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The perturbative invariants of rational homology 3-spheres can be recovered from the LMO invariant

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Abstract

We show that the perturbative $\mathfrak{g}$ invariant of rational homology 3-spheres can be recovered from the Le-Murakami-Ohtsuki (LMO) invariant for any simple Lie algebra $\mathfrak{g}$, that is, the LMO invariant is universal among the perturbative invariants. This universality was conjectured in Le, Murakami and Ohtsuki ['On a universal perturbative invariant of 3-manifolds', *Topology* 37 (1998) 539–574]. Since the perturbative invariants dominate the quantum invariants of integral homology 3-spheres [K. Habiro, 'On the quantum sl_2 invariants of knots and integral homology spheres', *Invariants of knots and 3-manifolds* (Kyoto 2001), Geometry and Topology Monographs 4 (Geometry and Topology Publications, Coventry, 2002) 161–181; K. Habiro, 'A unified Witten–Reshetikhin–Turaev invariant for integral homology spheres', *Invent. Math.* 171 (2008) 1–81; K. Habiro and T. T. Q. Le, in preparation], the LMO invariant dominates the quantum invariants of integral homology 3-spheres.

1. *Introduction*

In the late 1980s, Witten [35] proposed topological invariants of a closed 3-manifold M for a simple compact Lie group G, which is formally presented by a path integral whose Lagrangian is the Chern–Simons functional of G connections on M. There are two approaches to obtain mathematically rigorous information from a path integral: the operator formalism and the perturbative expansion. Motivated by the operator formalism of the Chern–Simons path integral, Reshetikhin and Turaev [33] gave the first rigorous mathematical construction of quantum invariants of 3-manifolds, and, after that, rigorous constructions of quantum invariants of 3-manifolds were obtained by various approaches. When M is obtained from S^3 by surgery along a framed knot K, the quantum G invariant $\tau_r^G(M)$ of M is defined to be a linear sum of the quantum $(\mathfrak{g}, V_\lambda)$ invariant $Q^{\mathfrak{g},V_\lambda}(K)$ of K at an rth root of unity, where $\mathfrak{g}$ is the Lie algebra of G and V_λ denotes the irreducible representation of $\mathfrak{g}$ whose highest weight is λ. On the other hand, the perturbative expansion of the Chern–Simons path integral suggests that we can obtain the perturbative $\mathfrak{g}$ invariant (a power series) when we fix $\mathfrak{g}$ and obtain the Le-Murakami-Ohtsuki (LMO) invariant (an infinite linear sum of trivalent graphs) when we make the perturbative expansion without fixing $\mathfrak{g}$. As a mathematical construction, we can define the perturbative $\mathfrak{g}$ invariant $\tau^{\mathfrak{g}}(M)$ of a rational homology 3-sphere M by arithmetic perturbative expansion of $\tau_r^{PG}(M)$ as $r \to \infty$ (see [25, 29, 34]), where PG denotes the quotient of G by its centre. Further, we can present the LMO invariant $\hat{Z}^{\mathrm{LMO}}(M)$ (see [27]) of a rational homology 3-sphere M by the Aarhus integral [4]. It was conjectured [27] that the perturbative $\mathfrak{g}$ invariant can be recovered from the LMO invariant by the weight system $\hat{W}_{\mathfrak{g}}$ for any simple Lie algebra $\mathfrak{g}$. In the $\mathfrak{sl}_2$ case, this has been shown in [30]; see Figures 1 and 2, for these invariants and the relations among them.

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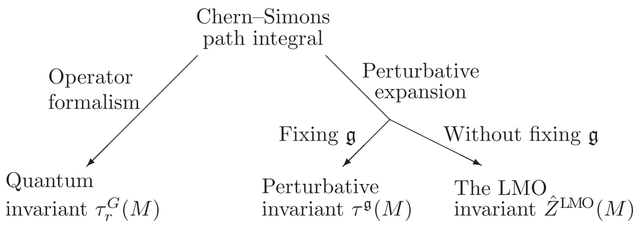


FIGURE 1. *Physical background.*

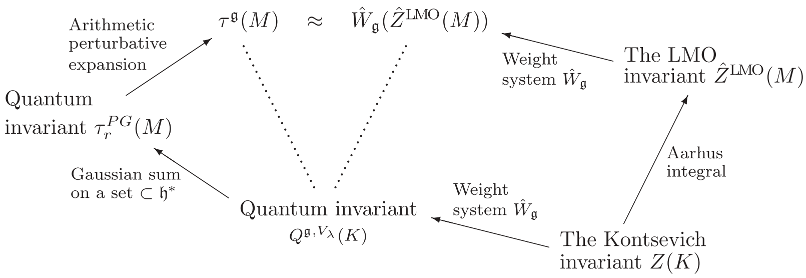


FIGURE 2. *Mathematical construction.*

The aim of this paper is to show Theorem 1.1; see also the first remark at the end of this section.

THEOREM 1.1 (see [**3**, **22**]). *Let $\mathfrak{g}$ be any simple Lie algebra. Then, for any rational homology 3-sphere M,*

$$\hat{W}_{\mathfrak{g}}(\hat{Z}^{\mathrm{LMO}}(M)) = |H_1(M;\mathbb{Z})|^{(\dim \mathfrak{g} - \mathrm{rank}\, \mathfrak{g})/2} \tau^{\mathfrak{g}}(M),$$

where $|H_1(M;\mathbb{Z})|$ denotes the cardinality of the first homology group $H_1(M;\mathbb{Z})$ of M.

