mu_m).$$

FIGURE 2. The A_1 case

Remark. One can drop the "primitive" in the statements of the theorem.

2.3. The second symmetry principle. Recall that $\bar{C}_r$ is a fundamental domain of the action of W on $\mathfrak{h}^*_{\mathbb{R}}$. Because of the first symmetry principle, at primitive rth roots of unity, it is enough to consider $J_L(\mu_1, \dots, \mu_m)$ with μ_j in $\bar{C}_r \cap X$, a finite set. It turns out that we can do better. There is a finite group G acting on $\bar{C}_r$, and although J_L is not really invariant under this action, it behaves quite nicely.

2.3.1. The extended affine Weyl group and the center group G. Recall that $W_r = W \ltimes rY$. Note that X is invariant under the action of the Weyl group. Let $\hat{W}_r$ be the group generated by W and translation by rX. Then $\hat{W}_r = W \ltimes rX$. If $\lambda \in X$ and $w \in W$, then $w(\lambda) - \lambda$ is in Y. This implies W_r is a *normal subgroup* of $\hat{W}_r$. We have an exact sequence

$$1 \longrightarrow W_r \longrightarrow \hat{W}_r \longrightarrow G \longrightarrow 1,$$

where $G = X/Y$ is the *fundamental group* of the root data. It is known that G is isomorphic to the center of the simply connected complex Lie group associated with $\mathfrak{g}$ and that $|G| = \det(a_{ij})$. The group G for various Lie algebras is listed in Table 1.

Taking the action of $\hat{W}_r$ modulo the action of W_r, we get an action of G on $\bar{C}_r$ that can be described explicitly as follows. Suppose $\tilde{g} \in X$ is a lift of $g \in G = X/Y$. There is a unique $w \in W_r$ such that $w(\bar{C}_r + r\tilde{g}) = \bar{C}_r$. Then, for $\mu \in \bar{C}_r$,

$$g(\mu) = w\big(\mu + r\tilde{g}\big) \in \bar{C}_r.$$

For $g \in D$ and $\mu \in X$, we define a scalar product $(g \mid \mu) := (\tilde{g} \mid \mu)$, which is well defined as an element in $(1/D)\mathbb{Z}/\mathbb{Z}$. Similarly, for $g_1, g_2 \in G$, let $(g_1 \mid g_2) = (\tilde{g}_1 \mid \tilde{g}_2) \in (1/D)\mathbb{Z}/\mathbb{Z}$.

2.3.2. Examples of G and its action. Here we give examples of the cases A_1, A_2, and B_2. When $\mathfrak{g} = \mathrm{sl}_2$ (the A_1 case), the space $\mathfrak{h}^*_{\mathbb{R}}$ is one-dimensional. The basis root and the weight are depicted in Figure 2(a). The simplex $\bar{C}_r$ is the interval $[0, r\lambda_1]$. The nontrivial element of $G = \mathbb{Z}_2$ acts as the reflection about the midpoint $r\lambda_1/2$; see Figure 2(b).

When $\mathfrak{g} = \mathrm{sl}_3$ (the A_2 case), the space $\mathfrak{h}^*_{\mathbb{R}}$ is two-dimensional and $d = 1$. The basis roots and weights are depicted in Figure 3(a). The simplex $\bar{C}_r$ is the equilateral

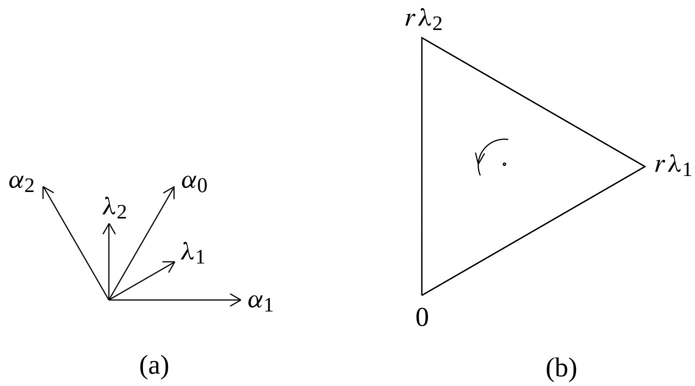


FIGURE 3. The A_2 case

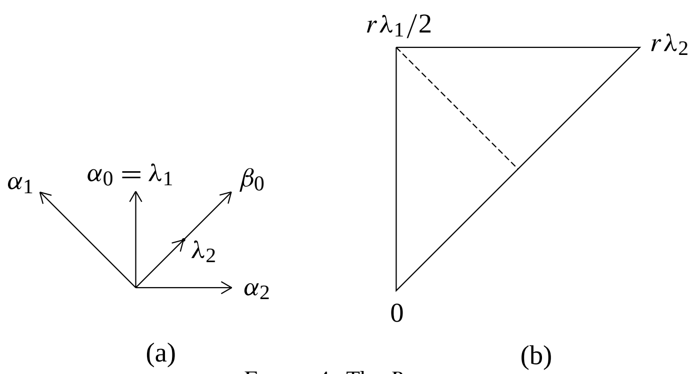


FIGURE 4. The B_2 case

triangle with vertices at $0, r\lambda_1$, and $r\lambda_2$. The group $G = \mathbb{Z}_3$ acts as rotations by $2\pi i k/3$ about the center point; see Figure 3(b).

When $\mathfrak{g}$ is of B_2 type, the space $\mathfrak{h}^*_{\mathbb{R}}$ is again two-dimensional, with $d_1 = 2$, $d_2 = 1$. The basis roots and weights are depicted in Figure 4(a). The simplex $\bar{C}_r$ is the triangle with vertices at 0, $r\lambda_1/2$, and $r\lambda_2$. The nontrivial element of $G = \mathbb{Z}_2$ acts as the reflection about the dashed line that separates $\bar{C}_r$ into two equal halves; see Figure 4(b).

2.3.3. Second symmetry principle

THEOREM 2.6. *Suppose $\mu_1, \dots, \mu_m \in \bar{C}_r$ and $g_1, \dots, g_m \in G$. Then*

$$Q_L\big(g_1(\mu_1), \dots, g_m(\mu_m)\big) = v^{rt} Q_L(\mu_1, \dots, \mu_m), \tag{2.3}$$

at primitive rth roots of unity. Here t depends only on the linking matrix (l_{ij}) of L:

$$t = (r-h) \sum_{1 \le i, j \le m} l_{ij}\big(g_i \mid g_j\big) + 2 \sum_{1 \le i, j \le m} l_{ij}\big(g_i \mid \mu_j - \rho\big),$$

with h being the Coxeter number of the Lie algebra $\mathfrak{g}$ (see Table 1).

Remark. The factor $v^{rt} = q^{rt/2}$ makes both sides of (2.3) belong to the same $q^a\mathbb{Z}[q^{\pm 1}]$.

For the special cases $\mathfrak{g} = \mathrm{sl}_2$ and $\mathfrak{g} = \mathrm{sl}_n$, the theorem was proved by Kirby and Melvin [KM] and Kohno and Takata [KT1]. In [KM] and [KT1], the *twisting factor* v^{rt} is derived by direct computations. Here we express the twisting factor through the scalar product in $\mathfrak{h}^*_{\mathbb{R}}$. We also have the result for every primitive rth root of unity.

Since $\hat{W}_r = W \ltimes rX$ and since Q_L is componentwise invariant under W, (2.3) is equivalent to the statement that for $x_1, \ldots, x_m \in X$,

$$\begin{aligned} &Q_L(\mu_1 + rx_1, \ldots, \mu_m + rx_m) \\ &\qquad \stackrel{(r)}{=} v^{r[(r-h)\sum l_{ij}(x_i|x_j) + 2\sum l_{ij}(x_i|\mu_j - \rho)]} Q_L(\mu_1, \ldots, \mu_m). \end{aligned} \tag{2.4}$$

COROLLARY 2.7. *Suppose L has zero linking matrix. Then $Q_L(\mu_1, \ldots, \mu_m)$, at primitive rth roots of unity, is invariant under componentwise action of $\hat{W}_r$.*

COROLLARY 2.8. *If $\mu_j - \rho$ is in the root lattice and $\mu_j \in \bar{C}_r$, then*

$$Q_L\big(g_1(\mu_1), \ldots, g_m(\mu_m)\big) \stackrel{(r)}{=} v^{r[\sum l_{ij}(g_i|g_j)]} Q_L(\mu_1, \ldots, \mu_m).$$

The corollary follows from Theorem 2.6, since the second term in the expression of t in the theorem is in $2\mathbb{Z}$. These corollaries have application in the study of quantum 3-manifold invariants. In a subsequent paper, we will use Corollary 2.8 to define the projective version of quantum invariants and its perturbative expansion.

