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ON THE COLORED JONES POLYNOMIAL AND THE KASHAEV INVARIANT

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Abstract. We express the colored Jones polynomial as the inverse of the quantum determinant of a matrix with entries in the q-Weyl algebra of q-operators, evaluated at the trivial function (plus simple substitutions). The Kashaev invariant is proved to be equal to another special evaluation of the determinant. We also discuss the similarity between our determinant formula of the Kashaev invariant and the determinant formula of the hyperbolic volume of knot complements, hoping it will lead to a proof of the volume conjecture.

1. Introduction

For a knot K in $\mathbb{R}^3$, the colored Jones polynomial $J'_K(N)$ is a Laurent polynomial, $J'_K(N) \in \mathcal{R} := \mathbb{Z}[q^{\pm 1}]$ (see [10, 18]). Here N is a positive integer standing for the N-dimensional prime sl_2-module. We use the unframed version and the normalization in which $J'_K(N) = 1$ when K is the unknot. The colored Jones polynomial $J'_K(N)$ is defined using the R-matrix of the quantized enveloping algebra of $\mathrm{sl}_2(\mathbb{C})$.

Here we present the colored Jones polynomial as the inverse of the quantum determinant of an almost quantum matrix whose entries are in the q-Weyl algebra of q-operators acting on the polynomial rings, evaluated at the constant function 1. The proof is based on the quantum MacMahon master theorem proved in [5]. Actually, it was an attempt to get a determinant formula for the colored Jones polynomial that led the second author to the conjecture that eventually became the quantum MacMahon master theorem in [5].

We will then give an application to the case of the Kashaev invariant

$$\langle K \rangle_N := J'_K(N)|_{q=\exp 2\pi i/N}.$$

We show that a special evaluation of the determinant will give the Kashaev invariant. Our interpretation of the Kashaev invariant suggests that the natural generalization of the Kashaev invariant for other simple Lie algebras should be the quantum invariant of knots colored by the Verma module of highest weight $-\delta$, where δ is the half-sum of positive roots.

Finally, we point out how the hyperbolic volume of the knot complement, through the theory of L^2-torsion, has a determinant formula that looks strikingly similar to the one of Kashaev invariants: In both we have noncommutative deformations of the Burau matrices, but in one case the quantum determinant is used and in the other the Fuglede–Kadison determinant is used. This suggests an approach to the volume conjecture using the quantum determinant as an approximation of the infinite-dimensional Fuglede–Kadison determinant.

1.1. A Determinant Formula for the Colored Jones Polynomial.

1.1.1. Right-quantum matrices and quantum determinants. A (2×2)-matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is *right-quantum* if

$$ac = qca \qquad (q\text{-commutation of the entries in a column}),$$

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$$bd = qdb \qquad \text{(q-commutation of the entries in a column),}$$
$$ad = da + qcb - q^{-1}bc \qquad \text{(cross commutation relation).}$$

An $(m \times m)$-matrix is *right-quantum* if any (2×2)-submatrix of it is right-quantum. The significance of a right-quantum matrix lies in the fact that it preserves the structure of quantum m-spaces (see [17]). The product of two right-quantum matrices is a right-quantum matrix, provided that every entry of the first commutes with every entry of the second. The quantum determinant of any right-quantum $A = (a_{ij})$ is defined by

$$\det_q(A) := \sum_{\pi} (-q)^{\mathrm{inv}(\pi)} a_{\pi_1,1} a_{\pi_2,2} \cdots a_{\pi_m,m},$$

where the sum ranges over all permutations of $\{1, \dots, m\}$ and $\mathrm{inv}(\pi)$ denotes the number of inversions.

Note that in general $I - A$, where I is the identity matrix, is not right-quantum any more. We will define its determinant, using an analog of the expansion in the case $q = 1$:

$$\widetilde{\det}_q(I - A) := 1 - C, \quad \text{where} \quad C := \sum_{\varnothing \neq J \subset \{1,2,\dots,r\}} (-1)^{|J|-1} \det_q(A_J),$$

where A_J is the $(J \times J)$-submatrix of A, which is always right-quantum.

1.1.2. Deformed Burau matrix. On the polynomial ring $\mathcal{R}[x^{\pm 1}, y^{\pm 1}, u^{\pm 1}]$ act operators $\hat{x}$ and τ_x and their inverses:

$$\hat{x} f(x, y, \dots) := x f(x, y, \dots), \quad \tau_x f(x, y, \dots) := f(qx, y, \dots).$$

It is easy to see that $\hat{x}\tau_x = q\tau_x\hat{x}$. For other variables, say y, there are similar operators $\hat{y}$ and τ_y, each of which commutes with each of the $\hat{x}$, τ_x. Let us define

$$a_+ = (\hat{u} - \hat{y}\tau_x^{-1})\tau_y^{-1}, \qquad b_+ = \hat{u}^2, \qquad c_+ = \hat{x}\tau_y^{-2}\tau_u^{-1}, \tag{1}$$
$$a_- = (\tau_y - \hat{x}^{-1})\tau_x^{-1}\tau_u, \qquad b_- = \hat{u}^2, \qquad c_- = \hat{y}^{-1}\tau_x^{-1}\tau_u. \tag{2}$$

Then it is easy to verify that the following matrices $S_\pm$ are right-quantum:

$$S_+ := \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}, \qquad S_- := \begin{pmatrix} 0 & c_- \\ b_- & a_- \end{pmatrix}.$$

Assume that P is a polynomial in the operators $a_\pm$, $b_\pm$, and $c_\pm$ with coefficients in $\mathcal{R} = \mathbb{Z}[q^{\pm 1}]$. Applying P to the constant function 1, then substituting u by 1 and x and y by z, one gets a polynomial $\mathcal{E}(P) \in \mathbb{Z}[q^{\pm 1}, z^{\pm 1}]$. Then it is readily seen that $\mathcal{E}(S_+)$ and $\mathcal{E}(S_-)$ are the transpose Burau matrix and its inverse:

$$\mathcal{E}(S_+) = \begin{pmatrix} 1 - z & 1 \\ z & 0 \end{pmatrix}, \qquad \mathcal{E}(S_+) = \begin{pmatrix} 0 & z^{-1} \\ 1 & 1 - z^{-1} \end{pmatrix}.$$

1.1.3. Determinant formula. Let σ_i, $1 \le i \le m - 1$, be the standard generators of the braid group on m strands (see, e.g., [3,10]). For a sequence $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_k)$ of pairs $\gamma_j = (i_j, \varepsilon_j)$, where $1 \le i_j \le m-1$ and $\varepsilon_j = \pm$, let $\beta = \beta(\gamma)$ be the braid

$$\beta := \sigma_{i_1}^{\varepsilon_1} \sigma_{i_2}^{\varepsilon_2} \cdots \sigma_{i_k}^{\varepsilon_k}.$$

Here $\sigma^\pm$ means $\sigma^{\pm 1}$. We will assume that the closure of β (see [3]) is a *knot*, i.e., it has only one connected component. Recall that in the Burau representation of the braid $\beta(\gamma)$, we associate to each $\sigma_{i_j}^{\varepsilon_j}$ an $(m \times m)$-matrix which is the same as the identity matrix everywhere except for the (2×2)-minor of rows i_j, $i_j + 1$ and columns i_j, $i_j + 1$, where we put the Burau (2×2)-matrix if $\varepsilon_j = +$, or its inverse if $\varepsilon_j = -$. Let us do the same, only now the Burau (2×2)-matrix and its inverse, for $\sigma_{i_j}^{\varepsilon_j}$, are replaced by $S_{+,j}$ and $S_{-,j}$. Here $S_{\pm,j}$ are the same as $S_\pm$ with x, y, and u replaced by x_j, y_j, and u_j (for the precise definition see Sec. 2.2.2). The result is a right-quantum matrix $\rho(\gamma)$ whose entries are operators acting on $\mathcal{P}_k = \bigotimes_{j=1}^{k} \mathcal{R}[x_j^{\pm 1}, y_j^{\pm 1}, u_j^{\pm 1}]$. Note that $\rho(\gamma)$ might not be an invariant of the braid $\beta(\gamma)$. We can define $\mathcal{E}(P)$, where P is an operator acting on $\mathcal{P}_k$, as before: first apply P to the constant function 1, then replace all the u_j with 1 and all the x_j and y_j with z. Further, let $\mathcal{E}_N(P)$ be obtained from $\mathcal{E}(P)$ by the substitution $z \to q^{N-1}$.