We give two proofs of the theorem: a geometric proof (Subsections 4.1 and 5.2) and an algebraic proof (Subsections 4.2 and 5.1). The theorem implies that the LMO invariant dominates the perturbative invariants. Further, since the perturbative invariants dominate the quantum Witten–Reshetikhin–Turaev invariants of integral homology 3-spheres [**12**, **13**] (see the forthcoming paper by Habiro and Le), it follows from the theorem that the LMO invariant dominates the quantum invariants of integral homology 3-spheres; see the second remark at the end of this section for rational homology 3-spheres.

Let us explain a sketch of the proof when M is obtained from S^3 by surgery along a knot. The LMO invariant $\hat{Z}^{\mathrm{LMO}}(M)$ can be presented by the Aarhus integral [**4**]. It is shown from this presentation that the image $\hat{W}_{\mathfrak{g}}(\hat{Z}^{\mathrm{LMO}}(M))$ can be presented by an integral of Gauss type over the dual $\mathfrak{g}^*$ or, alternatively, by an expansion given in terms of the Laplacian $\Delta_{\mathfrak{g}^*}$ of $\mathfrak{g}^*$. On the other hand, as we explain in Subsection 6.2, the perturbative invariant $\tau^{\mathfrak{g}}(M)$ is presented by a Gaussian integral over $\mathfrak{h}^*$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ or, alternatively, by an expansion given in terms of the Laplacian $\Delta_{\mathfrak{h}^*}$ of $\mathfrak{h}^*$. We then show that $\hat{W}_{\mathfrak{g}}(\hat{Z}^{\mathrm{LMO}}(M)) = \tau^{\mathfrak{g}}(M)$ by establishing a result relating integrals over $\mathfrak{g}^*$ and integrals over $\mathfrak{h}^*$, similar to the well-known Weyl reduction integration formula. Alternatively, we show $\hat{W}_{\mathfrak{g}}(\hat{Z}^{\mathrm{LMO}}(M)) = \tau^{\mathfrak{g}}(M)$ by using Harish-Chandra's radial component formula (also known as Harish-Chandra's restriction formula) that relates the Laplacian $\Delta_{\mathfrak{g}^*}$ on $\mathfrak{g}^*$ to the Laplacian $\Delta_{\mathfrak{h}^*}$ on $\mathfrak{h}^*$. For a sketch of the algebraic proof, see Figure 3.

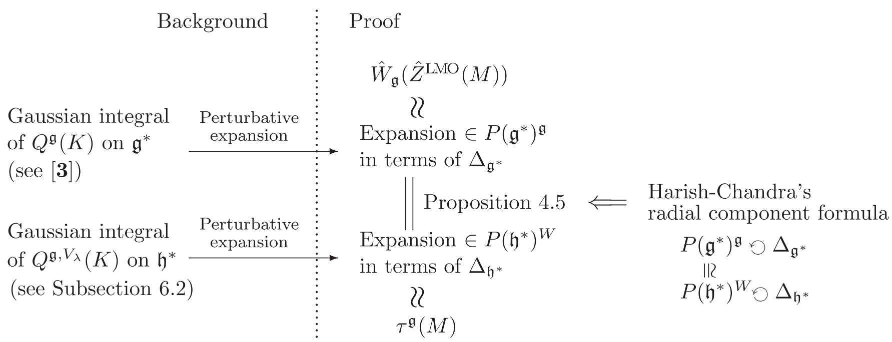


FIGURE 3. *Sketch of the algebraic proof of Theorem* 1.1, *when M is obtained from S^3 by surgery along a framed knot K.*

In the case when M is obtained from S^3 by surgery along a link, we present two proofs. The first one is more algebraic. We reduce the theorem to the case of surgery along knots by using the fact that the operators involved are invariant under the action of $\mathfrak{g}$. The other proof has quite a different flavour. We show that two multiplicative finite-type invariants of rational homology spheres are the same if they agree on the set of rational homology spheres obtained from S^3 by surgery along knots (for finer results, see Theorem 5.4). This result is also interesting by itself. The theorem then follows, since both $\hat{W}_{\mathfrak{g}}(\hat{Z}^{\mathrm{LMO}}(M))$ and $\tau^{\mathfrak{g}}(M)$, up to any degree, are finite type. This part relates the paper to the origin of the theory: the discovery of the perturbative invariant of homology 3-spheres for SO(3) case [**29**], leads the third author to define finite-type invariants of 3-manifolds.

The paper is organized as follows. In Section 2, we review definitions of terminologies and show some properties of Jacobi diagrams. In Section 3, we present the proof of the main theorem, based on the results proved later. We consider the knot case in Section 4 and the link case in Section 5. In Section 6, we discuss how the perturbative invariant can be obtained as an asymptotic expansion of the Witten–Reshetikhin–Turaev invariant and give a proof that our formula of the perturbative invariant is coincident with that given in [**25**]. We also show that finite parts of the perturbative invariant $\tau^{\mathfrak{g}}$ are of finite type.

