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Article Author: Le Ty Kuok Tkhang,

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Thang Le (tl117)
Skiles Bldg.
686 Cherry St.
Atlanta, GA 30332-0160

Faculty
Math

Росс. Акад. Наук Матем. Сборник
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VARIETIES OF REPRESENTATIONS AND THEIR COHOMOLOGY-JUMP SUBVARIETIES FOR KNOT GROUPS

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LE TY KUOK TKHANG

ABSTRACT. This paper studies the spaces of representations of knot groups into a linear group $\mathrm{GL}_n(\mathbb{C})$, their categorical factor-spaces (i.e., the spaces of all characters of the representations), and their cohomology-jump subspaces. Connections are established between the latter and the spaces of representations of dimension one greater. A complete description is given of these spaces for 2-bridge knots.

Bibliography: 12 titles.

0. INTRODUCTION

In recent years the spaces of all representations of a finitely generated group π into a linear group $\mathrm{GL}_n(\mathbb{C})$ have been studied intensively. These spaces are algebraic subsets in $\mathbb{C}^{n^2p}$ for some $p \in \mathbb{N}$, and with their help many invariants of the given group can be constructed.

Each representation of the given group determines cohomology groups of the latter with coefficients in the representation. The rank of the ith such cohomology group is constant throughout the space of representations except for a certain closed subset, which we call the ith cohomology-jump subvariety.

The study of these cohomology-jump subvarieties was begun by S. P. Novikov (see [1]). Already in the case of 1-dimensional representations there are nontrivial results. If π is a knot group, then the cohomology-jump subvariety in the space of representations of dimension 1 consists of the roots of the Alexander polynomial of the knot. This is a reformulation of classical results (see §2.3 below). A natural question to pose is that of the role of the cohomology-jump subvarieties for a knot group with coefficients in multidimensional representations. The object of this paper is to study these cohomology-jump subvarieties.

In §1 we lay out the theory of the spaces of representations of finitely generated groups and their categorical factor-spaces. In §2 we study the cohomology-jump subvariety, proving in particular that it is Zariski-closed, and we indicate the connection of this subvariety with the space of representations of the same group of dimension one greater. In §3 we describe the spaces of representations for 2-bridge knots; another description of these spaces is given in [8], but we use a different approach to the computation that in our view is more natural and more suitable for our purposes. In §4 we describe in explicit form the cohomology-jump subvarieties for 2-bridge knots and present some examples.

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1. The space of representations of a group, and the corresponding space of characters

1.1. An algebraic variety will be understood in the most naive meaning of the term—simply the set of zeros of a system of polynomial equations in $\mathbb{C}^n$ or an open subset thereof. The topology will always be understood in the sense of Zariski, unless otherwise stated.

Let π be a group, with presentation

$$\pi = \langle a_1, a_2, \dots, a_p \mid r_1, \dots, r_q \rangle;$$

if π is a knot group, we assume that $q = p-1$ and all the generators $a_1, a_2, \dots, a_p$ are conjugate to each other. We denote by $R_n(\pi)$ (resp. $\mathrm{SR}_n(\pi)$) the set of all homomorphisms of π into $\mathrm{GL}_n(\mathbb{C})$ (resp. $\mathrm{SL}_n(\mathbb{C})$). An action of the group $\mathrm{GL}_n(\mathbb{C})$ is defined on these spaces: if $g \in \mathrm{GL}_n$ and $\rho \in R_n(\pi)$, then $g(\rho) = g\rho g^{-1}$.

It is known (see [6]) that if two semisimple representations of the group π into a linear group have the same characters, they are conjugate to each other. The character of any representation coincides with that of its semisimple part. The set of all characters of representations of π into GL_n is somewhat smaller than the set of equivalence classes of representations, but is more convenient to use, because it can be parametrized as an algebraic variety (see [6]). We denote the set of all characters of representations of π into GL_n (resp. SL_n) by $\mathrm{X}_n(\pi)$ (resp. $\mathrm{SX}_n(\pi)$). There is a natural mapping pr: $R_n(\pi) \to \mathrm{X}_n(\pi)$, taking each representation into its character. The mapping pr (and its restriction to SR_n) is regular and also submersive, i.e., a set $Y \subset \mathrm{X}_n(\pi)$ is open if and only if $\mathrm{pr}^{-1}(Y)$ is open. $\mathrm{SR}_n(\pi)$ and $\mathrm{SX}_n(\pi)$ are closed algebraic varieties.