Let $\rho'(\gamma)$ be obtained from $\rho(\gamma)$ by removing the first row and column. Let $w(\beta)$ denote the writhe, $w(\beta) := \sum_j \varepsilon_j 1$. It is easy to show that when the closure of β is a knot, $w(\beta) - m + 1$ is always even.

Theorem 1. *Assume that the closure in the standard way of the m-strand braid $\beta(\gamma)$ is a knot K.*

(1) *For any positive integer N one has*

$$q^{(N-1)(w(\beta)-m+1)/2}\mathcal{E}_N\left(\frac{1}{\widetilde{\det}_q(I - q\,\rho'(\gamma))}\right) = J'_K(N).$$

(2) $\det \mathcal{E}(I - \rho'(\gamma))$ *is equal to the Alexander polynomial of K.*

Part (1) should be understood as follows. Suppose $\widetilde{\det}_q(I - \rho'(\gamma)) = 1 - C$; then, when applying $\mathcal{E}_N$ to

$$\frac{1}{1-C} := \sum_{n=0}^{\infty} C^n, \tag{3}$$

only a finite number of terms are nonzero, hence the sum is well-defined and is equal to the colored Jones polynomial. We would like to emphasize that here $N > 0$. If $N = 0$, when applying $\mathcal{E}_N$ to the right-hand side of (3), there might be infinitely many nonzero terms. From the theorem one can immediately get the Melvin–Morton conjecture, first proved by Bar-Natan and Garoufalidis [1].

Remark 1.1. Another determinant formula of the colored Jones polynomial using noncommutative variables was given in the independent work [6], also based on the quantum MacMahon master theorem. The main difference is that here our variables are explicit operators acting on a polynomial ring. This sometimes helps since operators can be composed. Another difference is that we derive our formula from the R-matrix, while [6] used cablings of the original Jones polynomial and graph theory. Our approach is a noncommutative analog of Rozansky's beautiful work [20].

1.1.4. An example. To see an application of our formula let us calculate the colored Jones polynomial of the right-handed trefoil. In this case, we need only two strands with $\beta = \sigma^3$. Thus, $\rho(\gamma) = S_{+,1}S_{+,2}S_{+,3}$ is easy to calculate, and we get $\rho'(\gamma) = c_1 a_2 b_3$. Hence, with K being the right-handed trefoil,

$$\begin{aligned} J'_K(N) = q^{N-1}\mathcal{E}_N\left(\frac{1}{1 - qc_1a_2b_3}\right) &= q^{N-1}\sum_{n=0}^{\infty}\mathcal{E}_N(q^n c_1^n a_2^n b_3^n) \\ &= q^{N-1}\sum_{n=0}^{\infty} q^{nN}(1-q^{N-1})(1-q^{N-2})\dots(1-q^{N-n}). \end{aligned} \tag{4}$$

Note that the sum is always finite, since the term on the right-hand side is 0 if $n \geq N$.

1.2. The Kashaev Invariant as the Invariant of Dimension 0. Kashaev [11] used the quantum dilogarithm to define a knot invariant $\langle K\rangle_N$, depending on a positive number N. Murakami and Murakami [19] showed that

$$\langle K\rangle_N = J'_K(N)|_{q=\exp(2\pi i/N)}.$$

The famous volume conjecture [11, 19] says that the growth rate of $\langle K\rangle_N$ is equal to the volume $V(K)$ (see the definition below) of the knot complement:

$$\lim_{N\to\infty}\frac{\ln|\langle K\rangle_N|}{N} = \frac{\mathrm{Vol}(K)}{2\pi}.$$

Working with varying N, i.e., working with varying sl_2-modules, might be difficult. Here we show that the value of $\langle K\rangle_N$ comes from just one sl_2-module, the Verma module of highest weight -1, and is a kind of analytic function in the following sense. Let us define the Habiro ring $\widehat{\mathbb{Z}[q]}$ by

$$\widehat{\mathbb{Z}[q]} := \varprojlim \mathbb{Z}[q]/((1-q)(1-q^2)\cdots(1-q^n)).$$

Habiro [7] called it the cyclotomic completion of $\mathbb{Z}[q]$. Formally, $\widehat{\mathbb{Z}[q]}$ is the set of all series of the form

$$f(q) = \sum_{n=0}^{\infty} f_n(q)(1-q)(1-q^2)\cdots(1-q^n), \quad \text{where} \quad f_n(q) \in \mathbb{Z}[q].$$

Assume that U is the set of roots of 1. If $\xi \in U$, then $(1-\xi)(1-\xi^2)\cdots(1-\xi^n) = 0$ if n is big enough, hence one can define $f(\xi)$ for $f \in \widehat{\mathbb{Z}[q]}$. One can consider every $f \in \widehat{\mathbb{Z}[q]}$ as a function with domain U. Note that $f(\xi) \in \mathbb{Z}[\xi]$ is always an algebraic integer. It turns out that $\widehat{\mathbb{Z}[q]}$ has remarkable properties and plays an important role in quantum topology. First, each $f \in \widehat{\mathbb{Z}[q]}$ has a natural Taylor series at every point of U, and if two functions $f, g \in \widehat{\mathbb{Z}[q]}$ have the same Taylor series at a point in U, then $f = g$. A consequence is that $\widehat{\mathbb{Z}[q]}$ is an integral domain. Second, if $f = g$ at infinitely many roots of prime power orders, then $f = g$ (see [7]). Hence one can consider $\widehat{\mathbb{Z}[q]}$ as a class of "analytic functions" with domain U. It was proved, by Habiro for sl_2 and by Habiro with the second author for general simple Lie algebras, that quantum invariants of integral homology 3-spheres belong to $\widehat{\mathbb{Z}[q]}$ and thus have remarkable integrality properties. Here we show that the Kashaev invariant also belongs to $\widehat{\mathbb{Z}[q]}$.

Theorem 2.

(1) $q^{(m-w(\beta)-1)/2}\mathcal{E}_0\left(\frac{1}{\widetilde{\det}_q(I-q\rho'(\gamma))}\right)$ *belongs to* $\widehat{\mathbb{Z}[q]}$ *and is an invariant of the knot* K *obtained by closing* $\beta(\gamma)$.

(2) *Kashaev's invariant is equal to*

$$\langle K\rangle_N = q^{(m-w(\beta)-1)/2}\mathcal{E}_0\left(\frac{1}{\widetilde{\det}_q(I-q\rho'(\gamma))}\right)\Bigg|_{q=\exp(2\pi i/N)}. \tag{5}$$

For example, when K is the left-handed trefoil, from (4), with $q \to q^{-1}$, we have

$$\langle K\rangle_N = q\sum_{n=0}^{\infty}(1-q)(1-q^2)\cdots(1-q^n),$$

where $q = \exp(2\pi i/N)$. The function given by the infinite sum on the right-hand side was first written down by M. Kontsevich, and its asymptotics was completely determined by Zagier [23]. We see that it has a nice geometric interpretation: It is the Kashaev invariant of the trefoil.

1.3. Hyperbolic Volume and L^2-Torsion. It is known that by cutting the knot complement $S^3 \setminus K$ along some embedded tori one gets connected components which are either Seifert-fibered or hyperbolic. Let $\mathrm{Vol}(K)$ be the sum of the hyperbolic volume of the hyperbolic pieces, ignoring the Seifert-fibered components. It is known that $\mathrm{Vol}(K)$ is proportional to the Gromov norm [2] and can be calculated using L^2-torsion as follows. Let the knot K again be the closure of the braid β. The fundamental group of the knot complement has a presentation

$$\pi_1 = \langle z_1, \ldots, z_m \mid r_1, \ldots, r_m\rangle,$$

where $r_i = \beta(z_i)z_i^{-1}$, with β considered as an automorphism of the free group on m generators $z_1, \ldots z_m$.