REMARK. It was announced in [**3**] that the perturbative $\mathfrak{g}$ invariant can be recovered from the LMO invariant. However, that proof is not yet published. The first author [**22**] showed a proof, but his proof is partially incomplete. The aim of this paper is to show a complete proof of the theorem.

REMARK. For rational homology 3-spheres, it is known [**7**] that the quantum Witten-Reshetikhin-Turaev (WRT) invariant $\tau_r^{\mathrm{SO}(3)}(M)$, at roots of unity of order co-prime to the order of the first homology group, can be obtained from the perturbative invariant $\tau^{\mathfrak{sl}_2}(M)$. Hence, the LMO invariant $\hat{Z}^{\mathrm{LMO}}(M)$ dominates $\tau_r^{\mathrm{SO}(3)}(M)$ for those roots of unity.

2. *Preliminaries*

We recall basic facts about Lie algebras in Subsection 2.1 and theory of the Kontsevich invariant in Subsection 2.3. We introduce Laplacian operators in Euclidean spaces in Subsection 2.2 and present the LMO invariant in Subsection 2.5.

2.1. *Lie algebra*

We review the known facts about simple compact Lie algebras, mainly to fix the notations.

In this paper, G is a compact, connected, simple Lie group, $\mathfrak{g}$ its Lie algebra and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, that is, the Lie algebra of a maximal torus in G. Let $\mathfrak{g}^*$ and $\mathfrak{h}^*$ be, respectively, the $\mathbb{R}$-dual of $\mathfrak{g}$ and $\mathfrak{h}$.

The complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g} \oplus \sqrt{-1}\,\mathfrak{g}$ is a simple complex Lie algebra. Let $\mathcal{J} : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ be the $\mathbb{R}$-linear map defined by $\mathcal{J}(x) = \sqrt{-1}\,x$. As vector spaces over $\mathbb{R}$, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus \mathcal{J}\mathfrak{g}$. The subalgebra $\mathfrak{h}_{\mathbb{C}} = \mathfrak{h} \oplus \mathcal{J}\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Every $\mathbb{R}$-linear functional on $\mathfrak{h}$ extends $\mathbb{C}$-linearly to a unique $\mathbb{C}$-valued functional on $\mathfrak{h}_{\mathbb{C}}$. Hence, we can consider $\mathfrak{h}^*$ as an $\mathbb{R}$-subspace of ${\mathfrak{h}_{\mathbb{C}}}^*$, and ${\mathfrak{h}_{\mathbb{C}}}^* = \mathfrak{h}^* \oplus \mathcal{J}(\mathfrak{h}^*)$, where again $\mathcal{J} : {\mathfrak{g}_{\mathbb{C}}}^* \to {\mathfrak{g}_{\mathbb{C}}}^*$ is the multiplication by $\sqrt{-1}$.

The Killing form B of $\mathfrak{g}_{\mathbb{C}}$ is negative definite on $\mathfrak{g}$ and positive definite on $\mathcal{J}\mathfrak{g}$. We define an inner product on $\mathfrak{g}$ by $(x, y) := -B(x, y)$. We also consider $\mathcal{J}\mathfrak{g}$ as a Euclidean space, where the inner product is the restriction of B. Then $\mathcal{J} : \mathfrak{g} \to \mathcal{J}\mathfrak{g}$ is an isomorphism of Euclidean spaces. Using $(\cdot, \cdot)$, we can identify $\mathfrak{h}^*$ with a subspace of $\mathfrak{g}^*$. For a function g on $\mathfrak{g}^*$, its restriction to $\mathfrak{h}^*$ will be denoted by $\mathcal{P}(g)$.

Let $\Phi \subset {\mathfrak{h}_{\mathbb{C}}}^*$ be the root system associated with $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. It is known that Φ is purely complex, that is, if $\alpha \in \Phi$ and $x \in \mathfrak{h}$, then $\alpha(x) \in \sqrt{-1}\,\mathbb{R}$. In other words, $\Phi \subset \mathcal{J}(\mathfrak{h}^*)$. In fact, the $\mathbb{R}$-span of Φ is $\mathcal{J}(\mathfrak{h}^*)$. Let Φ_+ be a set of positive roots of $\mathfrak{g}_{\mathbb{C}}$ and ϕ_+ be the number of positive roots of $\mathfrak{g}$. One has $\phi_+ = (\dim \mathfrak{g} - \dim \mathfrak{h})/2 = (\dim \mathfrak{g} - \operatorname{rank} \mathfrak{g})/2$. Following the common convention in Lie algebra theory (see, for example, [**14**]), we call $\beta \in \mathfrak{h}^*$, a *real root*, a *real positive root*, or a *real dominant weight*, if $\mathcal{J}(\beta)$ is, respectively, a root, a positive real root, or a dominant weight. Let $\Phi_+^{\mathbb{R}} = (\mathcal{J})^{-1}(\Phi_+)$ be the set of positive real roots, and let ρ be the half-sum of all positive real roots. For a dominant real weight $\lambda \in \mathfrak{h}^*$, let V_λ be the irreducible representation of $\mathfrak{g}_{\mathbb{C}}$ whose highest weight is $\mathcal{J}(\lambda)$.