2.4. Refined versions of symmetry principles. When $(r, d) \neq 1$, we can strengthen both symmetry principles. Actually, we formulate the result so that it includes the $(d, r) = 1$ case. In the refined versions, the symmetry groups are larger, actually, the largest possible. For example, if one wants to construct quantum invariants of 3-manifolds, one has to use the refined versions because of the nondegeneracy property in Proposition 2.10 below.

2.4.1. Refined versions of $C_r, W_r, \hat{W}_r$. If $(r, d) = 1$, let $X' = X$ and $Y' = Y$. If $(r, d) \neq 1$, then $(r, d) = d > 1$. In this case, let X' (resp., Y') be the $\mathbb{Z}$-lattice generated by λ_i/d_i (resp., α_i/d_i), $i = 1, \ldots, \ell$. In other words, if $(r, d) \neq 1$, then X' is the lattice dual to Y with respect to our scalar product, that is, $X' = \{x \in \mathfrak{h}^*_{\mathbb{R}} \mid (x \mid y) \in \mathbb{Z},$ for every $y \in Y\}$, and similarly, Y' is the lattice dual to X. If $(d, r) \neq 1$, then X', Y' is a realization of the root lattice and the weight lattice of the dual root system whose Cartan matrix is the transpose of the original one.

In any case, $X \subset X'$, $Y \subset Y'$. Note that X' and Y' are invariant under W, and if $x \in X'$, then $x - w(x) \in Y'$ for every $w \in W$. Let us define

$$W'_r = W \ltimes rY', \qquad \hat{W}'_r = W \ltimes rX'.$$

Then W'_r is a normal subgroup of $\hat{W}'_r$, and we have an exact sequence

$$1 \longrightarrow W'_r \longrightarrow \hat{W}'_r \longrightarrow G' \longrightarrow 1, \tag{2.5}$$

where $G' = X'/Y'$. Note that G' is always isomorphic to G.

The lattices X', Y' thus depend on the root data (of the Lie algebra $\mathfrak{g}$) and whether $(r, d) = 1$ or not. There is a unifying definition good for *every* r, as described in the following lemma, which is easy to prove.

LEMMA 2.9. *One has that*

$$rX' = \{x \in X \mid (x \mid y) \in r\mathbb{Z} \ \text{ for every } y \in Y\},$$
$$rY' = \{y \in Y \mid (x \mid y) \in r\mathbb{Z} \ \text{ for every } x \in X\}.$$

2.4.2. The fundamental domain of W'_r. If $(r, d) = 1$, let $C'_r = C_r$. Otherwise, let

$$C'_r = \{x \in C \mid (x \mid \beta_0) < r\},$$

where β_0 is the long highest root. The closure $\bar{C}'_r$ is a simplex and a fundamental domain of W'_r. In any case $C'_r \subset C_r$, and if $(d, r) \neq 1$, then C'_r is strictly less than C_r. In fact, if $(r, d) \neq 1$, then it can be shown that the volume of C_r is $\prod_{i=1}^{\ell}(d/d_i)$ times that of C'_r. One important property of C'_r is the following nondegeneracy property, proved in [AP].

PROPOSITION 2.10 (see [AP]). *Suppose $\mu \in \bar{C}'_r$ and ε^2 is a primitive rth root of unity. Then the quantum dimension $J_U(\mu)|_{v=\varepsilon} = 0$ if and only if μ is on the boundary of $\bar{C}'_r$.*

Proof. This follows from the explicit formula of the quantum dimension (2.1). □

This proposition shows that the set C'_r (but not C_r in general) is exactly what should be used in the construction of the topological quantum field theory.

2.4.3. Actions of G' and the quadratic form on G'. Again, the exact sequence (2.5) leads to an action of G' on $\bar{C}'_r$. Explicitly it can be described as follows. Suppose $\tilde{g} \in X$ is a lift of $g \in G' = X'/Y'$. There is a unique $w \in W'_r$ such that $w(\bar{C}'_r + r\tilde{g}) = \bar{C}'_r$. Then $g(\mu) := w(\mu + r\tilde{g})$ for every $\mu \in \bar{C}'_r$.

For $g \in G'$ and $\mu \in X$ (not X' here), let us define the product $(g \mid \mu) := (\tilde{g} \mid \mu)$, which is well defined as an element in $\mathbb{Q}$ modulo $\mathbb{Z}$. If ζ is a $2Dr$th root of unity, then $\zeta^{2Dr(g|\mu)}$ is well defined as a complex number.

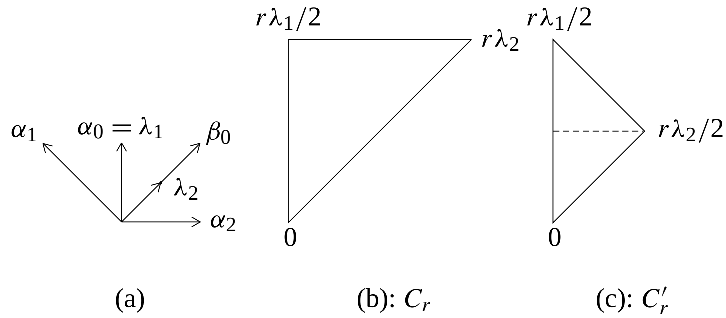


FIGURE 5. The B_2 case

Similarly, for $g_1, g_2 \in G'$, let $(g_1 \mid g_2) = (\tilde{g}_1 \mid \tilde{g}_2)$, defined as an element in $\mathbb{Q}$ modulo $(1/(d,r))\mathbb{Z}$. If ζ is a $2Dr$th root of unity and k an integer divisible by (d,r), then $\zeta^{2Drk(g_1|g_2)}$ is well defined as a complex number. A little bit more difficult to see is that $\zeta^{Dkr(g|g)}$ is well defined as a complex number for every $g \in G'$. But this follows from the evenness of the quadratic form $(\cdot \mid \cdot)$ on the root lattice.

When $(r,d) = 1$, we could drop all the primes. Note that all the "prime" versions, such as $X', Y', C'_r, W'_r, \ldots,$ depend on the root data and whether $(r,d) = 1$ or not. The group G' is isomorphic to G; however, the scalar product on G' depends on whether $(r,d) = 1$ or not.

Let us consider an example when $(r,d) \neq 1$. Then $d = 2$ or 3. If we want G' to not be trivial, then we are left with only two cases: the B_ℓ or C_ℓ series of Lie algebras. Let us consider the case of B_2 (see Figure 5). In this case C'_r is a half of C_r such that C'_r is the triangle with vertices at $0, r\lambda_1/2, r\lambda_2/2$. The nontrivial element of $G' = Z_2$ acts as the reflection about the dashed line in Figure 5(c).

2.4.4. Refined symmetry principles. Both symmetry principles remain valid if we replace W_r, C_r by W'_r, C'_r. However, one has to take care of the Coxeter numbers.

THEOREM 2.11 (Refined first symmetry principle). *At primitive rth roots of unity, Q_L is componentwise invariant under actions of W'_r: For $\mu_1, \ldots, \mu_m \in X$, $w_1, \ldots, w_m \in W'_r$,*

$$Q_L\big(w_1(\mu_1), \ldots, w_m(\mu_m)\big) \overset{(r)}{=} Q_L(\mu_1, \ldots, \mu_m). \tag{2.6}$$

If one of the μ_j is on the boundary of $\bar{C}'_r$, then $J_L(\mu_1, \ldots, \mu_m) \overset{(r)}{=} 0$.

Since $W_r \subset W'_r$, the refined version implies the nonrefined version. We also prove that J_L is componentwise *skew-invariant* under W'_r: For $w_1, \ldots, w_m \in W'_r$,

$$J_L\big(w_1(\mu_1), \ldots, w_m(\mu_m)\big) \overset{(r)}{=} \mathrm{sn}(w_1) \cdots \mathrm{sn}(w_m) J_L(\mu_1, \ldots, \mu_m). \tag{2.7}$$

Certainly this identity implies (2.6). It also implies the second statement of the theorem: If one of $\mu_1, \dots, \mu_m$, say, μ_1, is on the boundary of $\bar{C}'_r$, then there is reflection along a facet of $\bar{C}'_r$ that fixes μ_1. The sign of any reflection is -1. Hence, $J_L(\mu_1, \dots, \mu_m) \overset{(r)}{=} 0$. Another way to prove the second statement is the following. By (1.8), $J_L = \tilde{J}_T J_U(\mu_1)$. If μ_1 is on the boundary of $\bar{C}'_r$, then by Lemma 2.10, $J_U(\mu_1) \overset{(r)}{=} 0$. Hence $J_L \overset{(r)}{=} 0$.

Again, because of the first symmetry, it is enough to restrict the colors to $\bar{C}'_r$ when considering quantum invariants. Let $h^\vee$ be the dual Coxeter number of $\mathfrak{g}$ (see Table 1).