Let

$$\mathrm{Ja} = \left(\frac{\partial r_i}{\partial z_j}\right)$$

be the Jacobian matrix with entries in $\mathbb{Z}[\pi_1]$, where $\frac{\partial r_i}{\partial z_j}$ is the Fox derivative. For a matrix with entries in $\mathbb{Z}[\pi_1]$, one can define its Fuglede–Kadison determinant (see [16]), denoted by $\det_{\pi_1}$. A deep theorem of Luck and Schick [16] says that

$$\mathrm{Vol}(K) = 6\pi\ \ln(\det_{\pi_1}(\mathrm{Ja}')),$$

where Ja' is obtained from Ja by removing the first row and column. It is easy to see that

$$\mathrm{Ja} = \psi(\beta) - I, \quad \text{where} \quad \psi(\beta) = \left(\frac{\partial(\beta(z_i))}{\partial z_j}\right).$$

A simple property of Fuglede–Kadison determinant is that $\det_{\pi_1}(A) = \det_{\pi_1}(-A)$. Hence we have the following proposition.

Proposition 1.2. *Let $\psi'(\beta)$ be obtained from $\psi(\beta)$ by removing the first row and column. Then*

$$\exp(-\frac{\mathrm{Vol}(K)}{6\pi}) = \frac{1}{\det_{\pi_1}(I - \psi'(\beta))}. \tag{6}$$

Note that under the abelianization map $ab\colon \mathbb{Z}[\pi_1] \to \mathbb{Z}[\mathbb{Z}]$, the matrix $\psi(\beta)$ becomes the Burau representation of β. Hence both $\psi(\beta)$ and $\rho(\beta(\gamma))$ are two different kinds of quantization of the Burau representation. We hope that the similarity between (6) and (5) will help to solve the volume conjecture. One needs to relate the Fuglede–Kadison determinant $\det_{\pi_1}$ to the quantum determinant.

Also note that the abelianized version of the right-hand side of (6), i.e., $\det_{\mathbb{Z}}(I - ab(\psi'(\beta)))$, is equal to the Mahler measure of the Alexander polynomial (see [16]). This partially explains some similarity between the Mahler measure and the hyperbolic volume of a knot, as observed in [21].

1.4. Plan of the Paper. In Sec. 2 we prove Theorem 1. Section 3 contains a proof of Theorem 2 and a discussion on generalization to other Lie algebras of the Kashaev invariants.

2. Proof of Theorem 1

In Sec. 2.1, we recall the definition of the colored Jones polynomial using the R-matrix. We will follow Rozansky [20] to twist the R-matrix so that it has a "nice" form. Then in the subsequent subsections we show how the twisted R-matrix can be obtained from the deformed Burau matrix, giving a proof of Theorem 1.

We will use the variable $v^{1/2}$ such that $v^2 = q$. Note that our q is equal to q^2 in [9]. Recall that $\mathcal{R} = \mathbb{Z}[q^{\pm 1}]$, which is a subring of the field $\tilde{\mathcal{R}} := \mathbb{C}(v^{\pm 1/2})$. We will use the following standard notations:

$$[n] := \frac{v^n - v^{-n}}{v - v^{-1}}, \qquad [n]! := \prod_{i=1}^{n}[i], \qquad \begin{bmatrix} n \\ l \end{bmatrix} := \prod_{i=1}^{l} \frac{[n-i+1]}{[l-i+1]},$$

$$(n)_q := \frac{1-q^{-n}}{1-q^{-1}}, \qquad \begin{pmatrix} n \\ l \end{pmatrix}_q := \prod_{i=1}^{l} \frac{(n-i+1)_q}{(l-i+1)_q}, \qquad (1-x)_q^d := \prod_{i=0}^{d-1}(1 - xq^i).$$

2.1. The Colored Jones Polynomial through the *R*-Matrix.

2.1.1. The quantized enveloping algebra $U_v(\mathrm{sl}_2)$. Let $\mathcal{U}$ be the algebra over the field $\tilde{\mathcal{R}} = \mathbb{C}(v^{\pm 1/2})$ generated by $K^{\pm 1/2}$, E, and F, subject to the relation

$$K^{1/2}K^{-1/2} = 1, \quad K^{1/2}E = vEK^{1/2}, \quad K^{1/2}F = v^{-1}FK^{1/2}, \quad EF - FE = \frac{K - K^{-1}}{v - v^{-1}}.$$

Then $\mathcal{U}$ is a Hopf algebra with coproduct:

$$\Delta(K^{1/2}) = K^{1/2} \otimes K^{1/2}, \quad \Delta(E) = E \otimes 1 + K \otimes E, \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F.$$

Here we follow the definition of [9], only we add the square root $K^{1/2}$ for convenience. Note that $V \otimes W$ has a natural $\mathcal{U}$-module structure whenever V and W have, due to the co-algebra structure.

2.1.2. The quasi-R-matrix and braiding. The quasi-R-matrix Θ is an element of some completion of $\mathcal{U}$:

$$\Theta := \sum_{n=0}^{\infty} (-1)^n v^{-n(n-1)/2} \frac{(v-v^{-1})^n}{[n]!} F^n \otimes E^n.$$

A $\mathcal{U}$-module V is *E-locally-finite* if for every $u \in V$ there is n such that $E^n u = 0$. If V and W are E-locally-finite, then for every $u \otimes w \in V \otimes W$, there are only a finite number of terms in the sum of Θ that do not annihilate $u \otimes w$, hence we can define Θ as an $\tilde{\mathcal{R}}$-linear operator acting on $V \otimes W$. The inverse of Θ is given by

$$\Theta^{-1} := \sum_{n=0}^{\infty} v^{n(n-1)/2} \frac{(v-v^{-1})^n}{[n]!} F^n \otimes E^n.$$

An element u in a $\mathcal{U}$-module is said to have weight l if $Ku = v^l u$. We will consider only $\mathcal{U}$-modules that are spanned by weight vectors. For such modules V and W we define the diagonal operator D by

$$D(u \otimes w) = v^{-kl/2} u \otimes w,$$

where u has weight k and w has weight l. The braiding $\mathfrak{b}\colon V \otimes W \to W \otimes V$ is defined by

$$\mathfrak{b}(u \otimes w) := \Theta(D(w \otimes u)).$$

It is known that $\mathfrak{b}$ commutes with the action of $\mathcal{U}$, is invertible, and satisfies the braid relation. Suppose that V is an E-locally-finite $\mathcal{U}$-module. Let $\mathfrak{b}_{12} := \mathfrak{b} \otimes \mathrm{id}$ and $\mathfrak{b}_{23} := \mathrm{id} \otimes \mathfrak{b}$ be the operators acting on $V \otimes V \otimes V$. Then

$$\mathfrak{b}_{12}\, \mathfrak{b}_{23}\, \mathfrak{b}_{12} = \mathfrak{b}_{23}\, \mathfrak{b}_{12}\, \mathfrak{b}_{23}.$$

One can define a representation of the braid group on m strands into the group of linear operators acting on $V^{\otimes m}$ by putting

$$\tau(\sigma_i) = \mathrm{id}^{\otimes(i-1)} \otimes \mathfrak{b} \otimes \mathrm{id}^{\otimes m-i-1},$$

i.e., σ_i acts trivially on all components, except for the ith and $(i+1)$st, where it acts as $\mathfrak{b}$.

2.1.3. A modification of the Verma module V_N. For an integer N, not necessarily positive, let V_N be the $\tilde{\mathcal{R}}$-vector space freely spanned by e_i, $i \in \mathbb{Z}_{\geq 0}$. The following can be readily checked.