Let $\mathbb{Z}_n = \{z \mid z^n = 1\}$. If ρ is a representation of π into SL_n, then obviously $z\rho$ is likewise such a representation, where $(z\rho)(c) = z\rho(c)$; i.e., the group $\mathbb{Z}_n$ acts on $\mathrm{SR}_n(\pi)$. It also acts on $\mathrm{SX}_n(\pi)$. If π is a knot group, then $H^1(\pi, \mathbb{Z}) = \mathbb{Z}$ and $\mathrm{Hom}(\pi, \mathbb{C}^*) = \mathbb{C}^*$; and if $\rho_0 \in \mathrm{Hom}(\pi, \mathbb{C}^*)$, then $z\rho_0 \in \mathrm{Hom}(\pi, \mathbb{C}^*)$ where $(z\rho_0)(c) = z^{-1}(\rho_0(c))$; i.e., the group $\mathbb{Z}_n$ acts on $\mathbb{C}^*$, and therefore on $\mathrm{SX}_n \times \mathrm{Hom}(\pi, \mathbb{C}^*) = \mathrm{SX}_n \times \mathbb{C}^*$.

1.1.1. **Proposition.** *Let π be a knot group. Then the variety $\mathrm{X}_n(\pi)$ is the quotient of the variety $\mathrm{SX}_n \times \mathbb{C}^*$ by the action of the group $\mathbb{Z}_n$.*

Proof. Consider the mapping h: $\mathrm{SX}_n \times \mathbb{C}^* \to X_n$, $h(x, \rho_0) = \rho_0 x$, where $\rho_0 x = \mathrm{pr}(\rho_0\rho)$, $\rho \in \mathrm{pr}^{-1}(x)$. Note that $\mathrm{tr}[(\rho_0\rho)(c)] = \rho_0(c)\,\mathrm{tr}(\rho(c))$.

Obviously h is surjective, and if $z(x, \rho_0) = (x', \rho_0')$, then $h(x, \rho_0) = h(x', \rho_0')$.

Conversely, suppose $h(x, \rho_0) = h(x', \rho_0')$. Let ρ, ρ' be two representations such that $\mathrm{pr}(\rho) = x$, $\mathrm{pr}(\rho') = x'$. Then $\rho\rho_0 = \rho'\rho_0'$, $\det(\rho\rho_0) = \det(\rho'\rho_0')$, $\det(\rho_0\rho) = \rho_0^n \det\rho$, and therefore $\rho_0^n = (\rho_0')^n$. This means that there exists a $z \in \mathbb{Z}_n$ such that $\rho_0 = z^{-1}\rho_0'$; then $\rho = z\rho'$, i.e., $z(x', \rho_0') = (x, \rho_0)$.

1.1.2. **Proposition.** *Suppose $V \subset R_n$ is closed and invariant under the action of the group GL_n. Then $\mathrm{pr}(V)$ is a closed subset of X_n.*

Proof. Consider an $x \in \mathrm{X}_n$ that does not belong to $\mathrm{pr}(V)$. Then both sets $\mathrm{pr}^{-1}(x)$ and V are closed and invariant. Hence by Lemma 1 in Chapter 4, §1, of [10], there exists an invariant regular function $\tilde{f}$: $R_n \to \mathbb{C}$ such that $\tilde{f}|_V = 0$ and $\tilde{f}|_{\mathrm{pr}^{-1}(x)} = 1$. Dropping to X_n, we obtain a function f: $\mathrm{X}_n \to \mathbb{C}$ such that $f(x) = 1$, $f(\mathrm{pr}(V)) = 0$, and f is regular on X_n (since the ring of regular functions on X_n coincides with the ring of invariant functions on R_n; see [6]). This means that there exists a Zariski-open neighborhood containing the point x and not intersecting the set $\operatorname{pr}(V)$. Thus, $\operatorname{pr}(V)$ is Zariski-closed.

1.2. **The subspaces of reducible representations.** Let τ be a partition of the number n, $\tau = (n_1, n_2, \dots, n_l)$, $n_1 + \dots + n_l = n$, and let μ be a set $(m_1, \dots, m_k)$ such that $0 < m_1 < \dots < m_k = n$. Every such set determines a partition of n, by putting $n_1 = m_1$, $n_2 = m_2 - m_1, \dots, n_k = m_k - m_{k-1}$; we call this the partition induced by the set μ. In general, many different sets of increasing numbers determine the same partition; for example, the sets $(1, 3)$ and $(2, 3)$ determine the same partition of the number 3: $(1, 2) = (2, 1)$.