THEOREM 2.12 (Refined second symmetry principle). *Suppose $\mu_1, \dots, \mu_m \in \bar{C}'_r$ and $g_1, \dots, g_m \in G'$. Then, at primitive rth roots of unity,*

$$Q_L\big(g_1(\mu_1), \dots, g_m(\mu_m)\big) = v^{rt'} Q_L(\mu_1, \dots, \mu_m), \tag{2.8}$$

where t' is determined by the linking matrix of L:

$$t' = (r - h') \sum_{i,j} l_{ij}(g_i \mid g_j) + 2 \sum_{i,j} l_{ij}(g_i \mid \mu_j - \rho),$$

with

$$h' = \begin{cases} h & \textit{if } (d, r) = 1, \\ dh^\vee & \textit{if } (d, r) \neq 1. \end{cases}$$

Note that $v^{rt'}$ in (2.8) is well defined as a complex number; see §2.4.3. Again, the factor $v^{rt'} = q^{rt'/2}$ makes both sides of (2.8) belong to the same $q^a \mathbb{Z}[q^{\pm 1}]$.

Since the action of G' is obtained from the action of the extended affine Weyl group $\hat{W}'_r$, let us describe how Q_L behaves under the action of $\hat{W}'_r$. Recall that $\hat{W}'_r = W \ltimes rX'$ and that Q_L is componentwise invariant under W. We need only to describe how Q_L behaves under the translation group rX'. Suppose $x_1, \dots, x_m \in X'$; then,

$$\begin{aligned} &Q_L(\mu_1 + rx_1, \dots, \mu_m + rx_m) \\ &\qquad \overset{(r)}{=} Q_L(\mu_1, \dots, \mu_m) v^{r[(r-h')\sum l_{ij}(x_i|x_j) + 2\sum l_{ij}(x_i|\mu_j - \rho)]}. \end{aligned} \tag{2.9}$$

The theorem certainly follows from this statement.

3. Proofs

3.1. Quantum groups at roots of unity. We recall the theory of quantum groups at roots of unity, following [An], [AP], and [Lu2] and then prove some auxiliary facts.

3.1.1. Quantum group at roots of unity and its category of modules. Suppose $\varepsilon \in \mathbb{C}$ is a number such that ε^2 is an rth primitive root of unity. Then ε is either a primitive $2r$th root of unity or a primitive rth root of unity. The latter can happen only when r is odd. Fix a number ζ such that $\zeta^D = \varepsilon$. If $a \in (1/D)\mathbb{Z}$, then by ε^a we mean ζ^{Da}.

Let ${}_\varepsilon\mathfrak{U}$ be the algebra ${}_\mathcal{A}\mathfrak{U}\otimes_\mathcal{A}\mathbb{C}$, where $\mathbb{C}$ is considered as an $\mathcal{A}$-algebra by mapping v to ε. Then ${}_\varepsilon\mathfrak{U}$ is a Hopf $\mathbb{C}$-algebra, called a quantum group at a root of unity (Lusztig's version). The Cartan subalgebra of ${}_\mathcal{A}\mathfrak{U}$ is not generated (over $\mathcal{A}$) by the K_α alone; one needs the elements

$$\begin{bmatrix} K_{\alpha_i} \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{\left(v^{1-s}K_{\alpha_i}\right)^{d_i} - \left(v^{1-s}K_{\alpha_i}\right)^{-d_i}}{v^{sd_i} - v^{-sd_i}} \in {}_\mathcal{A}\mathfrak{U},$$

where $i = 1, \dots, \ell$ and $t = 1, 2, 3 \dots.$

For a ${}_\varepsilon\mathfrak{U}$-module M and a weight $\nu \in X$, let

$$M^\nu = \left\{ x \in M \mid K_\alpha(x) = \varepsilon^{\langle \nu,\alpha\rangle} x \text{ and } \begin{bmatrix} K_{\alpha_i} \\ t \end{bmatrix}(x) = \begin{bmatrix} \langle \nu, \alpha_i\rangle \\ t \end{bmatrix}_i x \right\},$$

where for $x \in \mathbb{Z}$, $t \in \mathbb{Z}$, $t > 0$,

$$\begin{bmatrix} x \\ t \end{bmatrix}_i := \prod_{s=1}^{t} \frac{\varepsilon^{d_i(x-s+1)} - \varepsilon^{-d_i(x-s+1)}}{\varepsilon^{sd_i} - \varepsilon^{-sd_i}}.$$

Let ${}_\varepsilon\mathcal{C}$ be the category of finite-dimensional ${}_\varepsilon\mathfrak{U}$-modules M such that

$$M = \oplus_{\nu\in X} M^\nu$$

and that $E_i^{(p)}(x) = F_i^{(p)}(x) = 0$ on M for sufficiently large p.

Using the same formulas as in the case over $\mathbb{Q}(v)$, we define dual modules and the evaluation and coevaluation maps. We define the twist ${}_\varepsilon\theta$ and the braiding ${}_\varepsilon c$ using the same formulas of θ and c, replacing $v^{1/D}$ by ζ. Then $({}_\varepsilon\mathcal{C}, {}_\varepsilon\theta, {}_\varepsilon c)$ is a ribbon category. In particular,

$$ {}_\varepsilon\theta(M,N)\left[{}_\varepsilon\theta(M) \otimes {}_\varepsilon\theta(N)\right]^{-1} = {}_\varepsilon c(N,M) \times {}_\varepsilon c(M,N). \tag{3.1}$$

3.1.2. Simple modules. In general, ${}_\varepsilon\mathcal{C}$ is not semisimple: there are modules in ${}_\varepsilon\mathcal{C}$ that are indecomposable, but not simple.

Since ${}_\mathcal{A}\Lambda_\lambda$ is invariant under ${}_\mathcal{A}\mathfrak{U}$, there is defined ${}_\varepsilon\Lambda_\lambda =_\mathcal{A} \Lambda_\lambda \otimes \mathbb{C}$, which is a ${}_\varepsilon\mathfrak{U}$-module in ${}_\varepsilon\mathcal{C}$. Here λ is in X_+. Since ε is a root of unity, ${}_\varepsilon\Lambda_\lambda$ may not be simple. But ${}_\varepsilon\Lambda_\lambda$ always has a unique quotient L_λ that is a *simple* ${}_\varepsilon\mathfrak{U}$-module. This ${}_\varepsilon\mathfrak{U}$-module $L_\lambda \in {}_\varepsilon\mathcal{C}$ is also of highest weight λ. If $\lambda \neq \mu$, then L_λ is not isomorphic to L_ν. If $\lambda \in C'_r$, then ${}_\varepsilon\Lambda_\lambda$ is a simple ${}_\varepsilon\mathfrak{U}$-module, that is, $L_\lambda = {}_\varepsilon\Lambda_\lambda$.

3.1.3. Composition factors and the twist ${}_\varepsilon\theta$. In general, if $M \in {}_\varepsilon\mathcal{C}$, then M may not be a direct sum of simple modules. However, there is a decreasing sequence of submodules $M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$ such that M_i/M_{i+1} is simple. The quotient M_i/M_{i+1} is called a *composition* factor of M.

In [AP], as a corollary of the linkage principle, it was proved that for every $M \in {}_\varepsilon\mathscr{C}$,

$$M = \bigoplus_{\mu \in C'_r} M_{[\mu]}. \tag{3.2}$$

Here $M_{[\mu]}$ is the maximal submodule of M such that each composition factor of it is isomorphic to L_ν with ν in the W'_r-orbit of μ under the dot action, that is, $\nu = w(\mu+\rho)-\rho$ for some $w \in W'_r$.

LEMMA 3.1. *If ν is in the W'_r-orbit of μ (under the dot action), then*

$$\varepsilon^{(\mu+2\rho|\mu)} = \varepsilon^{(\nu+2\rho|\lambda)}.$$

Proof. Suppose $\nu = w(\mu+\rho)-\rho$. Using the fact that $\varepsilon^{1/D}$ is a $2rD$th root of unity, it is easy to check the statement for the case when w is in W and the case when w is a translation by a vector in rY'. □

Note that the twist ${}_\varepsilon\theta$ acts as the scalar $\varepsilon^{(\mu+2\rho|\mu)}$ on ${}_\varepsilon\Lambda_\mu$ (see §1.3.3); hence, it acts as the same scalar on any composition factor of ${}_\varepsilon\Lambda_\mu$. Thus we get the following.

PROPOSITION 3.2. *In the above notation, the twist ${}_\varepsilon\theta$ acts as scalar $\varepsilon^{(\mu+2\rho|\mu)}$ on $M_{[\mu]}$.*

3.1.4. Quantum link invariants. Recall that ${}_\varepsilon\mathscr{C}$ is a ribbon category. Thus if L is a framed link, then there is defined the invariant ${}_\varepsilon J_L(M_1,\dots,M_m) \in \mathbb{C}$, where $M_1,\dots,M_m$ are in ${}_\varepsilon\mathscr{C}$. Although we use the notation with ε, it is understood that ${}_\varepsilon J_T$ depends on the choice of ζ, a Dth root of ε, since the twist and the braiding do. However, this is not essential, since one can always get rid of fractional powers of v by a suitable normalization.