Proposition 2.1. *The space V_N has the structure of an E-locally-finite $\mathcal{U}$-module given by*

$$\begin{aligned} Ke_i &= v^{N-1-2i} e_i, \\ Ee_i &= (i)_{q^{-1}} e_{i-1}, \\ Fe_i &= v^i [N-1-i] e_{i+1} = \frac{v^{1-N}}{v-v^{-1}} (q^{N-1} - q^i) e_{i+1}. \end{aligned}$$

For $N > 0$, let W_N be the $\tilde{\mathcal{R}}$-subspace of V_N spanned by e_i, $0 \leq i \leq N-1$. It is easy to see that W_N is a simple $\mathcal{U}$-submodule of V_N. Every simple finite-dimensional $\mathcal{U}$-module is isomorphic to one of W_N.

Remark 2.2. The traditional basis $e'_i := F^i(e_0)/[i]!$ is related to the basis e_i by

$$\begin{bmatrix} N-1 \\ i \end{bmatrix} e_i = v^{-i(i-1)/2} e'_i.$$

2.1.4. The colored Jones polynomial. If the closure of the m-strand braid β is the knot K, then the colored Jones polynomial $J_K(N)$ can be defined as the quantum trace of $\tau(\beta)$ on $(W_N)^{\otimes m}$:

$$J_K(N) = v^{w(\beta)\frac{N^2-1}{2}} \mathrm{tr}_q(\tau(\beta), (W_N)^{\otimes m}) := v^{w(\beta)\frac{N^2-1}{2}} \mathrm{tr}(\tau(\beta) K^{-1}, (W_N)^{\otimes m}).$$

Here $w(\beta) := \sum_j \varepsilon_j 1$ is the writhe of β. The factor $v^{w(\beta)\frac{N^2-1}{2}}$ will make $J_K(N)$ not depend on the framing. If K is the unknot, then $J_K(N) = [N]$. The normalized version $J'_K(N) := J_K(N)/[N]$ can be calculated using the partial trace as follows. Recall that $\tau(\beta)$ acts on $(W_N)^{\otimes m}$. Taking the quantum trace of $\tau(\beta)$ in only the $m-1$ last components, we get an operator acting on the first W_N, which is known to be a scalar times the identity operator, with the scalar being exactly $J'_K(N)$. This can be written in the formula form as follows. Let $p_0\colon (V_N)^{\otimes m} \to (V_N)^{\otimes m}$ be the projection onto $e_0 \otimes (V_N)^{\otimes(m-1)}$, i.e.,

$$p_0(e_{n_1} \otimes e_{n_2} \otimes \cdots \otimes e_{n_m}) = \delta_{0,i_1}\, e_{n_2} \otimes \cdots \otimes e_{n_m}.$$

Then p_0 also restricts to a projection from $(W_N)^{\otimes m}$ onto $e_0 \otimes (W_N)^{\otimes(m-1)}$, and

$$J'_K(N) = v^{w(\beta)\frac{N^2-1}{2}} \operatorname{tr}\bigl(p_0(\tau(\beta)\, K^{-1}), e_0 \otimes (W_N)^{\otimes(m-1)}\bigr). \tag{7}$$

2.1.5. Twisting the braiding. It is straightforward to calculate the action of the braiding $\mathfrak{b}$ on $V_N \otimes V_N$, using the basis $e_{n_1} \otimes e_{n_2}$, $n_1, n_2 \in \mathbb{Z}_{\geq 0}$. However, to get a better, more convenient form we will follow Rozansky [20] and use the twisted braiding

$$\check{\mathfrak{b}} := Q^{-1}\mathfrak{b}Q, \quad \text{where} \quad Q = \mathrm{id} \otimes K^{(1-N)/2}.$$

Then direct calculation shows that on $V_N \otimes V_N$ the action of the twisted braiding $\check{\mathfrak{b}}_\pm$ is given by

$$\check{\mathfrak{b}}_\pm(e_{n_1} \otimes e_{n_2}) = \sum_{l=0}^{\max(n_1,n_2)} \check{\mathfrak{b}}_\pm(n_1, n_2; l)(e_{n_2 \pm l} \otimes e_{n_1 \mp l}),$$

where, with $z = q^{N-1}$,

$$(\check{\mathfrak{b}}_+)(n_1, n_2; l) = q^{-\frac{(N-1)^2}{4}} \binom{n_1}{l}_{q^{-1}} q^{n_2(l-n_1)} z^{n_2} (1 - zq^{-n_2})^l_{q^{-1}}, \tag{8}$$

$$(\check{\mathfrak{b}}_-)(n_1, n_2; l) = q^{\frac{(N-1)^2}{4}} \binom{n_2}{l}_{q} q^{n_1(n_2-l)} z^{-n_1} (1 - z^{-1}q^{n_1})^l_{q}. \tag{9}$$

Note that our formulas differ from those in [20] by $q \to q^{-1}$, since we derived our formula directly from the quantized enveloping algebra, which differs from the one implicitly used by Rozansky. (The co-products are opposite; "implicitly" since Rozansky never used a quantized enveloping algebra but just took the formula of the R-matrix from [13].)

To justify the use of the twisted braiding we argue as follows. First note that $\mathfrak{b}_\pm$ commutes with $K^{1/2}$, the action of which on $V_N \otimes V_N$ is given by $\Delta(K^{1/2}) = K^{1/2} \otimes K^{1/2}$. Thus, $K^l \otimes K^l$ commutes with $\mathfrak{b}_\pm$ for every half-integer l. Hence

$$(Q')^{-1}\mathfrak{b}_\pm Q' = Q^{-1}\mathfrak{b}_\pm Q = \check{\mathfrak{b}}_\pm \tag{10}$$

if

$$Q' = Q(K^{i(1-N)/2} \otimes K^{i(1-N)/2}) = K^{i(1-N)/2} \otimes K^{(i+1)(1-N)/2}.$$

Let us define the operator Q_m acting on $(W_N)^{\otimes m}$ by

$$Q_m := K^{(1-N)/2} \otimes K^{2(1-N)/2} \otimes \cdots \otimes K^{m(1-N)/2}$$

and let

$$\check{\tau}(\beta) = Q_m^{-1}\tau(\beta)Q_m.$$

Then $\check{\tau}$ is also a representation of the braid group. Since the action of K^{-1} on $(W_N)^{\otimes m}$ commutes with the action of Q_m, one sees that in formula (7) we can use $\check{\tau}(\beta)$ instead of $\tau(\beta)$:

$$J'_K(N) = v^{w(\beta)(N^2-1)/2} \operatorname{tr}\bigl(p_0(\check{\tau}(\beta)K^{-1}), e_0 \otimes (W_N)^{\otimes(m-1)}\bigr). \tag{11}$$

Assume that $\beta = \sigma_{i_1}^{\varepsilon_1} \dots \sigma_{i_k}^{\varepsilon_k}$. Then $\check{\tau}(\beta) = \check{\tau}(\sigma_{i_1})^{\varepsilon_1} \dots \check{\tau}(\sigma_{i_k})^{\varepsilon_k}$. Let us calculate $\check{\tau}(\sigma_i)$:

$$\begin{aligned}\check{\tau}(\sigma_i^{\pm 1}) &= Q_m^{-1}\tau(\sigma_i)Q_m = Q_m^{-1}\bigl(\mathrm{id}^{\otimes(i-1)} \otimes \mathfrak{b}_\pm \otimes \mathrm{id}^{\otimes m-i-1}\bigr)Q_m \\ &= \mathrm{id}^{\otimes(i-1)} \otimes \bigl(\bigl(K^{i(1-N)/2} \otimes K^{(i+1)(1-N)/2}\bigr)^{-1}\mathfrak{b}_\pm\bigl(K^{i(1-N)/2} \otimes K^{(i+1)(1-N)/2}\bigr)\bigr) \otimes \mathrm{id}^{\otimes m-i-1} \\ &= \mathrm{id}^{\otimes(i-1)} \otimes \check{\mathfrak{b}}_\pm \otimes \mathrm{id}^{\otimes m-i-1}\end{aligned}$$

by (10).