The Weyl group of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ is the group acting on $\mathbb{R}\Phi = \mathcal{J}(\mathfrak{h}^*)$ generated by reflections in the walls perpendicular to root vectors $\alpha \in \Phi$. Using the isomorphism $(\mathcal{J})^{-1} : \mathcal{J}(\mathfrak{h}^*) \to \mathfrak{h}^*$, we define the action of W on $\mathfrak{h}^*$, and then on $\mathfrak{h}$ by the isomorphism $\mathfrak{h}^* \to \mathfrak{h}$ induced by the inner product. This action of W on $\mathfrak{h}$ coincides with the action defined by the normalizer of H (in G) on $\mathfrak{h}$, where H is the maximal torus of G whose Lie algebra is $\mathfrak{h}$; see, for example, [**20**].

The following W-skew-invariant function $\mathcal{D}$ on $\mathfrak{h}^*$ is important to us:

$$\mathcal{D}(\lambda) := \prod_{\alpha \in \Phi_+^{\mathbb{R}}} \frac{(\lambda, \alpha)}{(\rho, \alpha)}.$$

When $\lambda - \rho$ is a dominant real weight, $\mathcal{D}(\lambda)$ is the dimension of $V_{\lambda-\rho}$.

A source of functions on $\mathfrak{h}^*$ is given by the enveloping algebra $U(\mathfrak{g})$. For $g \in U(\mathfrak{g})$, we define a polynomial function on $\mathfrak{h}^*$ as follows: suppose that $\lambda - \rho$ is a dominant real weight. There is a unique polynomial function, denoted also by g, on $\mathfrak{h}^*$ such that $g(\lambda)$ is equal to the trace of the action of g on the $\mathfrak{g}_{\mathbb{C}}$-module $V_{\lambda-\rho}$. A proof of this fact is given in the Appendix.

Let $S(\mathfrak{g})$ be the symmetric tensor algebra of $\mathfrak{g}$, which is graded by the degree. Using the Poincare–Birkhoff–Witt isomorphism $U(\mathfrak{g}) \cong S(\mathfrak{g})$, we transfer the degree on $S(\mathfrak{g})$ to a degree on $U(\mathfrak{g})$, that is, the degree of $x \in U(\mathfrak{g})$ is the degree of its image under the Poincare–Birkhoff–Witt isomorphism.

Let $\Upsilon_{\mathfrak{g}} : S(\mathfrak{g}) \to U(\mathfrak{g})$ be the Duflo–Kirillov map (see [**2**, **6**, **8**]), which is an isomorphism of vector spaces. We can extend $\Upsilon_{\mathfrak{g}}$ multi-linearly to a vector space isomorphism $\Upsilon_{\mathfrak{g}} : S(\mathfrak{g})^{\otimes \ell} \to U(\mathfrak{g})^{\otimes \ell}$. When restricted to the $\mathfrak{g}$-invariant parts, $\Upsilon_g : S(\mathfrak{g})^{\mathfrak{g}} \to U(\mathfrak{g})^{\mathfrak{g}}$ is an algebra isomorphism. Note that $U(\mathfrak{g})^{\mathfrak{g}}$ is the centre of the algebra $U(\mathfrak{g})$.

2.2. *Laplacian on a Euclidean space*

Suppose that V is a Euclidean space. In our applications, we will always have $V = \mathfrak{g}$ or $V = \mathfrak{h}$, with the Euclidean structure coming from the invariant inner product.

Let $V^{*\ell}$ be the direct sum of ℓ copies of V^*, the dual of V. As usual, one identifies the symmetric algebra $S(V)$ with $P(V^*)$, the algebra of polynomial functions on V^*. More generally, one identifies $S(V)^{\otimes \ell}$ with $P(V^{*\ell})$, the algebra of polynomial functions on $V^{*\ell}$.

The Laplacian Δ_{V^*}, associated with the Euclidean structure of V, acts on $S(V) = P(V^*)$ and is defined by

$$\Delta_{V^*} = \sum_i \partial^2_{e_i},$$

where $\{e_i\}$ is an orthonormal basis of V. It is shown by direct calculation that, for $x, y \in V$, $\frac{1}{2}\Delta_V(xy) = (x, y)$, the inner product of x and y.

Let $\hbar$ be a formal parameter. For a non-zero integer f, let us consider the following operator $\mathcal{E}_V^{(f)} : S(V) = P(V^*) \to \mathbb{R}[1/\hbar]$ expressed through an exponential of the Laplacian and the evaluation at 0:

$$\mathcal{E}_V^{(f)}(g) = \exp\left(-\frac{\Delta_{V^*}}{2f\hbar}\right)(g)\bigg|_{x=0} \in \mathbb{R}[1/\hbar].$$

Because Δ_{V^*} is a second-order differential operator, it is easy to see that if g is a homogeneous polynomial of degree $\deg(g)$, then

$$\mathcal{E}_V^{(f)}(g) = \begin{cases} 0 & \text{if } \deg(g) \text{ is odd,} \\ \dfrac{\text{scalar}}{\hbar^{\deg(g)/2}} & \text{if } \deg(g) \text{ is even.} \end{cases} \tag{2.1}$$

For an ℓ-tuple $\boldsymbol{f} := (f_1, \ldots, f_\ell)$ of non-zero integers, let

$$\mathcal{E}_V^{(\boldsymbol{f})} := \bigotimes_j \mathcal{E}_V^{(f_j)} : S(V)^{\otimes \ell} = P(V^{*\ell}) \longrightarrow \mathbb{R}[1/\hbar].$$

In other words, if $g_1 \otimes \ldots \otimes g_\ell \in S(V)^{\otimes \ell}$, then $\mathcal{E}_V^{(\boldsymbol{f})}(g_1 \otimes \ldots \otimes g_\ell) = \prod_{j=1}^{\ell} \mathcal{E}_V^{(f_j)}(g_j)$.