We say that a representation ρ is subordinate to the set $\mu = (m_1, \dots, m_k)$ if there exists a basis in which ρ has the block form

$$\rho(a_i) = \begin{pmatrix} * & & & \\ & * & & \\ & & \ddots & \\ 0 & & & * \end{pmatrix}$$

where the elements below the diagonal are zero and the dimensions of the blocks are, respectively, $m_1 \times m_1$, $(m_2 - m_1) \times (m_2 - m_1), \dots, (m_k - m_{k-1}) \times (m_k - m_{k-1})$. We shall write $\rho \le \mu$.

Let $R_\mu(\pi) = \{\rho \in R_n(\pi),\ \rho \le \mu\}$.

1.2.1. **Proposition.** *The set $R_\mu(\pi)$ is closed in the Zariski topology of $R_n(\pi)$.*

Proof. Let E be the set of flags of dimensions $(m_1, m_2, \dots, m_k)$ in $\mathbb{C}^n$. Then E is a (closed) projective variety. Consider the mapping

$$f\colon E \times R_n(\pi) \to E^{p+1},$$
$$f(\alpha, \rho) = (\alpha, \rho(a_1)\alpha, \rho(a_2)\alpha, \dots, \rho(a_p)\alpha).$$

The condition $\rho \le \mu$ is equivalent to the requirement that all the mappings $\rho(a_i)$, $i = 1, \dots, p$, preserve some point of E. Let D be the diagonal in E^{p+1}: $D = \{(\alpha, \alpha, \dots, \alpha) \in E^{p+1}\}$; D is closed in E^{p+1}. Let $p_2\colon E \times R_n(\pi) \to R_n(\pi)$ be a projection onto the second factor. Then $R_\mu = p_2(f^{-1}(D))$, since E is a projective variety, p_2 is a closed mapping, and so R_μ is a closed subset of $R_n(\pi)$.

Let τ be a partition of the number n. We say that a representation ρ is subordinate to τ if ρ is subordinate to some set μ that induces τ. Let X_τ be the set of characters of all the representations subordinate to τ. Then $X_\tau = \operatorname{pr}(\bigcup R_\mu)$, where μ runs over the collection of sets that induce the partition τ.

1.2.2. **Corollary.** *X_τ is closed in the space $X_n(\pi)$.*

This follows from the fact that pr is a submersion.

1.2.3. **Corollary.** *The set R_n^s of irreducible representations is open in R_n. Similarly, the set X_n^s is open in X_n.*

2. The cohomology-jump subvariety

2.1. Every representation $\rho\colon \pi \to \mathrm{GL}_n$ turns the vector space $\mathbb{C}^n$ into a left $\mathbb{Z}[\pi]$-module. We can therefore define the cohomology groups $H^i(\pi, \rho)$ with coefficients in this module. They are the cohomology groups $H^i(K(\pi, 1), \rho)$ of the space $K(\pi, 1)$ with coefficients in ρ. The groups $H^i(\pi, \rho)$ do not change if ρ is replaced by a conjugate representation. Hence the function $\mathrm{rk}_i(x) = \mathrm{rk}\, H^i(\pi, \rho)$, where $\mathrm{pr}(\rho) = x$, is well defined on the set $\mathrm{X}_n^s(\pi)$ of characters of all simple (i.e., irreducible) representations. On the rest of the variety $\mathrm{X}_n(\pi)$ this function is no longer well defined, since every point of this part of $\mathrm{X}_n(\pi)$ corresponds, in general, to many nonequivalent representations.

We define $\mathrm{rk}_i(x)$ (for any $x \in \mathrm{X}_n(\pi)$) as the greatest of the ranks of the groups $H^i(\pi, \rho)$, where $\mathrm{pr}(\rho) = x$.