This means that in the definition of $\check{\tau}$ one just uses $\check{\mathfrak{b}}_\pm$ instead of $\mathfrak{b}_\pm$, and then $\check{\tau}$ is obtained from τ by the global twist Q_m.

2.1.6. From W_N to V_N. Thus far we have taken the trace using the finite-dimensional module W_N. For the infinite-dimensional V_N, we define the trace of an operator if only a finite number of diagonal entries are nonzero. The following was observed in [20].

Lemma 2.3. *Assume that the closure of the braid β is a knot; then*

$$J'_K(N) = v^{w(\beta)(N^2-1)/2}\,\mathrm{tr}\big(p_0(\check{\tau}(\beta)K^{-1}), e_0 \otimes (W_N)^{\otimes(m-1)}\big)$$
$$= v^{w(\beta)(N^2-1)/2}\,\mathrm{tr}\big(p_0(\check{\tau}(\beta)K^{-1}), e_0 \otimes (V_N)^{\otimes(m-1)}\big).$$

Proof. One important observation is that if $n < N$ and $n + l \geq N$, then $F^l e_n = 0$. Hence $\check{\mathfrak{b}}_\pm(e_{n_1} \otimes e_{n_2})$ is a linear combination of $e_{m_1} \otimes e_{m_2}$ (with $m_1 + m_2 = n_1 + n_2$), and if $n_1 < N$, then $m_2 < N$, or if $n_2 < N$, then $m_1 < N$.

Let $(\tau(\beta)K^{-1})^{s_1,s_2\dots s_m}_{n_1,n_2\dots n_m}$ be the matrix of $\tau(\beta)K^{-1}$ with respect to the basis $e_{n_1} \otimes e_{n_2} \otimes \cdots \otimes e_{n_m}$ in $(V_N)^{\otimes m}$. Note that K^{-1} acts diagonally in this basis. The above observation shows that if $n_i < N$, then $s_{\bar{\beta}(i)} < N$ for the matrix entry $(\tau(\beta)K^{-1})^{s_1,s_2\dots s_m}_{n_1,n_2\dots n_m}$ not to be 0, where $\bar{\beta}$ is the permutation corresponding to β. To take the trace we only have to concern ourselves with the case $s_i = n_i$. We have already had $n_1 = 0$, which is less than N. Thus, we must have $n_j < N$ for $j = 1, \bar{\beta}(1), (\bar{\beta})^2(1), \dots$. The fact that the closure of β is a knot implies that $\{(\bar{\beta})^l(1),\ 1 \leq l \leq m\}$ is the whole set $\{1, 2, \dots, m\}$. Hence taking the trace over $e_0 \otimes (V_N)^{\otimes(m-1)}$ is the same as over $e_0 \otimes (W_N)^{\otimes(m-1)}$. □

2.2. Algebra of the Deformed Burau Matrix.

2.2.1. Algebra $\mathcal{A}_\varepsilon$. Let us define

$$\mathcal{A}_+ := \mathcal{R}\langle a_+, b_+, c_+\rangle / (a_+b_+ = b_+a_+,\ a_+c_+ = qc_+a_+,\ b_+c_+ = q^2c_+b_+),$$
$$\mathcal{A}_- := \mathcal{R}\langle a_-, b_-, c_-\rangle / (a_-b_- = q^2b_-a_-,\ c_-a_- = qa_-c_-,\ c_-b_- = q^2b_-c_-).$$

It is easy to verify that the $a_\pm$, $b_\pm$, and $c_\pm$ of Sec. 1.1.2 satisfy the commutation relations of the algebras $\mathcal{A}_\pm$.

For a sequence $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k)$, where each ε_j is either $+$ or $-$, let $\mathcal{A}_\varepsilon = \mathcal{A}_{\varepsilon_1} \otimes \mathcal{A}_{\varepsilon_2} \otimes \cdots \otimes \mathcal{A}_{\varepsilon_k}$. We can consider $\mathcal{A}_\varepsilon$ as the algebra over $\mathcal{R}$ freely generated by a_j, b_j, and c_j subject to the commutation relations: if $i \neq j$, then each of the a_i, b_i, and c_i commutes with each of the a_j, b_j, and c_j, if $\varepsilon_j = +$, then the commutations among a_j, b_j, and c_j are the same as those of a_+, b_+, and c_+, and if $\varepsilon_j = -$, then the commutations among a_j, b_j, and c_j are the same as those of a_-, b_-, and c_-. Note that the algebra $\mathcal{A}_\varepsilon$ is a generalized quantum space in the sense that for any a and b among the generators, one has the almost q-commutation relation $ab = q^l ba$ for some integer l.

Replacing x, y, u, $a_\pm$, $b_\pm$, and $c_\pm$ with respectively x_j, y_j, u_j, a_j, b_j, and c_j in (1) if $\varepsilon_j = +$, or in (2) if $\varepsilon_j = -$, we identify a_j, b_j, and c_j with operators acting on $\mathcal{R}[x_j^{\pm1}, y_j^{\pm1}, u_j^{\pm1}]$. We assume that a_j, b_j, and c_j leave alone x_i, y_i, and u_i if $i \neq j$. Thus, $\mathcal{A}_\varepsilon$ acts on the algebra $\mathcal{P}_k$ of Laurent polynomials in x_j, y_j, and u_j, $1 \leq j \leq k$, with coefficients in $\mathcal{R}$. The map $\mathcal{E}\colon \mathcal{A}_\varepsilon \to \mathcal{R}[z^{\pm1}]$ is defined as in Sec. 1.1.2.

Lemma 2.4.

1. *If $f, g \in \mathcal{A}_\varepsilon$ are separate, i.e., f contains only a_j, b_j, and c_j with $j \leq r$ and g contains only a_l, b_l, and c_l with $r < l$ (for some r), then $\mathcal{E}(fg) = \mathcal{E}(f)\mathcal{E}(g)$.*
2. *The following equalities hold:*

$$\mathcal{E}(b_+^s c_+^r a_+^d) = q^{-rd} z^r (1 - zq^{-r})^d_{q^{-1}}, \tag{12}$$
$$\mathcal{E}(b_-^s c_-^r a_-^d) = z^{-r} (1 - z^{-1}q^r)^d_q. \tag{13}$$

Proof. (1) follows directly from the definition. (2) follows from an easy induction. □

2.2.2. Definition of $\rho(\gamma)$. Let us give here the precise definition of $\rho(\gamma)$ for $\gamma = ((i_1,\varepsilon_1),\dots,(i_k,\varepsilon_k))$. Recall that β is the braid

$$\beta = \beta(\gamma) := \sigma_{i_1}^{\varepsilon_1}\sigma_{i_2}^{\varepsilon_2}\cdots\sigma_{i_k}^{\varepsilon_k}.$$

If $\varepsilon_j = +$ ($\varepsilon_j = -$), let S_j be the matrix S_+ (respectively, S_-) with a_+, b_+, and c_+ (respectively, a_-, b_-, and c_-) replaced by a_j, b_j, and c_j. For the jth factor $\sigma_{i_j}^{\varepsilon_j}$, let us define a right-quantum $(m\times m)$-matrix A_j by the block sum, just as in the Burau representation, only the nontrivial (2×2)-block now is S_j instead of the Burau matrix:

$$A_j := I_{i_j-1}\oplus S_j \oplus I_{m-i_j-1}.$$

Here I_l is the identity $(l\times l)$-matrix.

Let $\rho(\gamma) := A_1A_2\dots A_k$. Then $\rho(\gamma)$ is a right-quantum $(m\times m)$-matrix with entries polynomials in a_j, b_j, and c_j.