We want to extend $\mathcal{E}_V^{(\boldsymbol{f})}$ to a formal power series in $P(V^{*\ell})[[\hbar]]$. For the convergence of the images of the extension, we will restrict ourselves to the following subalgebra. A formal power series $\sum_{n=0}^{\infty} g_n \hbar^n \in P(V^{*\ell})[[\hbar]]$ is *tame* if $\deg(g_n) \leqslant 3n/2$. Let $P_\hbar(V^{*\ell}) \subset P(V^{*\ell})[[\hbar]]$ be the set of all tame formal power series. Then $P_\hbar(V^{*\ell})$ is a subalgebra and $\mathcal{E}_V^{(\boldsymbol{f})}$ extends to a linear operator from $P_\hbar(V^{*\ell})$ to $\mathbb{R}[[\hbar]]$, also denoted by $\mathcal{E}_V^{(\boldsymbol{f})}$, as follows

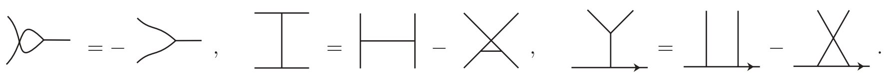
$$\mathcal{E}_V^{(\boldsymbol{f})}\left(\sum g_n \hbar^n\right) = \sum \mathcal{E}_V^{(\boldsymbol{f})}(g_n)\hbar^n.$$

Tameness and equation (2.1) guarantee that the right-hand side is in $\mathbb{R}[[\hbar]]$.

2.3. *Jacobi diagrams and weight systems*

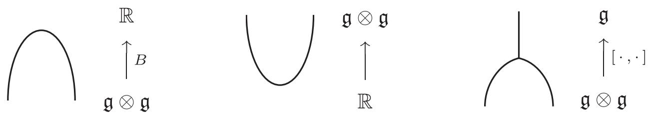
Here we review Jacobi diagrams and weight systems. For details, see, for example, [**31**].

A *uni-trivalent graph* is a graph every vertex of which is either univalent or trivalent. A uni-trivalent graph is *vertex oriented* if at each trivalent vertex a cyclic order of edges is fixed. For a 1-manifold Y, a *Jacobi diagram* on Y is the manifold Y together with a vertex-oriented uni-trivalent graph such that univalent vertices of the graph are distinct points on Y. In figures, we draw Y by thick lines and the uni-trivalent graphs by thin lines, in such a way that each trivalent vertex is vertex oriented in the counterclockwise order. We define the *degree* of a Jacobi diagram to be half the number of univalent and trivalent vertices of the uni-trivalent graph of the Jacobi diagram. We denote by $\mathcal{A}(Y)$ the quotient vector space spanned by Jacobi diagrams on Y subject to the following relations, called *the AS, IHX,* and *STU relations*, respectively,

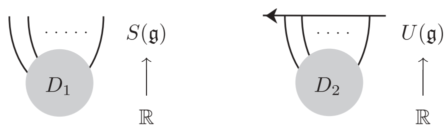


For $S = \{x_1, \dots, x_\ell\}$, a *Jacobi diagram* on S is a vertex-oriented uni-trivalent graph whose univalent vertices are labelled by elements of S. We denote by $\mathcal{A}(*_S)$ the quotient vector space spanned by Jacobi diagrams on S subject to the AS and IHX relations. In particular, when S consists of a single element, we denote $\mathcal{A}(*_S)$ by $\mathcal{A}(*)$. Both $\mathcal{A}(\emptyset)$ and $\mathcal{A}(*_S)$ form algebras with respect to the disjoint union of Jacobi diagrams, and $\mathcal{A}(\sqcup^\ell \downarrow)$ forms an algebra with respect to the vertical composition of copies of $\sqcup^\ell \downarrow$.

We briefly review weight systems; for details, see [**1**, **31**]. We define the *weight system* $W_{\mathfrak{g}}(D)$ of a Jacobi diagram D by 'substituting' $\mathfrak{g}$ into D, that is, putting D in a plane, $W_{\mathfrak{g}}(D)$ is defined to be the composition of intertwiners, each of which is given at each local part of D as follows.



Here, the first map is the invariant form of $\mathfrak{g}$, the second is the map taking 1 to $\sum_i X_i \otimes X_i$, where $\{X_i\}_{i \in I}$ is an orthonormal basis of $\mathfrak{g}$ with respect to the invariant form, and the third is the Lie bracket of $\mathfrak{g}$. For $D_1 \in \mathcal{A}(*)$ and $D_2 \in \mathcal{A}(\downarrow)$, we have the following intertwiners as the compositions of the above maps, and we can define $W_{\mathfrak{g}}(D_1) \in S(\mathfrak{g})$ and $W_{\mathfrak{g}}(D_2) \in U(\mathfrak{g})$ as the images of 1 by these maps.