Obviously when $M_j = {}_\varepsilon\Lambda_{\mu_j}$, then

$${}_\varepsilon J_L\big({}_\varepsilon\Lambda_{\mu_1},\dots,{}_\varepsilon\Lambda_{\mu_m}\big) = J_L\big(\Lambda_{\mu_1},\dots,\Lambda_{\mu_m}\big)\big|_{v^{1/D}=\zeta},$$

where the right-hand side means the value of $J_L(\Lambda_{\mu_1},\dots,\Lambda_{\mu_m})$ at $v^{1/D}=\zeta$.

Many modules are not direct sums of ${}_\varepsilon\Lambda_\lambda$. We can also define invariants of framed links colored by these modules. The presence of these modules helps us to relate values of quantum invariants at various μ.

The simplest argument goes as follows. Suppose T is a framed $(1,1)$-tangle whose open component is colored by Λ_λ. Then $J_T(\Lambda_\lambda)$ is a scalar operator from Λ_λ to Λ_λ,

$$J_T(\Lambda_\lambda) = \tilde{J}_T(\Lambda_\lambda)\,\mathrm{id}, \quad \text{with } \tilde{J}_T(\Lambda_\lambda) \in \mathbb{Z}\big[v^{\pm 1/D}\big].$$

Hence, when specialized at $v^{1/D}=\zeta$, the map ${}_\varepsilon J_T({}_\varepsilon\Lambda_\lambda) : {}_\varepsilon\Lambda_\lambda \to {}_\varepsilon\Lambda_\lambda$ is also a scalar operator (although ${}_\varepsilon\Lambda_\lambda$ may not be irreducible):

$${}_\varepsilon J_T\big({}_\varepsilon\Lambda_\lambda\big) = {}_\varepsilon\tilde{J}_T\big({}_\varepsilon\Lambda_\lambda\big)\,\mathrm{id}.$$

Here ${}_\varepsilon\tilde{J}_T({}_\varepsilon\Lambda_\lambda)$ is a complex number obtained from $\tilde{J}_L(\Lambda_\lambda)$ by putting $v^{1/D} = \zeta$.

It follows that if M is a composition factor of ${}_\varepsilon\Lambda_\lambda$, then ${}_\varepsilon J_T(M) : M \to M$ is also a scalar operator with the *same* scalar ${}_\varepsilon\tilde{J}_T({}_\varepsilon\Lambda_\lambda)$. In particular,

$$ {}_\varepsilon\tilde{J}_T\big(L_\lambda\big) = {}_\varepsilon\tilde{J}_T\big({}_\varepsilon\Lambda_\lambda\big). \tag{3.3} $$

PROPOSITION 3.3. *If ${}_\varepsilon\Lambda_\lambda$ and ${}_\varepsilon\Lambda_\mu$ have a common composition factor and if T is a $(1,1)$-tangle, then*

$$ {}_\varepsilon\tilde{J}_T\big({}_\varepsilon\Lambda_\lambda\big) = {}_\varepsilon\tilde{J}_T\big({}_\varepsilon\Lambda_\mu\big). $$

3.2. Lemmas on quantum dimensions and signs

LEMMA 3.4. *Recall that $h' = h$ if $(d,r) = 1$ and $h' = dh^\vee$ if $(d,h) \neq 1$.*
(a) *For every $x_1, x_2 \in X'$, the number $h'(x_1 \mid x_2)$ is an integer.*
(b) *For every $x \in X'$, one has*

$$ (2\rho \mid x) \equiv h'(x \mid x) \pmod 2 \qquad \textit{and} \qquad (2\rho \mid x) \equiv dh^\vee(x \mid x) \pmod 2. $$

(c) *For every $x_1, x_2 \in X$ (not X'), one has*

$$ (h' - h)(x_1 \mid x_2) \in \mathbb{Z}, \qquad \textit{and} \qquad (h' - h)(x_1 \mid x_1) \in 2\mathbb{Z}. $$

Proof. (a) Suppose $(d,r) = 1$. Then $X' = X$, and $(x_1 \mid x_2) \in (1/D)\mathbb{Z}$. The values of $h' = h$ and D in Table 1 show that h is divisible by D. Hence $h'(x_1 \mid x_2) \in \mathbb{Z}$.

Now suppose $(d,r) \neq 1$; then $(d,r) = d > 1$. This means $\mathfrak{g}$ is of type B, C, F_4, or G_2. Note that X' is the $\mathbb{Z}$-lattice generated by λ_i/d_i, and $(\lambda_i/d_i \mid \lambda_j/d_j) = (1/d_i)(A^{-1})_{ij}$, where A^{-1} is the inverse of the Cartan matrix. Explicit calculation shows that $(\lambda_i/d_i \mid \lambda_j/d_j) \in (1/D')\mathbb{Z}$, where $D' = 2$ for B_ℓ, F_2, G_2, and C_ℓ with ℓ even, and $D' = 4$ for C_ℓ with ℓ odd. In any case, D' divides $dh^\vee$, and hence $dh^\vee/D' \in \mathbb{Z}$.

(b) Note that if the statement is true for $x = x_1$ and $x = x_2$, then it is true for $x = x_1 + x_2$. Hence, it is enough to restrict oneself to the case when x is in a basis set. If $(d,r) = 1$, a basis set is $\{\lambda_i, i = 1, \dots, \ell\}$; if $(d,r) \neq 1$, a basis set is $\{\lambda_i/d_i, i = 1, \dots, \ell\}$. One can easily check the statement for each simple Lie algebra.

(c) If $(r,d) = 1$, then $h' = h$ and both statements are trivial.

Suppose $(r,d) \neq 1$; then $h' = dh^\vee$. Again one needs only to verify the statements when x_1, x_2 is in a basis set of X, say, $x_1 = \lambda_i, x_2 = \lambda_j$. Recalling that $(\lambda_i \mid \lambda_j) = (A^{-1})_{ij} d_j$, one can easily check both statements. □

LEMMA 3.5. *Recall that U is the unknot. Let $\mu \in X$. At primitive rth roots of unity, one has*

$$ J_U(\mu + ry) = J_U(\mu) \qquad \textit{and} \qquad Q_U(\mu + ry) = Q_U(\mu) \quad \textit{for every } y \in Y', $$
$$ J_U(\mu + rx) = (-1)^{(2\rho|x)} J_U(\mu) \qquad \textit{and} \qquad Q_U(\mu + rx) = Q_U(\mu) \quad \textit{for every } x \in X', $$
$$ J_U(w(\mu)) = \operatorname{sn}(w) J_U(\mu) \qquad \textit{and} \qquad Q_U(w(\mu)) = Q_U(\mu) \quad \textit{for every } w \in W'_r. $$

For the Q_L version, the statements are much simpler, since there is no sign.

Proof. The first two identities for J_U follow from the formula (2.1), if we remember that $r(x \mid \alpha) \in r\mathbb{Z}$ for every $x \in X', \alpha \in Y$ (see Lemma 2.9). The third identity for J_U follows from the first one and the fact that J_U is skew-invariant under the action of the Weyl group W. All the identities for Q_U follow from the corresponding ones for J_U. □

3.3. Proof of refined first symmetry principle. The proof utilizes results from [AP] in the theory of quantum groups. We focus only on the first component of L. Suppose the color of this component is μ. Cut the link at a point on the component to get a $(1, 1)$-tangle T. Then by formula (1.8) we have

$$J_L(\mu) = \tilde{J}_T(\mu) J_U(\mu).$$

Since at primitive rth roots of unity we have (see Lemma 3.5)

$$J_U\big(w(\mu)\big) = \mathrm{sn}(w) J_U(\mu),$$

it is enough to show that $\tilde{J}_T(w(\mu)) = \tilde{J}_T(\mu)$ at primitive rth roots of unity. Here $w \in W_r'$.

From [AP, Section 3] and [Ja, Chapter II] we know that if $\lambda = w(\mu)$, then there is a sequence of $\mu_1, \dots, \mu_s$ such that $L_{\lambda-\rho}$ and $L_{\mu-\rho}$ are composition factors of ${}_\varepsilon\Lambda_{\mu_1}$ and ${}_\varepsilon\Lambda_{\mu_s}$, respectively, and two consecutive ${}_\varepsilon\Lambda_{\mu_j}, {}_\varepsilon\Lambda_{\mu_{j+1}}$ have a common composition factor. It follows from Proposition 3.3 that $\tilde{J}(\lambda) = \tilde{J}(\mu)$ at primitive rth roots of unity. This proves the refined first symmetry principle.