2.3. Quantum MacMahon Master Theorem.

2.3.1. Co-actions of right-quantum matrices on the quantum space. The quantum plane $\mathbb{C}_q[z_1,z_2,\dots,z_m]$, considered as the space of q-polynomial in the variables $z_1,\dots,z_m$, is defined as

$$\mathbb{C}_q[z_1,z_2,\dots,z_m] := \tilde{\mathcal{R}}\langle z_1,\dots,z_m\rangle/(z_iz_j = qz_jz_i \text{ if } i<j).$$

Remark 2.5. Our definitions of quantum spaces, quantum matrices... differ from those in [5,12] by the involution $q\to q^{-1}$ but agree with those in [9].

If $A = (a_{ij})_{i,j=1}^m$ is right-quantum and all a_{ij}'s commute with all $z_1,\dots,z_m$, then it is known that the $Z_i := \sum\limits_j a_{ij}z_j$, i.e.,

$$\begin{pmatrix} Z_1\\ Z_2\\ \dots\\ Z_m\end{pmatrix} = A\begin{pmatrix} z_1\\ z_2\\ \dots\\ z_m\end{pmatrix},$$

also satisfy $Z_iZ_j = qZ_jZ_i$ if $i<j$. Let $\mathcal{W} = \mathcal{W}(A)$ be the algebra generated by a_{ij}, $1\le i,j\le m$, subject to the commutation relations of a_{ij}. Then we have an *algebra* homomorphism

$$\Phi_A\colon \mathbb{C}_q[z_1\dots,z_m]\to \mathcal{W}\otimes\mathbb{C}_q[z_1,\dots,z_m]$$

defined by $\Phi_A(z_i) = Z_i$. Informally, one could look at Φ_A as the degree-preserving *algebra* homomorphism on the q-polynomial ring $\mathbb{C}_q[z_1,z_2,\dots,z_m]$ defined by matrix A. Here we assume that the degree of each z_i is 1 and the degree of each a_{ij} is 0.

We will consider the case $A=\rho(\gamma)$, and, in particular, $A = \check{\mathfrak{b}}_\pm$. In this case, we define $\mathcal{E}_N(\Phi_A) := (\mathcal{E}_N\otimes id)\circ\Phi_A$, which is a *linear* operator acting on $\mathbb{C}_q[z_1\dots,z_m]$, not necessarily an algebra homomorphism.

2.3.2. Quantum MacMahon master theorem. Let $\mathbb{C}_q[z_1,\dots,z_m]^{(n)}$ be the part of total degree n in $\mathbb{C}_q[z_1,\dots,z_m]$. Since Φ_A preserves the total degree, it restricts to a linear map

$$\Phi_A\colon \mathbb{C}_q[z_1\dots,z_m]^{(n)}\to\mathcal{W}\otimes\mathbb{C}_q[z_1,\dots,z_m]^{(n)}.$$

Let us define the trace by

$$\operatorname{tr}\bigl(\Phi_A,\mathbb{C}_q[z_1\dots,z_m]^{(n)}\bigr) = \sum_{n_1+\dots+n_m=n}(\Phi_A)_{n_1,\dots,n_m}^{n_1,\dots,n_m},$$

where $(\Phi_A)_{n_1,\dots,n_m}^{n_1,\dots,n_m}$ represents the coefficients of $z_1^{n_1}\cdots z_m^{n_m}$ in $Z_1^{n_1}\cdots Z_m^{n_m}$. Let us consider

$$\operatorname{tr}\bigl(\Phi_A,\mathbb{C}_q[z_1\dots,z_m]^{(n)}\bigr)$$

as the trace of Φ_A acting on the part of total degree n. The quantum MacMahon master theorem, proved in [5], says that

$$\frac{1}{\widetilde{\det}_q(I-A)} = \operatorname{tr}(\Phi_A, \mathbb{C}_q[z_1 \dots, z_m]) := \sum_{n=0}^{\infty} \operatorname{tr}\big(\Phi_A, \mathbb{C}_q[z_1 \dots, z_m]^{(n)}\big).$$

It is the q-analog of the identity

$$\frac{1}{\det(I-C)} = \sum_{n=0}^{\infty} \operatorname{tr}(S^n C),$$

where C is a linear operator acting on a finite-dimensional $\mathbb{C}$-space V and S^nC is the action of C on the nth symmetric power of V.

2.4. From Deformed Burau Matrices $S_\pm$ to R-matrices $\check{\mathfrak{b}}_\pm$. Let

$$\mathcal{F}_m\colon (V_N)^{\otimes m} \to \mathbb{C}_q[z_1, \dots, z_m]$$

be the $\tilde{\mathcal{R}}$-linear isomorphism defined by

$$\mathcal{F}(e_{n_1} \otimes \cdots \otimes e_{n_m}) := z_1^{n_1} \cdots z_m^{n_m}.$$

The following is important to us.

Proposition 2.6.

(1) *Under the isomorphism $\mathcal{F}_2$, the twisted braiding matrices $\check{\mathfrak{b}}_\pm$ acting on $V_N \otimes V_N$ map to $v^{\mp(N-1)^2/2}\mathcal{E}_N(S_\pm)$, i.e.,*

$$\check{\mathfrak{b}}_\pm = v^{\mp(N-1)^2/2}\mathcal{F}_2^{-1}\mathcal{E}_N(\Phi_{S_\pm})\mathcal{F}_2.$$

(2) *Under the isomorphism $\mathcal{F}_m$, the linear automorphism $\check{\tau}(\beta(\gamma))$ of $(V_N)^{\otimes m}$ maps to $v^{\mp w(\beta)(N-1)^2/2}\mathcal{E}_N(\Phi_{\rho(\gamma)})$.*

Proof.

(1) Assume that for two variables X and Y we have $YX = qXY$; then the Gauss q-binomial formula [12] says that

$$(X+Y)^n = \sum_{l=0}^{n} \binom{n}{l}_q X^l Y^{n-l}.$$

Let us first consider the case of S_+. Then $\Phi_{S_+}(z_1) = a_+z_1 + b_+z_2$ and $\Phi_{S_-}(z_2) = c_+z_1$. Note that $(b_+z_2)(a_+z_1) = q^{-1}(a_+z_1)b_+(z_2)$, hence using the Gauss binomial formula we have

$$\begin{aligned}\Phi_{S_+}(z_1^{n_1}z_2^{n_2}) &= (a_+z_1 + b_+z_2)^{n_1}(c_+z_1)^{n_2}\\ &= \sum_{l=0}^{n_1}\binom{n_1}{l}_{q^{-1}}(a_+z_1)^l(b_+z_2)^{n_1-l}(c_+z_1)^{n_2} = \sum_{l=0}^{n_1}\binom{n_1}{l}_{q^{-1}} q^{n_2(n_1-l)}a_+^l b_+^{n_1-l}c_+^{n_2}(z_1)^{n_2+l}(z_2)^{n_1-l}.\end{aligned}$$

Using formulas (8) and (12) one sees that

$$\check{\mathfrak{b}}_+ = v^{-(N-1)^2/2}\mathcal{F}_2^{-1}\mathcal{E}_N(\Phi_{S_+})\mathcal{F}_2.$$

The proof for S_- is quite similar, using formulas (9) and (13).

(2) Since the variables x_j, y_j, and u_j are separated, we have that

$$\mathcal{E}(\rho(\gamma)) = \mathcal{E}(\rho(\sigma_{i_1}^{\varepsilon_1}) \cdots \mathcal{E}(\rho(\sigma_{i_k}^{\varepsilon_k}),$$

and the statement follows from part (1). □

2.4.1. Under the isomorphism $\mathcal{F}_m$, the projection $p_0\colon (V_N)^{\otimes m} \to (V_N)^{\otimes m}$ maps to a projection, also denoted by p_0, of $\mathbb{C}_q[z_1, z_2, \ldots, z_m]$, which can be defined as

$$p_0(z_1^{n_1} z_2^{n_2} \cdots z_m^{n_m}) = \delta_{0,n_1} z_2^{n_2} \cdots z_m^{n_m}.$$

Note that the kernel of p_0 is the ideal generated by z_1.

Lemma 2.7.