2.1.1. **Proposition.** *Let* $\Pi_n^{i,k}$ *be the subset (in* $\mathrm{X}_n(\pi)$*) of all points* x *such that* $\mathrm{rk}_i(x) \geq k$, $i, k \in \mathbb{Z}$, $i, k \geq 0$. *Then* $\Pi_n^{i,k}$ *is a closed algebraic subset. In other words, the function* rk_i *is upper semicontinuous.*

Proof. Let $\widetilde{\Pi}_n^{i,k} = \{\rho \in R_n(x), \mathrm{rk}\, H^i(\pi, \rho) \geq k\}$. Then $\widetilde{\Pi}_n^{i,k}$ is a closed subset. This fact appears (in a different form) in [1]. The subset $\widetilde{\Pi}_n^{i,k}$ is obviously invariant under the action of GL_n; therefore, $\mathrm{pr}(\widetilde{\Pi}_n^{i,k})$ is closed in X_n, by Proposition 1.1.3. By definition, $\mathrm{pr}(\widetilde{\Pi}_n^{i,k})$ is precisely $\Pi_n^{i,k}$. Thus, $\Pi_n^{i,k}$ is closed in X_n.

Let k be the smallest value of the function rk_i on the variety X_n. The subset $\Pi_n^{i,k+1}$ will be called the jump subvariety of the ith cohomology group.

2.1.2. **Proposition.** *Suppose* $x \in \mathrm{X}_n(\pi)$, *and let* $\rho \in R_n(\pi)$ *be a representative of the unique class of semisimple representations with character* x. *Then* $\mathrm{rk}_i(x) = \mathrm{rk}_i(\rho)$.

Proof. We must show that if $\tau \in \mathrm{pr}^{-1}(x)$, then $\mathrm{rk}_i(\tau) \leq \mathrm{rk}_i(\rho)$. Consider the orbits $O(\rho)$ and $O(\tau)$ in $R_n(\pi)$ (remember that the group GL_n acts on $R_n(\pi)$). By Lemma 1.2.6 of [6], $O(\rho)$ is closed; $O(\tau)$ is not closed if τ is nonsemisimple, but $\overline{O(\tau)} \supset O(\rho)$, so that $\rho \in \overline{O(\tau)}$. Since the function rk_i is upper semicontinuous and its restriction to $O(\tau)$ is constant, we have $\mathrm{rk}_i(\tau) \leq \mathrm{rk}_i(\rho)$.

2.2. **Cohomology in small dimensions; zero-dimensional cohomology.** We recall the construction for computing the cohomology of our group in small dimensions. Consider a 2-dimensional cell complex X consisting of a single 0-cell O, p 1-cells $a_1, a_2, \dots, a_p$, and q 2-cells $c_1, c_2, \dots, c_q$ such that $\partial c_i = r_i$. Let $\widetilde{X}$ be the universal covering of X, $\widetilde{O}$ a fixed point over O, and $\tilde{a}_i$, $\tilde{c}_j$ liftings of the cells a_i, c_j from $\widetilde{O}$. Then $\widetilde{X}$ is a free complex over the ring $\mathbb{Z}[\pi]$, $C_0(\widetilde{X}) = \mathbb{Z}[\pi]$, $C_1(\widetilde{X}) = (\mathbb{Z}[\pi])^p$ with generators $\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_p$, and $C_2(\widetilde{X}) = (\mathbb{Z}[\pi])^q$ with generators $\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_q$. We have the sequence

$$0 \to C_2(\widetilde{X}) \xrightarrow{\partial_1} C_1(\widetilde{X}) \xrightarrow{\partial_0} C_0(\widetilde{X}) \to 0.$$

The boundary operator is given by the formulas

$$\partial_0(\tilde{a}_i) = (a_i - 1)\widetilde{O}, \qquad \partial_1(\tilde{c}_j) = \sum_{i=1}^{p} \left(\frac{\partial r_j}{\partial a_i}\right) \tilde{a}_i$$

($\partial r_j / \partial a_i$ is an element of $\mathbb{Z}[\pi]$), where $\partial / \partial a_i$ is Fox differentiation (see [4]).

Now, if $\rho\colon \pi \to \mathrm{GL}_n$ is a representation of the group, $\mathbb{C}^n$ becomes a left $\mathbb{Z}[\pi]$-module. If $\alpha \in \mathbb{Z}[\pi]$, then $\rho(\alpha)$ is an endomorphism of the module $\mathbb{C}^n$. Consider the complex

$$0 \leftarrow (\mathbb{C}^n)^q \xleftarrow{\rho(\partial_1)} (\mathbb{C}^n)^p \xleftarrow{\rho(\partial_0)} \mathbb{C}^n \leftarrow 0,$$

where $\rho(\partial_1)$ is an $nq \times np$ matrix, and $\rho(\partial_0)$ an $np \times n$ matrix. The first two cohomology groups of this complex are the groups $H^0(\pi, \rho)$ and $H^1(\pi, \rho)$.