3.4. Proof of refined second symmetry principle. As argued in §2.4.4, we need to prove (2.9). We use a result of Lusztig, which we first recall.

3.4.1. A tensor product theorem. Recall that ε^2 is a primitive rth root of unity. Let

$$X_r = \left\{ x \in X_+ \mid \langle x, \alpha_i \rangle < \frac{r}{(r, d_i)}, i = 1, \dots, \ell \right\}.$$

One can check that $(C_r \cap X_+) \subset X_r$.

It is easy to check that if $\xi \in X_+$, then there exists unique $\lambda \in X_r$ and $\nu \in rX' \cap X_+$ such that $x = \lambda + \nu$. We denote $\xi^{(0)} = \lambda \in X_r$ and $\xi^{(1)} = \nu/r \in X'$. Lusztig [Lu1] proved that, as ${}_\varepsilon\mathcal{U}$-modules,

$$L_\xi \cong L_\lambda \otimes L_\nu.$$

This is quite nontrivial and very different from the classical case. It is similar to Steinberg's tensor product theorem for algebraic groups over fields of positive characteristic. In [Lu1] the proof is given only for the case $(r, d) = 1$. However, the proof can be generalized to the case $(r, d) \neq 1$ (see the arguments of [AP, Theorem 3.12]).

3.4.2. The square of the braiding on $L_\lambda \otimes L_\nu$. Assume that, as in §3.4.1, $\lambda \in X_r$ and $\nu \in (rX' \cap X_+)$. The square of the braiding ${}_\varepsilon c(L_\nu, L_\lambda)\, {}_\varepsilon c(L_\lambda, L_\nu)$ is an operator acting on $L_\lambda \otimes L_\nu$, commuting with the action of ${}_\varepsilon\mathcal{U}$. But $L_\lambda \otimes L_\nu = L_{\lambda+\nu}$ is a *simple* module. Hence, the square of the braiding is a scalar operator,

$$ {}_\varepsilon c(L_\nu, L_\lambda)\, {}_\varepsilon c(L_\lambda, L_\nu) = b_{\lambda,\nu}\, \mathrm{id}, $$

where $b_{\lambda,\nu} \in \mathbb{C}$ is a constant.

For the tangle diagrams D_3, D_4 of Figure 1 corresponding to c, c^{-1}, we have $D_3 = D_3^2 D_4^{-1}$. It follows that

$$ {}_\varepsilon J_{D_3}(L_\lambda, L_\nu) = b_{\lambda,\nu}\, {}_\varepsilon J_{D_4}(L_\lambda, L_\nu). \tag{3.4} $$

This means the operator of a positive crossing and the one of a negative crossing are proportional. The proportional factor $b_{\lambda,\nu}$ can be calculated as follows. By (3.1),

$$ ({}_\varepsilon c)^2 = {}_\varepsilon\theta(L_\lambda \otimes L_\nu)\big[{}_\varepsilon\theta(L_\lambda) \otimes {}_\varepsilon\theta(L_\nu)\big]^{-1}. $$

Using $L_\lambda \otimes L_\nu = L_{\lambda+\nu}$ and the fact that ${}_\varepsilon\theta$ acts on L_μ as the scalar $\varepsilon^{(\mu+2\rho|\mu)}$ (see §1.3.3), we see that

$$ b_{\lambda,\nu} = \varepsilon^{2(\lambda|\nu)}. \tag{3.5} $$

3.4.3. The square of the braiding on $L_\nu \otimes L_\nu$. We continue to assume that, as in the previous subsection, $\nu \in rX' \cap X_+$. We show that $({}_\varepsilon c)^2$ acts as a scalar operator on $L_\nu \otimes L_\nu$. It is enough to show that ${}_\varepsilon\theta(L_\nu \otimes L_\nu)$ is a scalar operator, since

$$ c^2 = \theta(L_\nu \otimes L_\nu)\big[\theta(L_\nu) \otimes \theta(L_\nu)\big]^{-1}. $$

The structure of the module L_ν and its tensor powers can be understood by classical Lie theory, via the quantum Frobenius map (see [Lu2, Chapter 35]). Every weight of L_ν must be of the form $\nu - \alpha$, where $\alpha \in rY'$. The tensor product $L_\nu \otimes L_\nu$ is completely reducible:

$$ L_\nu \otimes L_\nu = \oplus_\tau L_\tau, $$

where $\tau \in 2\nu - rY'$.

The twist θ acts on L_τ as a scalar operator, with the scalar $\varepsilon^{(\tau+2\rho|\tau)}$ (see Proposition 3.2). When $\tau \in 2\nu - rY'$, it is easy to see that the scalar is always equal to $\varepsilon^{(2\nu+2\rho|2\nu)}$. This means $\theta(L_\nu \otimes L_\nu)$ is a scalar operator with the scalar $\varepsilon^{(2\nu+2\rho|2\nu)}$.

So we have $c^2 = b_\nu\, \mathrm{id}$, where the value of b_ν can be calculated ($b_\nu = \varepsilon^{2(\nu|\nu)}$; we do not need this value). It follows that $J_{D_3}(L_\nu, L_\nu) = b_\nu J_{D_4}(L_\nu, L_\nu)$.

For example, suppose T is a $(1,1)$-tangle with framing zero. Let us calculate ${}_\varepsilon\tilde{J}_T(L_\nu)$. We switch over- or undercrossing at some points in a good diagram of T to get the trivial $(1,1)$-tangle. With each switching we have to multiply the quantum invariant by b_ν or $(b_\nu)^{-1}$. Since the framing is zero, we conclude that ${}_\varepsilon\tilde{J}_T(L_\nu) = 1$.

3.4.4. Reduction to the framing-zero case. We show here that if (2.9) is true for a link L, then it holds true for any link obtained from L by altering the framing of the components. It is sufficient to consider the case when we increase the framing of the first component by 1. Then the left-hand side of (2.9) is multiplied by

$$a_{\text{LHS}} = \varepsilon^{(\mu_1+rx_1+\rho|\mu_1+rx_1-\rho)}$$

and the right-hand side by

$$a_{\text{RHS}} = \varepsilon^{(\mu_1+\rho|\mu_1-\rho)}\varepsilon^{r[(r-h')(x_1|x_1)+2(x_1|\mu_1-\rho)]}.$$

Hence, to show that $a_{\text{LHS}} = a_{\text{RHS}}$ it is enough to prove that

$$1 = \varepsilon^{r[h'(x_1|x_1)+2(x_1|\rho)]},$$

which follows from Lemma 3.4. (The term in the square bracket of the exponent is divisible by 2 by Lemma 3.4.)

3.5. A special case. By virtue of the result of the previous subsection, we assume from now on that $0 = l_{11} = l_{22} = \cdots$. Suppose $\xi \in X_+$. Then $\xi = \xi^{(0)} + r\xi^{(1)}$ (see the notation in §3.4.1). In this subsection we assume that $\mu_2, \ldots, \mu_m \in X_+$. We show that

$$Q_L\big(\Lambda_\xi, \Lambda_{\mu_2}, \ldots, \Lambda_{\mu_m}\big) \overset{(r)}{=} v^{2r\kappa} Q_L\big(\Lambda_{\xi^{(0)}}, \Lambda_{\mu_2}, \ldots, \Lambda_{\mu_m}\big), \tag{3.6}$$

where

$$\kappa = \sum_j l_{1j}\big(\xi^{(1)} \mid \mu_j\big).$$

By Lemma 2.9, $r(\xi^{(1)} \mid \alpha) \in r\mathbb{Z}$ for every $\alpha \in Y$. Since $w(\mu_j) - \mu_j$ is in Y for every $w \in W'_r$, we have that $\varepsilon^{2r(\xi^{(1)}|\mu_j)} = \varepsilon^{2r(\xi^{(1)}|w(\mu_j))}$. It follows that $\varepsilon^{2r\kappa}$ is invariant under the action of W'_r. Thus using the refined first symmetry principle, we see that to prove (3.6) we can assume that $\mu_2, \ldots, \mu_m$ are in C_r. In this case ${}_\varepsilon\Lambda_{\mu_j} = L_{\mu_j}$, $j = 2, \ldots, m$. Hence, to prove (3.6) one just needs to show that

$${}_\varepsilon Q_L\big({}_\varepsilon\Lambda_\xi, L_{\mu_2}, \ldots, L_{\mu_m}\big) = \varepsilon^{2r\kappa}\,{}_\varepsilon Q_L\big({}_\varepsilon\Lambda_{\xi^{(0)}}, L_{\mu_2}, \ldots, L_{\mu_m}\big). \tag{3.7}$$