In a similar way, we can also define $W_{\mathfrak{g}} : \mathcal{A}(\sqcup^\ell \downarrow) \to (U(\mathfrak{g})^{\otimes \ell})^{\mathfrak{g}}$ and $W_{\mathfrak{g}} : \mathcal{A}(*_S) \to (S(\mathfrak{g})^{\otimes \ell})^{\mathfrak{g}}$; they are algebra homomorphisms. If D is a diagram with k univalent vertices, then $W_{\mathfrak{g}}(D)$ has degree at most k. The weight system $W_{\mathfrak{g}_{\mathbb{C}}}$ is defined in the same way. Since $W_{\mathfrak{g}_{\mathbb{C}}} = W_{\mathfrak{g}}$ by definition, we denote $W_{\mathfrak{g}_{\mathbb{C}}}$ by $W_{\mathfrak{g}}$. Further, we define $\hat{W}_{\mathfrak{g}}$ by $\hat{W}_{\mathfrak{g}}(D) = W_{\mathfrak{g}}(D)\, \hbar^d$ for a Jacobi diagram D of degree d.

There is a formal Duflo–Kirillov algebra isomorphism $\Upsilon : \mathcal{A}(*) \to \mathcal{A}(\downarrow)$ (see [**2**, **6**]). The obvious multi-linearly extension $\Upsilon : \mathcal{A}(\{x_1, \dots, x_\ell\}) \to \mathcal{A}(\sqcup^\ell \downarrow)$ is not an algebra isomorphism, but a vector space isomorphism. The following diagram is commutative [**2**, Theorem 3].

$$\begin{array}{ccccc}
\mathcal{A}(*) & \xrightarrow{\hat{W}_{\mathfrak{g}}} & S(\mathfrak{g})^{\mathfrak{g}}[[\hbar]] & = & P(\mathfrak{g}^*)^{\mathfrak{g}}[[\hbar]] \\
\Upsilon \downarrow \cong & & \Upsilon_{\mathfrak{g}} \downarrow \cong & & \cong \downarrow \mathcal{P} \\
\mathcal{A}(\downarrow) & \xrightarrow[\hat{W}_{\mathfrak{g}}]{} & U(\mathfrak{g})^{\mathfrak{g}}[[\hbar]] & \xrightarrow[\psi_{\mathfrak{g}}]{\cong} & P(\mathfrak{h}^*)^W[[\hbar]]
\end{array} \tag{2.2}$$

Here $P(\mathfrak{h}^*)^W$ denotes the algebra of W-invariant polynomial functions on $\mathfrak{h}^*$, and $\psi_{\mathfrak{g}}$ is the composition of the Harish-Chandra isomorphism $U(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\cong} S(\mathfrak{h})^W$ (see [**17**, Section 23.3]) and the isomorphism $S(\mathfrak{h})^W \xrightarrow{\cong} P(\mathfrak{h}^*)^W$. The operator $\psi_{\mathfrak{g}}$ can also be described as follows: suppose $z \in U(\mathfrak{g})^{\mathfrak{g}}$ and $\lambda - \rho$ is a real dominant weight. Then z acts as a scalar operator on $V_{\lambda - \rho}$, with the scalar being $\psi_{\mathfrak{g}}(z)(\lambda)$. In other words,

$$\psi_{\mathfrak{g}}(z)(\lambda) = z(\lambda)/\mathcal{D}(\lambda). \tag{2.3}$$

Actually, in [**2**], the commutativity of diagram (2.2) is proved for the complexification of the involved spaces. In the complexified version of diagram (2.2), all the upper horizontal maps and the two boundary vertical maps preserve the real parts; it follows that all the other maps must also preserve the real parts, and we still have commutativity for diagram (2.2).

2.4. *The Kontsevich invariant*

A *string link* is an embedding φ of ℓ copies of the unit interval, $[0,1]\times\{1\},\ldots,[0,1]\times\{\ell\}$, into $[0,1]\times\mathbb{C}$, so that $\varphi((\varepsilon,j))=(\varepsilon,j)$ for all $\varepsilon\in\{0,1\}$ and $1\leqslant j\leqslant\ell$. We obtain a link from a string link by closing each component of $\sqcup^{\ell}\downarrow$. A (string) link is called *algebraically split* if the linking number of each pair of components is 0.

The *Kontsevich invariant* $Z(T)$ (see [**21**, **26**]) of an ℓ-component framed string link T is defined to be in $\mathcal{A}(\sqcup^{\ell}\downarrow)$; for its construction, see, for example, [**26**, **31**]. Let $\nu=Z(U)$, the Kontsevich invariant of the unknot U with framing 0; the exact value of ν is calculated in [**6**]. Using the Poincare–Birkhoff–Witt isomorphism $\mathcal{A}(S^1)\cong\mathcal{A}(\downarrow)$ (see [**1**]), we will consider ν as an element in $\mathcal{A}(\downarrow)$.