(1) *For every* $u \in \mathbb{C}_q[z_2, \ldots, z_m]$,

$$p_0(\Phi_{\rho(\beta)}(u)) = \Phi_{\rho'(\beta)}(u).$$

(2) *The operators* p_0 *and* $\mathcal{E}_N$ *commute*:

$$p_0(\mathcal{E}_N(\Phi_{\rho(\beta)}(u))) = \mathcal{E}_N(p_0(\Phi_{\rho(\beta)}(u))).$$

Proof.

(1) Recall that $\rho'(\gamma)$ is obtained from $\rho(\gamma)$ by removing the first row and column. Suppose that $u \in \mathbb{C}_q[z_2, \ldots, z_m]$; then $\Phi_{\rho(\beta)}(u) - \Phi_{\rho'(\beta)}(u)$ is divisible by z_1 and hence annihilated by p_0.

(2) follows trivially from the definition. □

The following is trivial.

Lemma 2.8. *Under* $\mathcal{F}_m$, *the action of* K^{-1} *on* $\mathbb{C}_q[z_2, \ldots, z_m]^{(n)}$ *is the scalar operator, with scalar* $v^{(m-1)(1-N)+2n} = v^{(m-1)(1-N)} q^n$.

2.5. Proof of Theorem 1.

$$\begin{aligned}
J'_K(N) &\overset{\text{by Lemma 2.3}}{=} v^{\frac{w(\beta)(N-1)^2}{2}} \operatorname{tr}\big(p_0(\check{\tau}(\beta)K^{-1}), e_0 \otimes (W_N)^{\otimes(m-1)}\big) \\
&\overset{\text{under } \mathcal{F}_m,\text{ by Proposition 2.6}}{=} v^{w(\beta)(N-1)} \operatorname{tr}(p_0(\mathcal{E}_N(\Phi_{\rho(\gamma)})K^{-1}), \mathbb{C}_q[z_2, \ldots, z_m]) \\
&\overset{\text{by Lemma 2.7}}{=} v^{w(\beta)(N-1)} \operatorname{tr}(\mathcal{E}_N(\Phi_{\rho'(\gamma)})K^{-1}, \mathbb{C}_q[z_2, \ldots, z_m]) \\
&= v^{w(\beta)(N-1)} \sum_{n=0}^{\infty} \operatorname{tr}(\mathcal{E}_N(\rho'(\gamma))K^{-1}, \mathbb{C}_q[z_2, \ldots, z_m]^{(n)}) \\
&\overset{\text{by Lemma 2.8}}{=} v^{(w(\beta)-m+1)(N-1)} \sum_{n=0}^{\infty} q^n \operatorname{tr}\big(\mathcal{E}_N(\Phi_{\rho'(\gamma)}), \mathbb{C}_q[z_2, \ldots, z_m]^{(n)}\big) \\
&= v^{(w(\beta)-m+1)(N-1)} \sum_{n=0}^{\infty} \operatorname{tr}\big(\mathcal{E}_N(\Phi_{q\rho'(\gamma)}), \mathbb{C}_q[z_2, \ldots, z_m]^{(n)}\big) \\
&= v^{(w(\beta)-m+1)(N-1)} \mathcal{E}_N \sum_{n=0}^{\infty} \operatorname{tr}\big(\Phi_{q\rho'(\gamma)}, \mathbb{C}_q[z_2, \ldots, z_m]^{(n)}\big) \\
&\overset{\text{by the quantum MacMahon master theorem}}{=} v^{(w(\beta)-m+1)(N-1)} \mathcal{E}_N \frac{1}{\widetilde{\det}_q(I - q\rho'(\gamma))}.
\end{aligned}$$

This proves part (1) of Theorem 1. As for part (2), first note that the braid

$$\overleftarrow{\beta} := \sigma_{i_k}^{\varepsilon_k} \sigma_{i_{k-1}}^{\varepsilon_{k-1}} \ldots \sigma_{i_1}^{\varepsilon_1}$$

has closure knot the same as that of β. The Alexander polynomial of K is known to be equal to $\det(I - \bar{\rho}'(\overleftarrow{\beta}))$, where $\bar{\rho}$ is the Burau representation, and $\bar{\rho}'(\overleftarrow{\beta})$ is obtained from $\bar{\rho}(\overleftarrow{\beta})$ by removing the first row and column. We know that $\mathcal{E}(S_\pm)$ are the transpose Burau matrices; hence $\bar{\rho}(\overleftarrow{\beta}) = \mathcal{E}(\rho(\beta))^T$, the transpose of $\mathcal{E}(\rho(\beta))$. The statement now follows.

3. The Kashaev Invariant

3.1. Proof of Theorem 2.

3.1.1. Completion of $\mathcal{A}_\varepsilon$. Let $\mathcal{I}$ be the left ideal in $\mathcal{A}_\varepsilon$ generated by $a_1, a_2, \dots, a_k$, i.e.,

$$\mathcal{I} := a_1\mathcal{A}_\varepsilon + a_2\mathcal{A}_\varepsilon + \cdots + a_k\mathcal{A}_\varepsilon,$$

and let $\hat{\mathcal{A}}_\varepsilon$ be the $\mathcal{I}$-adic completion of $\mathcal{A}_\varepsilon$. Using the almost q-commutativity, it is easy to see that $\mathcal{I}$ is a two-sided ideal.

Lemma 3.1. *When the closure of $\beta(\gamma)$ is a knot, $\widetilde{\det}_q(I - q\rho'(\gamma))$ belongs to $1+\mathcal{I}$, and hence $\frac{1}{\widetilde{\det}_q(I-q\rho'(\gamma))}$ belongs to $\hat{\mathcal{A}}_\varepsilon$.*

Proof. It is enough to show that when $a_1 = a_2 = \dots a_k = 0$, $\widetilde{\det}_q(I - q\rho'(\gamma)) = 1$, or $\det_q(C) = 0$ for any main minor C of $\rho'(\gamma)$.

Let us call a *permutation-like* matrix a matrix C where on each row and on each column there is at most one nonzero entry. If, in addition, on each row and on each column there is exactly one nonzero entry, we say that C is nondegenerate. Every nondegenerate permutation-like square matrix C gives rise to a permutation matrix $p(C)$ by replacing all the nonzero entries with 1. It is clear that the product of (nondegenerate) permutation-like matrices is a (nondegenerate) permutation-like one. If C is a permutation-like $(m \times m)$-matrix, and D is a main minor, i.e., a submatrix of type $(J \times J)$, then D is also permutation-like. If, in addition, both C and D are nondegenerate, then $p(C)$ leaves J stable, i.e., $p(C)(J) = J$, since the restriction of $p(C)$ on J is equal to $p(D)$ which leaves J stable.

Also note that if C is a degenerate, permutation-like, right-quantum matrix, then $\det_q(C) = 0$.

When $a_1 = a_2 = \cdots = a_k = 0$, each of the matrices A_j (whose definition is in Sec. 2.2.2) is a nondegenerate permutation-like matrix. Hence $C = \rho(\gamma)$ is permutation-like. Note that $p(C)$ is exactly $\bar{\beta}$, the permutation corresponding to β. Because the closure of β is a *knot*, $\bar{\beta} = p(C)$ does not leave any proper subset of $\{1, 2, \dots, m\}$ stable. Hence any main minor D of $\rho'(\gamma)$, which itself is a proper main minor of $C = \rho(\gamma)$, is a degenerate permutation-like matrix. Hence $\det_q(D) = 0$. □

3.1.2. $\hat{\mathcal{A}}_\varepsilon$ and the Habiro ring.

Lemma 3.2.

1. *If $f \in \mathcal{A}_\varepsilon$ is divisible by a_j^d for some $1 \le j \le k$ and a positive integer d, then $\mathcal{E}(f)$ is divisible by $(1 - zq^r)_q^d$, and hence $\mathcal{E}_N(f)$ is divisible by $(1-q)_q^d$ for every integer N, not necessarily positive.*
2. *Suppose $n > dk$. Then $\mathcal{E}_N(f)$is divisible by $(1-q)_q^d$ for every integer N and every $f \in \mathcal{I}^n$. Hence $\mathcal{E}_N \hat{\mathcal{A}}_\varepsilon \in \widehat{\mathbb{Z}[q]}$.*

Proof.