Cut L at a point on the first component to get a $(1,1)$-tangle T. From Lemma 3.5 we know that ${}_\varepsilon Q_U({}_\varepsilon\Lambda_\xi) = {}_\varepsilon Q_U({}_\varepsilon\Lambda_{\xi^{(0)}})$. Hence, by formula (1.8), identity (3.7) is equivalent to

$${}_\varepsilon\tilde{J}_T\big({}_\varepsilon\Lambda_\xi, L_{\mu_2}, \ldots, L_{\mu_m}\big) = \varepsilon^{2r\kappa}\,{}_\varepsilon\tilde{J}_T\big({}_\varepsilon\Lambda_{\xi^{(0)}}, L_{\mu_2}, \ldots, L_{\mu_m}\big). \tag{3.8}$$

Using (3.3) we can replace ${}_\varepsilon\Lambda_\xi$ and ${}_\varepsilon\Lambda_{\xi^{(0)}}$ by, respectively, L_ξ and $L_{\xi^{(0)}}$ in (3.8). The Lusztig theorem says $L_\xi = L_\nu \otimes L_{\xi^{(0)}}$, where $\nu = r\xi^{(1)}$. By the tensor product formula (1.6), we have

$${}_\varepsilon\tilde{J}_T\big(L_\xi, L_{\mu_2}, \ldots, L_{\mu_m}\big) = {}_\varepsilon\tilde{J}_{T^{(2)}}\big(L_\nu, L_{\xi^{(0)}}, L_{\mu_2}, \ldots, L_{\mu_m}\big).$$

There are two parallel push-offs of the first component of T; let us denote the one colored with L_ν by K. If we remove K, then from $T^{(2)}$ we get T.

In the tangle diagram $T^{(2)}$, consider a crossing point of K with the jth component whose color is L_{μ_j}. By formula (3.4), switching over- or undercrossing results in a factor $b_{\mu_j,\nu}$ or its inverse. Switching over- or undercrossing to unlink the component K from other components, from $T^{(2)}$ we get T'. Then we have

$$\begin{aligned} {}_\varepsilon\tilde{J}_{T^{(2)}}\big(L_\nu, L_{\xi^{(0)}}, L_{\mu_2}, \dots, L_{\mu_m}\big) &= \prod_{j=1}^{\ell} \big(b_{\mu_j,\nu}\big)^{l_{1j}} \times {}_\varepsilon\tilde{J}_{T'}\big(L_\nu, L_{\xi^{(0)}}, L_{\mu_2}, \dots, L_{\mu_m}\big) \\ &= \prod_{j=1}^{\ell} \big(b_{\mu_j,\nu}\big)^{l_{1j}} {}_\varepsilon\tilde{J}_K(L_\nu) {}_\varepsilon\tilde{J}_L\big(L_{\xi^{(0)}}, L_{\mu_2}, \dots, L_{\mu_m}\big). \end{aligned} \tag{3.9}$$

In §3.4.3 we showed that ${}_\varepsilon\tilde{J}_K(L_\nu) = 1$. Using the values of $b_{\mu,\nu}$ in (3.5), from (3.9) we get (3.8).

3.5.1. The case when $x_2 = \cdots = x_m = 0$. We prove (2.9) by assuming that $x_2 = \cdots = x_m = 0$. Recall that the framings of L are zero. In this case, (2.9) reads

$$Q_L\big(\mu_1 + r x_1, \mu_2, \dots, \mu_m\big) \overset{(r)}{=} v^{2r[\sum_j l_{1j}(x_1|\mu_j-\rho)]} Q_L\big(\mu_1, \mu_2, \dots, \mu_m\big). \tag{3.10}$$

By the refined first symmetry principle, Q_L is invariant under the translation by rY'. Hence, we can further assume that $\mu_1 + r x_1$ and all $\mu_1, \dots, \mu_m$ are in the interior of the fundamental chamber C, that is, they are in $\rho + X_+$.

Replacing μ_j by $\mu_j - \rho$, we see that (3.10) is equivalent to

$$Q_L\big(\Lambda_{\mu_1+rx_1}, \Lambda_{\mu_2}, \dots, \Lambda_{\mu_m}\big) \overset{(r)}{=} v^{2r[\sum_j l_{1j}(x_1|\mu_j)]} Q_L\big(\Lambda_{\mu_1}, \Lambda_{\mu_2}, \dots, \Lambda_{\mu_m}\big). \tag{3.11}$$

Note that $(\mu_1 + r x_1)^{(0)} = (\mu_1)^{(0)}$ and $(\mu_1 + r x_1)^{(1)} = (\mu_1)^{(1)} + x_1$. Applying formula (3.6) for $\xi = \mu_1 + r x_1$ and for $\xi = \mu_1$ and then comparing the right-hand sides of the resulting identities, we get (3.11).

3.5.2. End of proof of second symmetry principle. We continue to assume that the framings are 0, $l_{11} = \cdots = l_{mm} = 0$. The result of the previous subsection certainly holds true if we replace the first component by any component. Successively adding $r x_1$ to μ_1, $r x_2$ to μ_2, and so on, we get

$$Q_L\big(\mu_1 + r x_1, \dots, \mu_m + r x_m\big) \overset{(r)}{=} v^{2r\tau} Q_L\big(\mu_1, \dots, \mu_m\big), \tag{3.12}$$

where

$$\tau = \sum_{i,j} l_{ij}\big(x_i \mid \mu_j - \rho\big) + \sum_{i>j} l_{ij}\big(x_i \mid r x_j\big).$$

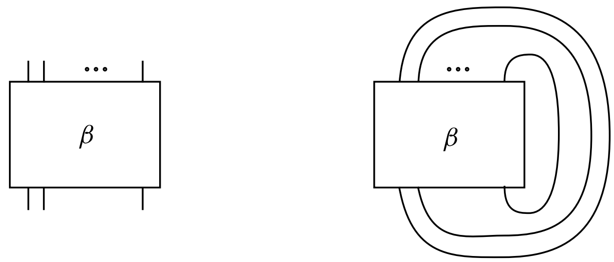


FIGURE 6. Braid closure

By Lemma 3.4, $h'(x_i \mid x_j) \in \mathbb{Z}$, and hence

$$2r\big(x_i \mid x_j\big) \equiv 2\big(r-h'\big)\big(x_i \mid x_j\big) \pmod 2,$$

and eventually

$$2r\sum_{i>j}\big(x_i \mid x_j\big) \equiv (r-h')\sum_{i,j} l_{ij}\big(x_i \mid x_j\big) \pmod 2.$$

This means that when v^2 is an rth root of unity, the second term in the formula of τ can be replaced by $(r-h')\sum_{i,j} l_{ij}(x_i \mid x_j)$, and (3.12) becomes (2.9). This completes the proof of the refined second symmetry principle.

Consider the nonrefined version. Dividing the right-hand side of (2.9) by the right-hand side of (2.4), the quotient is $v^{r(h'-h)\sum l_{ij}(x_i|x_j)}$. By Lemma 3.4(c), if all the x_i's are in X and $v^{2r}=1$, then $v^{r(h'-h)\sum l_{ij}(x_i|x_j)}=1$. Hence (2.9) implies (2.4), which, in turn, implies the nonrefined version of the second symmetry principle.

3.6. Proof of the strong integrality

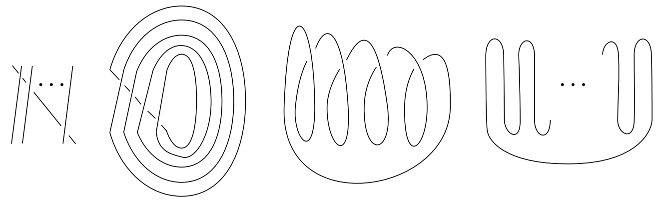
3.6.1. Presentation of links as plat closures of pure braids

PROPOSITION 3.6. *Every nonframed link has a diagram of the form $T_u \circ T \circ T_l$, where T_u and T_l do not have any crossing and T is a pure braid.*

Proof (W. Menasco). First consider the case when L has only one component, that is, L is a knot. Then L is the *braid closure* of a braid β (see Figure 6).

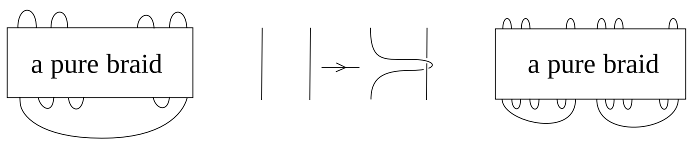
The natural projection from the braid group to the symmetric group maps β to an element with only one cycle, since L is a knot. Any two such elements are conjugate in the symmetric group. Since braids of the same conjugacy class have the same closure, we may assume that the projection of β onto the symmetric group is the permutation $(12\cdots n)$. This means, after some over- or undercrossing switchings, from β we get a braid isotopic to β' described in Figure 7(a). The isotopy can be assumed to be *horizontal*.