Let $\Delta^{(\ell)}:\mathcal{A}(\downarrow)\to\mathcal{A}(\sqcup^{\ell}\downarrow)$ be the cabling operation that replaces an arrow by ℓ parallel copies (see, for example, [**26**, Section 1]). The modification $\check{Z}(T)$ of $Z(T)$ used in the definition of the LMO invariant is

$$\check{Z}(T):=\nu^{\otimes\ell}(\Delta^{\ell}(\nu))Z(T).$$

Applying Υ^{-1} followed by the weight map, we define the following element:

$$\check{Q}^{\mathfrak{g}}(T)=\hat{W}_{\mathfrak{g}}(\Upsilon^{-1}(\check{Z}(T)))\in(S(\mathfrak{g})^{\otimes\ell})^{\mathfrak{g}}[[\hbar]]. \tag{2.4}$$

Let $P_{\hbar}(\mathfrak{g}^{*\ell})^{\mathfrak{g}}=(S(\mathfrak{g})^{\otimes\ell})^{\mathfrak{g}}[[\hbar]]\bigcap P_{\hbar}(\mathfrak{g}^{*\ell})$.

Lemma 2.1. *For an algebraically split* 0*-framed string link* T*, one has* $\check{Q}^{\mathfrak{g}}(T)\in P_{\hbar}(\mathfrak{g}^{*\ell})^{\mathfrak{g}}$.

Proof. A *strut* is a Jacobi diagram on $S=\{x_1,\ldots,x_\ell\}$ without a trivalent vertex; it is homeomorphic to an interval. An element in $\mathcal{A}(*_S)$ is *strutless* if it is a linear combination of diagrams which does not contain a strut. For a framed string link T, the strut part of the Kontsevich invariant is given by the linking matrix.

Suppose that D is a strutless Jacobi diagram on S. Let v_1 denote the number of univalent vertices and v_3 the number of trivalent vertices of D. As in graph theory, we say that two vertices of D are *adjacent* if they are connected by an edge. Because D is strutless, every univalent vertex of D must be adjacent to some trivalent vertex. On the other hand, each trivalent vertex is adjacent to at most three univalent vertices. It follows that $v_1\leqslant 3v_3$ and hence

$$\deg(W_{\mathfrak{g}}(D))\leqslant v_1=v_1/4+3v_1/4\leqslant 3v_3/4+3v_1/4=3\deg(D)/2. \tag{2.5}$$

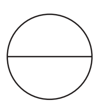
Since T is algebraically split and has 0-framing, $\Upsilon^{-1}(\check{Z}(T))$ is strutless. From (2.5), it follows that $\hat{W}_{\mathfrak{g}}(\Upsilon^{-1}(\check{Z}(T)))=\sum_n g_n\hbar^n$, where $\deg(g_n)\leqslant 3n/2$. Hence, $\check{Q}^{\mathfrak{g}}(T)\in P_{\hbar}(\mathfrak{g}^{*\ell})^{\mathfrak{g}}$. □

2.5. *Presentations of the LMO invariant*

In this section, we recall and modify a formula of the LMO invariant [**27**] of a rational homology 3-sphere M using the Aarhus integral [**4**] for the case when M is obtained from S^3 by surgery along an algebraically split link.

Suppose that T is an algebraically split ℓ-component string link with 0-framing on each component, and L is its closure. Suppose that the components of T are ordered. Let $\boldsymbol{f}=(f_1,\ldots,f_\ell)$ be an ℓ-tuple of ℓ non-zero integers, and let M be the rational homology 3-sphere obtained from S^3 by surgery along L with framing $f_1,\ldots,f_\ell$.

Let $\theta\in\mathcal{A}(\emptyset)$ be the following Jacobi diagram

$\theta=$  $\in\mathcal{A}(\emptyset)$. (2.6)

Define

$$\mathcal{I}(T,\boldsymbol{f}) := \exp\left(-\frac{\sum_j f_j}{48}\theta\right)\left\langle \prod_j \exp\left(-\frac{1}{2f_j}\overset{\partial_{x_j}\ \ \partial_{x_j}}{\bigcup}\right), (\Upsilon^{-1})\check{Z}(T)\right\rangle \in \mathcal{A}(\emptyset). \tag{2.7}$$

Here, for a Jacobi diagram D_1 whose univalent vertices are labelled by $\partial_{x_1},\dots,\partial_{x_\ell}$ and a Jacobi diagram D_2 whose univalent vertices are labelled by $x_1,\dots,x_\ell$ we define the bracket by

$$\langle D_1, D_2\rangle = \left(\begin{array}{l}\text{the sum of all ways of gluing the } \partial_{x_j}\text{-labelled univalent}\\ \text{vertices of } D_1 \text{ to the } x_j\text{-labelled univalent vertices of}\\ D_2 \text{ for each } j\end{array}\right) \in \mathcal{A}(\emptyset),$$

if the number of ∂_{x_j}-labelled univalent vertices of D_1 are equal to the number of x_j-labelled univalent vertices of D_2 for each j, and put $\langle D_1, D_2\rangle = 0$ otherwise. In particular, when T is the trivial string link $\downarrow$, one has

$$\mathcal{I}(\downarrow,\pm 1) = \exp\left(\mp\frac{1}{48}\theta\right)\left\langle \exp\left(\mp\frac{1}{2}\overset{\partial_{x}\ \ \partial_{x}}{\bigcup}\right), \Upsilon^{-1}(\nu^2)\right\rangle \in \mathcal{A}(\emptyset).$$

Then, the LMO invariant of M is presented by

$$\hat{Z}^{\mathrm{LMO}}(M) = \frac{\mathcal{I}(T,\boldsymbol{f})}{\prod_{j=1}^{\ell}\mathcal{I}(\downarrow, \operatorname{sign}(f_j))} \in \mathcal{A}(\emptyset). \tag{2.8}$$

Here, the bracket of this presentation is called the Aarhus "integral", since its corresponding Lie algebra version is actually an integral on $(\mathfrak{g}^*)^{\oplus\ell}$ [3].