(1) We assume that f is a *monomial* in the variables $a_1, b_1, c_1, a_2, \dots$. Using the almost q-commutativity, we move all a_j, b_j, and c_j to the right of f, so that $f = g b_j^s c_j^r a_j^d$ for some $g \in \mathcal{A}_\varepsilon$ not containing a_j, b_j, and c_j. Note that by Lemma 2.4

$$\mathcal{E}(f) = \mathcal{E}(g)\mathcal{E}(b_j^s c_j^r a_j^d)$$

is divisible by $\mathcal{E}(b_j^s c_j^r a_j^d)$. Note that a_j, b_j, and c_j are either a_+, b_+, and c_+ or a_-, b_-, and c_-. Using (12) and (13), we see that $\mathcal{E}_N(f)$ is divisible by $(1-q^l)_q^d$ for some integer l, which, in turn, is always divisible by $(1-q)_q^d$.

(2) Using the fact that generators a_j, b_j, c_j, $1 \le j \le k$, almost q-commute, it is easy to see that $\mathcal{I}^n$ is a two-sided ideal generated by $a_{s_1} a_{s_2} \cdots a_{s_n}$, where each s_i is one of $\{1, 2, \dots, k\}$. If $n > dk$, by the pigeon-hole principle, there is an index j such $a_{s_1} a_{s_2} \cdots a_{s_n}$ is divisible by a_j^d. Now the result follows from part (1). □

From Lemmas 3.2 and 3.1 we get the following.

Corollary 3.3. *Assume that N is an integer, not necessarily positive. Then*

$$\mathcal{E}_N\left(\frac{1}{\widetilde{\det}_q(I-q\rho'(\gamma))}\right)\in\widehat{\mathbb{Z}[q]}.$$

3.1.3. Proof of Theorem 2. Part (1) is a special case of Corollary 3.3, with $N=0$.

For part (2), first recall that $\langle K\rangle_N = J'_K(N)|_{q=\exp(2\pi i/N)}$. When $q=\exp(2\pi i/N)$, one has $q^N=1=q^0$. Thus, $\mathcal{E}_N=\mathcal{E}_0$ when $q=\exp(2\pi i/N)$. One has

$$J'_K(N)|_{q=\exp(2\pi i/N)} = v^{m-1-w(\beta)}\mathcal{E}_N(T)|_{q=\exp(2\pi i/N)} = v^{m-1-w(\beta)}\mathcal{E}_0(T)|_{q=\exp(2\pi i/N)},$$

where

$$T=\frac{1}{\widetilde{\det}_q(I-q\rho'(\gamma))}.$$

3.2. The Kashaev Invariant for Other Simple Lie Algebras. Fix a simple Lie algebra $\mathfrak{g}$. For every long knot K, presented by a $1-1$ tangle, one can define the $\mathfrak{g}$-universal invariant $\mathcal{J}_K^{\mathfrak{g}}$, which is a central element in an appropriate completion of the quantized universal enveloping algebra $U_v(\mathfrak{g})$ (see [14, 22]). Formally, $\mathcal{J}_{K,\mathfrak{g}}$ is an infinite sum of central elements in $U_v(\mathfrak{g})$:

$$\mathcal{J}_{K,\mathfrak{g}}=\sum_{n=0}^{\infty}\mathcal{J}_{K,\mathfrak{g}}^{(n)} \tag{14}$$

such that for any finite-dimensional simple $U_v(\mathfrak{g})$-module only the action of a finite number of terms is nonzero. Hence for a finite-dimensional simple module $U_v(\mathfrak{g})$-module V, $\mathcal{J}_K^{\mathfrak{g}}$ acts as a scalar times the identity. It can be shown that the scalar is a Laurent polynomial in q. Denote this scalar by $J'_{K,\mathfrak{g}}(V)$. One always has $J_{K,\mathfrak{g}}(V)=J'_{K,\mathfrak{g}}(V)\dim_q(V)$, when $J_{K,\mathfrak{g}}(V)$ is the usual quantum invariant of K colored by V and $\dim_q(V)$ is the quantum dimension, i.e., the invariant of the unknot colored by V.

For any Verma module V_λ of highest weight λ (an element in the weight lattice), the action of each of the $\mathcal{J}_{K,\mathfrak{g}}^{(n)}$ is still in $\mathcal{R}=\mathbb{Z}[q^{\pm 1}]$, but, in general, infinitely many of them are nonzero. In this case, $J'_{K,\mathfrak{g}}(V_\lambda)$ is an infinite series (sum). In a forthcoming work, we will show that $J'_{K,\mathfrak{g}}(V_\lambda)\in\widehat{\mathbb{Z}[q]}$; the special case where $\mathfrak{g}=\mathrm{sl}_2$ has been proved here by Corollary 3.3.

Note that if the weight λ is dominant, then $J'_{K,\mathfrak{g}}(V_\lambda)=J'_{K,\mathfrak{g}}(W_\lambda)$, where W_λ is the finite-dimensional $U_v(\mathfrak{g})$ module with highest weight λ. The reason is that both are the scalar of the same scalar operator acting on V_λ and its quotient W_λ. In this case, $J'_{K,\mathfrak{g}}(V_\lambda)$ is a Laurent polynomial in q. It is known that $\mathcal{R}=\mathbb{Z}[q^{\pm 1}]\subset\widehat{\mathbb{Z}[q]}$ (see [7]).

Due to the Weyl symmetry, we see that if w is in the Weyl group, then $J'_{K,\mathfrak{g}}(V_\lambda)=J'_{K,\mathfrak{g}}(V_{w\cdot\lambda})$, where $w\cdot\lambda$ is the dot action of the Weyl group (see [8]). If λ is not fixed (under the dot action) by any element of the Weyl group, then $\lambda=w\cdot\mu$ for some dominant μ, and hence $J'_{K,\mathfrak{g}}(V_\lambda)=J'_{K,\mathfrak{g}}(V_\mu)$. In this case, $J'_{K,\mathfrak{g}}(V_\lambda)$ might still be an infinite series, but it is equal to a Laurent polynomial, which is $J'_{K,\mathfrak{g}}(V_\mu)$ in the Habiro ring $\widehat{\mathbb{Z}[q]}$.

The more interesting and less understood case is when λ is fixed by an element of the Weyl group, i.e., λ is on a wall of a *shifted* Weyl chamber. Among them there is one special weight, namely $\lambda=-\delta$, where δ is the half-sum of positive roots, since $-\delta$ is the only element invariant by *all* elements of the Weyl group. When $\mathfrak{g}=\mathrm{sl}_2$, $V_{-\delta}$ is V_0 in Sec. 2, and $J'_{K,\mathfrak{g}}(V_{-\delta})$ is the Kashaev invariant in this case, according to Theorem 2. Note that $V_{-\delta}$ is always infinite-dimensional and irreducible; it is certainly a very special $U_v(\mathfrak{g})$-module.

Thus, a natural generalization of the Kashaev invariant to other simple Lie algebras is $J'_{K,\mathfrak{g}}(V_{-\delta})$. More precisely, let us define the $\mathfrak{g}$-Kashaev invariant by

$$\langle K\rangle_N^{\mathfrak{g}} := J'_{K,\mathfrak{g}}(V_{-\delta})|_{q=\exp(2\pi i/N)}$$

and suggest the following $\mathfrak{g}$-volume conjecture:

$$\lim_{N\to\infty} \frac{|\langle K\rangle_N^{\mathfrak{g}}|}{N} = c_{\mathfrak{g}}\,\mathrm{Vol}(K),$$

where $c_{\mathfrak{g}}$ is a constant depending only on the simple Lie algebra $\mathfrak{g}$.

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