The closure of β' is presented in Figure 7(b); it is a trivial knot. It could be horizontally isotoped into the diagram in Figure 7(c) and, eventually, into the one in



(a) (b) (c) (d)

FIGURE 7. The trivial knot



(a) (b) (c)

FIGURE 8. The plat closure

Figure 7(d). Now from this picture we go back to L by horizontal isotopy *inside the parallel strip*, as indicated in Figure 8(a), and undo the over- or undercrossing switchings using finger moves (see Figure 8(b)). We get the desired presentation (see Figure 8(a)).

The proof for the case when L has many components is quite similar. The result, for a link of two components, is described in Figure 8(c). □

3.6.2. Quantum invariants of pure braid. Suppose T is a pure braid whose components are colored by $M_1, \ldots, M_n \in \mathscr{C}$. Then $J_T(M_1, \ldots, M_n)$ is an operator acting on the vector space $M_1 \otimes \cdots \otimes M_n$. We show here that J_T can be expressed though the twist θ alone.

If T is the square of D_1, that is, T is a full twist (see Figure 9(a)), then (see §1.3.3)

$$J_T(M_1, M_2) = \big[\theta(M_1) \otimes \theta(M_2)\big]^{-1} \theta(M_1 \otimes M_2).$$

Hence, in this case J_T can be expressed through θ alone. Similarly, if $T = D_2^2$, then J_T can be expressed through θ.

If T is the tangle in Figure 9(b), which is obtained from the one in Figure 9(a) by taking parallels, then J_T can also be expressed through θ alone, by the tensor product formula. Here the band stands for a bunch of parallel lines.

Now we claim that every pure braid can be obtained from those in Figure 9(b) and their mirror images by using composition and tensor product. In fact, the pure

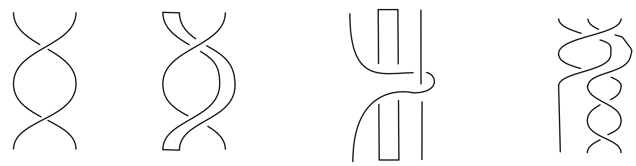


FIGURE 9. The full twist and its parallels

braids depicted in Figure 9(c) and their mirror images generate every pure braid (using composition and tensor product). But these pure braids can be expressed through the ones of Figure 9(b), as shown in Figure 9(d) for a simple case.

Hence J_T, when T is a pure braid, can be expressed through θ. Using Lemma 3.6 we see that for a link L, up to a framing factor, J_L can be expressed through θ and $K_{\pm 2\rho}$.

3.6.3. The map φ. Let $\varphi : \mathbb{Z}[v^{\pm 1/D}] \to \mathbb{Z}[v^{\pm 1/D}]$ be the algebra homomorphism defined by $\varphi(v^{1/D}) = e^{\pi i/D} v^{1/D}$. Then $\varphi(v) = -v$ and $\varphi^{2D} = \mathrm{id}$. Hence, the space $\mathbb{Z}[v^{\pm 1/D}]$ decomposes into eigenspaces of φ, whose eigenvalues are $2D$th roots of unity, $e^{a\pi i}$, with $a = 0, 1/D, \ldots, (2D-1)/D$. The eigenspaces of φ are $v^a \mathbb{Z}[v^{\pm 2}]$. Also $x \in \mathbb{Z}[v^{\pm 1/D}]$ is in the eigenspace $v^a \mathbb{Z}[v^{\pm 2}]$ if and only if $\varphi(x) = e^{a\pi i} x$.

Thus to prove the strong integrality theorem, one needs to show that

$$\varphi\big(J_L\big(\Lambda_{\mu_1}, \ldots, \Lambda_{\mu_m}\big)\big) = e^{a\pi i} J_L\big(\Lambda_{\mu_1}, \ldots, \Lambda_{\mu_m}\big), \tag{3.13}$$

with

$$a = \sum_{i,j} l_{ij}\big(\mu_i \mid \mu_j\big) + \sum_i (l_{ii}+1)(2\rho \mid \mu_i) \in \frac{1}{D}\mathbb{Z}.$$

Since both sides of (3.13) are Laurent polynomials in $v^{1/D}$, it is enough to prove (3.13) when $v^{1/D} = e^{2\pi i/2Dr}$ for every sufficiently large odd r.

3.6.4. The algebra homomorphism $\bar{\varphi}$. Let us fix an *odd* integer r. Let ε be a primitive $2r$th root of unity. Then $-\varepsilon$ is a primitive rth root of unity.

Andersen in [An] showed that there is an *algebra* homomorphism

$$\bar{\varphi} : {}_{-\varepsilon}\mathcal{U} \longrightarrow {}_{\varepsilon}\mathcal{U}$$

with the following properties. If M is a ${}_{\varepsilon}\mathcal{U}$-module, then pulling back via $\bar{\varphi}$, we get a ${}_{-\varepsilon}\mathcal{U}$-module $\bar{\varphi}^*(M)$. If M is the highest-weight module of highest weight μ, that is, $M = {}_{\varepsilon}\Lambda_\mu$, then $\bar{\varphi}^*(M)$ is also the highest-weight ${}_{-\varepsilon}\mathcal{U}$-module of highest weight μ, that is, $\bar{\varphi}^*({}_{\varepsilon}\Lambda_\mu) = {}_{-\varepsilon}\Lambda_\mu$. Note that both M and $\bar{\varphi}^*(M)$ have the same underlying vector space over $\mathbb{C}$.

In order to consider quantum invariants of links, we need to fix a Dth root of ε and a Dth root of $-\varepsilon$. Fix an arbitrary Dth root ζ of ε, and choose $\zeta' = e^{\pi i/D}\zeta = \varphi(v)|_{v^{1/D}=\zeta}$ as the Dth root of $-\varepsilon$. Now we can define ${}_{\varepsilon}\theta, {}_{\varepsilon}c, {}_{-\varepsilon}\theta, {}_{-\varepsilon}c$.

In general, $\bar{\varphi}$ does not commute with the coproduct. If M and N are two ${}_{\varepsilon}\mathcal{U}$-modules, there may be two different ${}_{-\varepsilon}\mathcal{U}$-module structures on $M \otimes_{\mathbb{C}} N$, one via the coproduct of ${}_{-\varepsilon}\mathcal{U}$ (the usual one) and one via $\bar{\varphi}$ and the coproduct of ${}_{\varepsilon}\mathcal{U}$. However, we have the following.

LEMMA 3.7. *Let $M_j = {}_{\varepsilon}\Lambda_{\mu_j}$, $j = 1, \dots, n$. The space $M_1 \otimes \cdots \otimes M_n$ has two ${}_{-\varepsilon}\mathcal{U}$-module structures as described above. Then the twist ${}_{-\varepsilon}\theta$ acts the same way in the two different module structures. Similarly, every $K_\beta, \beta \in Y$ acts the same way in the two different module structures.*

Proof. The statement for K_β follows from the fact that for K_β, the map $\bar{\varphi}$ commutes with Δ, $\bar{\varphi}(\Delta(K_\beta)) = \Delta(\bar{\varphi}(K_\beta))$, which, in turns, follows from the definition of $\bar{\varphi} : \bar{\varphi}(K_\beta) = K_\beta^{l+1}$ (see [An]).

The statement for θ follows from the fact that the action of θ is totally determined by the highest weight (see Proposition 3.2). One needs to decompose $M_1 \otimes \cdots \otimes M_n$ using (3.2) and applying Proposition 3.2. □

3.6.5. The action of the twist. Again let $M_j = {}_{\varepsilon}\Lambda_{\mu_j}$, $j = 1, \dots, n$. There are two actions of ${}_{-\varepsilon}\mathcal{U}$ on $M_1 \otimes \cdots \otimes M_n$. By the result of the previous subsection, the twist ${}_{-\varepsilon}\theta$ acts the same way in the two structures. On this same vector space, $M_1 \otimes \cdots \otimes M_n$ acts the twist ${}_{\varepsilon}\theta$ of ${}_{\varepsilon}\mathcal{U}$.

PROPOSITION 3.8. *Let $M_j = {}_{\varepsilon}\Lambda_{\mu_j}$. On $M_1 \otimes \cdots \otimes M_n$, the two operators ${}_{-\varepsilon}\theta$ and ${}_{\varepsilon}\theta$ are proportional:*

$$ {}_{-\varepsilon}\theta = e^{\pi i(\mu_1 + \cdots + \mu_n + 2\rho | \mu_1 + \cdots + \mu_n)}\, {}_{\varepsilon}\theta. $$

Proof. As ${}_{\varepsilon}\mathcal{U}$-modules, one has (see (3.2))

$$ M_1 \otimes \cdots \otimes M_n = \sum_{\nu \in C_r'} M_{[\nu]}. $$

On $M_{[\nu]}$, ${}_{\varepsilon}\theta$ acts as the scalar $\zeta^{D(\nu+2\rho|\nu)}$ (see Proposition 3.2). On that same subspace $M_{[\nu]}$, ${}_{-\varepsilon}\theta$ acts (through $\bar{\varphi}$) as the scalar $(\zeta')^{D(\nu+2\rho|\nu)} = e^{\pi i(\nu+2\rho|\nu)}\zeta^{D(\nu+2\rho|\nu)}$. Hence on $M_{[\nu]}$,

$$ {}_{-\varepsilon}\theta = e^{\pi i(\nu+2\rho|\nu)}\, {}_{\varepsilon}\theta. \tag{3.14} $$

Note that $\mu_1 + \cdots + \mu_n - \nu$ is in the root lattice. Using the fact that the scalar product of a vector in the root lattice and a vector in the weight lattice is always in $\mathbb{Z}$, one can easily show that

$$ (\nu + 2\rho \mid \nu) \equiv \big(\mu_1 + \cdots + \mu_n + 2\rho \mid \mu_1 + \cdots + \mu_n\big) \pmod{2\mathbb{Z}}. $$

It follows that the proportional factor $e^{\pi i(\nu+2\rho|\nu)}$ in (3.14) does not depend on ν and is always equal to $e^{\pi i(\mu_1+\cdots+\mu_n+2\rho|\mu_1+\cdots+\mu_n)}$. This proves the proposition. □

3.6.6. Pure braid. Suppose that a framed tangle T has a good diagram that is a *pure braid* on n strands, and suppose that the strands are colored by M_j, which are ${}_\varepsilon\mathcal{U}$-modules. Then ${}_\varepsilon J_T(M_1,\ldots,M_n)$ is an operator acting on $M_1\otimes\cdots\otimes M_n$.

Using $\bar\varphi$, one can consider T to be colored by ${}_{-\varepsilon}\mathcal{U}$-modules $\bar\varphi^*(M_j)$. Hence, there is defined the operator ${}_{-\varepsilon}J_T$ acting on the same space $M_1\otimes\cdots\otimes M_n$.

Proposition 3.9. *Suppose t_{ij} is the linking number of the ith and the jth components of the pure braid T. Suppose also that $M_j = {}_\varepsilon\Lambda_{\mu_j}$. Then on $M_1\otimes\cdots\otimes M_n$ the two operators ${}_{-\varepsilon}J_T$ and ${}_\varepsilon J_T$ are proportional:*

$$ {}_{-\varepsilon}J_T = e^{b\pi i}\,{}_\varepsilon J_T, $$

where $b=\sum_{1\le i<j\le n}2t_{ij}(\mu_i\mid\mu_j)$.

Proof. Since the pure braids in Figure 9(b) generate every pure braid, we assume T is as in Figure 9(b). We suppose that the band has $n-1$ parallel lines whose colors are $M_1,\ldots,M_{n-1}$, and the remaining line has color M_n. Then by the tensor product formula (1.6),

$$ {}_{\pm\varepsilon}J_T(M_1,\ldots,M_{n-1},M_n) = \big[{}_{\pm\varepsilon}\theta(M_1\otimes\cdots\otimes M_{n-1})\otimes{}_{\pm\varepsilon}\theta(M_n)\big]^{-1}\,{}_{\pm\varepsilon}\theta(M_1\otimes\cdots\otimes M_{n-1}\otimes M_n). $$

Using the relation between ${}_\varepsilon\theta$ and ${}_{-\varepsilon}\theta$ in Proposition 3.8, we get the desired result. □

3.6.7. End of proof of the strong integrality theorem. As noted in §3.6.3, we need to prove (3.13). Using the framing formula (1.7), it is easy to check that if (3.13) holds true for a framed link L, then it does for every framed link obtained from L by altering the framing. Hence, we may assume L has any framing we wish.

By Lemma 3.6, we can assume that L, as a nonframed link, has a diagram $D = T_u\circ T\circ T_l$, where T is a pure braid diagram, and T_u and T_l do not have any crossing points. Alter the framing of L so that D is a good diagram of it. Then the framing l_{jj} of the jth component is always even, since T is a pure braid.

We know that the operators ${}_{-\varepsilon}J_T$ and ${}_\varepsilon J_T$ are proportional. Now we prove that ${}_\varepsilon J_{T_u}$ and ${}_\varepsilon J_{T_l}$ are proportional to ${}_{-\varepsilon}J_{T_u}$ and ${}_{-\varepsilon}J_{T_l}$, respectively. Let us consider a diagram corresponding to a maximal or a minimal point. The corresponding operator involves only $\tilde K_{\pm 2\rho}$. Suppose the component is colored by ${}_\varepsilon\Lambda_\lambda$. Then ${}_\varepsilon\tilde K_{\pm2\rho}(x) = \varepsilon^{\pm(2\rho|\nu)}x$ if $x\in({}_\varepsilon\Lambda_\lambda)^\nu$. Similarly, ${}_{-\varepsilon}\tilde K_{\pm2\rho}(x) = (-\varepsilon)^{\pm(2\rho|\nu)}x$. Note that if ν is a weight, then $\mu-\nu\in Y$. Using the fact that $(2\rho\mid\alpha)\in 2\mathbb{Z}$ for every $\alpha\in Y$, we see that ${}_{-\varepsilon}\tilde K_{\pm2\rho}$ is proportional to ${}_\varepsilon\tilde K_{\pm2\rho}$ on ${}_\varepsilon\Lambda_\lambda$, with the proportional factor $(-1)^{(2\rho|\mu)}$. The proportional factor does not depend on the sign of $\pm2\rho$. Thus ${}_\varepsilon J_{T_u}$ and ${}_\varepsilon J_{T_l}$ are proportional to ${}_{-\varepsilon}J_{T_u}$ and ${}_{-\varepsilon}J_{T_l}$, respectively.

Now it is clear that $\varkappa := {}_{-\varepsilon}J_L/{}_{\varepsilon}J_L$ can be presented as the product of three scalar factors:

$$\frac{{}_{-\varepsilon}J_L}{{}_{\varepsilon}J_L} = \frac{{}_{-\varepsilon}J_T}{{}_{\varepsilon}J_T} \times \frac{{}_{-\varepsilon}J_{T_u}}{{}_{\varepsilon}J_{T_u}} \times \frac{{}_{-\varepsilon}J_{T_l}}{{}_{\varepsilon}J_{T_l}}. \tag{3.15}$$

The first factor can be calculated using Proposition 3.9. Let us calculate the product of the second and third factors. If we replace T by the trivial pure braid T', then we get a trivial link L' of m components. The value of $J_{L'}$ is known, and one has

$$\frac{{}_{-\varepsilon}J_{L'}}{{}_{\varepsilon}J_{L'}} = (-1)^{(2\rho|\mu_1+\cdots+\mu_m)} = e^{\pi i(2\rho|\mu_1+\cdots+\mu_m)}.$$

Here $\Lambda_{\mu_1}, \ldots, \Lambda_{\mu_m}$ are colors of the link. Applying (3.15), with T replaced by T', we see that the product of the second and the third factor is $e^{\pi i(2\rho|\mu_1+\cdots+\mu_m)}$.

Using Proposition 3.9 to calculate the first factor, we see that

$$\varkappa = e^{\pi i[\sum_{1\le i,\, j\le m} l_{ij}(\mu_i|\mu_j)+(2\rho|\mu_1+\cdots+\mu_m)]}.$$

Remember that l_{ii} is even, and $(2\rho \mid \mu_i)$ is always an integer. We can alter the second term in the square bracket to get the value

$$\varkappa = e^{\pi i[\sum_{1\le i,\, j\le m} l_{ij}(\mu_i|\mu_j)+(2\rho|(l_{11}+1)\mu_1+\cdots+(l_{mm}+1)\mu_m)]} = e^{a\pi i},$$

where $e^{\pi ia}$ is the one in (3.13). Thus we have

$$_{-\varepsilon}J_L = e^{\pi ia}\,{}_{\varepsilon}J_L. \tag{3.16}$$

If we replace $v^{1/D}$ by ζ in (3.13), then the right-hand side becomes $e^{\pi ia}\,{}_{\varepsilon}J_L$. The left-hand side, remembering that φ is an algebra homomorphism, is $J_L|_{v^{1/D}=\zeta'}$. The latter is ${}_{-\varepsilon}J_L$. Hence (3.16) implies that (3.13) holds true if $v^{1/D} = \zeta$. Since ζ can take any $2Dr$th root of unity with r odd, (3.13) must hold true for every $v^{1/D}$. This completes the proof of the strong integrality theorem.

Remark. As noted earlier, the use of roots of unity in the proof of the strong integrality seems very artificial. One could avoid roots of unity if the following question has an affirmative answer.

Question. Is it true that in the product of the canonical bases of $\Lambda_{\mu_1} \otimes \cdots \otimes \Lambda_{\mu_m}$, the twist θ has entries in $v^{(\mu_1+\cdots+\mu_m+2\rho|\mu_1+\cdots+\mu_m)}\mathbb{Z}[v^{\pm 2}]$?

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