We remark that the presentation (2.8) is obtained from [5, Theorem 6], noting that (with notations from [5])

$$\mathring{\mathrm{A}}_0(L) = \int\left(\prod_j \Upsilon_{x_j}^{-1}\right)(\check{Z}(L))\,dX,$$

$$\left(\prod_j \Upsilon_{x_j}^{-1}\right)(\check{Z}(L)) = \left(\prod_j \Upsilon_{x_j}^{-1}\right)(\check{Z}(T))\exp\left(-\frac{\sum_j f_j}{48}\theta\right)\prod_j \exp\left(\frac{f_j}{2}\underset{x_j\ \ x_j}{\bigcap}\right),$$

which are obtained from [5, Lemma 3.8 and Corollaries 3.11 and 3.12].

3. *Proof of the main theorem*

In this section, we show the proof of the main theorem in Subsection 3.3 based on Propositions 3.4 and 3.5, proved in later sections.

3.1. *Some lemmas on weights of Jacobi diagrams*

In this section, we show some lemmas on Jacobi diagrams that are used in the proof of Proposition 3.3.

LEMMA 3.1. *For a Jacobi diagram $D \in \mathcal{A}(*)$ and a non-zero integer f,*

$$\hat{W}_{\mathfrak{g}}\left(\left\langle \exp\left(\frac{-1}{2f}\overset{\partial_{x}\ \ \partial_{x}}{\bigcup}\right), D\right\rangle\right) = \mathcal{E}_{\mathfrak{g}}^{(f)}(\hat{W}_{\mathfrak{g}}(D)).$$

Proof. The bracket can be presented in terms of differentials as explained in [3, Appendix]. We verify this for the required formula concretely.

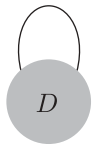
By expanding the exponential, it is sufficient to show that

$$W_{\mathfrak{g}}\left(\left\langle\left(\overset{\partial_x \quad \partial_x}{\bigcup}\right)^{d}, D\right\rangle\right) = \Delta_{\mathfrak{g}^*}^{d}(W_{\mathfrak{g}}(D))|_{X_i=0}. \tag{3.1}$$

Since both sides are equal to 0 unless D has $2d$ legs, we can assume that D has $2d$ legs.

When $d = 1$, (3.1) is shown by

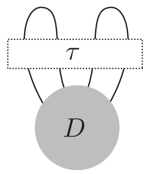
$$W_{\mathfrak{g}}\left(\left\langle \overset{\partial_x \quad \partial_x}{\bigcup}, D\right\rangle\right) = W_{\mathfrak{g}}\left(2\right.$$



$$\left.\right) = 2B(W_{\mathfrak{g}}(D)) = \Delta_{\mathfrak{g}^*}(W_{\mathfrak{g}}(D)),$$

where B is the invariant form. When $d = 2$, putting $W_{\mathfrak{g}}(D) = \sum_k Y_{1,k}Y_{2,k}Y_{3,k}Y_{4,k}$ for $Y_{i,j} \in \mathfrak{g}$, (3.1) is shown by

$$W_{\mathfrak{g}}\left(\left\langle\left(\overset{\partial_x \quad \partial_x}{\bigcup}\right)^{2}, D\right\rangle\right) = \sum_{\tau} W_{\mathfrak{g}}\left(\right.$$



$$\begin{aligned} &\left.\right) \\ &= \sum_{\tau,k} B(Y_{\tau(1),k}, Y_{\tau(2),k})B(Y_{\tau(3),k}, Y_{\tau(4),k}) \\ &= \sum_{\tau,i,j,k} \partial_{X_i}(Y_{\tau(1),k})\partial_{X_i}(Y_{\tau(2),k})\partial_{X_j}(Y_{\tau(3),k})\partial_{X_j}(Y_{\tau(4),k}) \\ &= \Delta_{\mathfrak{g}^*}^{2}(W_{\mathfrak{g}}(D)), \end{aligned}$$

where the sum of τ runs over all permutations on $\{1, 2, 3, 4\}$. For a general d, we can show (3.1) in the same way as above. □

LEMMA 3.2 [**22**]. *For the Jacobi diagram θ given in (2.6), $W_{\mathfrak{g}}(\theta) = 24|\rho|^2$, where ρ is the half-sum of positive real roots.*

Proof. It is shown from the definition of the weight system (see, for example, [**31**]) that $W_{\mathfrak{g}}(\theta) = \mathrm{Tr}(\mathcal{C}, \mathfrak{g})$, the trace of the Casimir element $\mathcal{C}$ acting on $\mathfrak{g}$ via the adjoint representation.

It is known that $\mathrm{Tr}(\mathcal{C}, \mathfrak{g}) = \dim \mathfrak{g}$ (see [**17**, Section 6.2]), and by the Freudenthal–de Vries strange formula [**9**, 47.11], $\dim \mathfrak{g} = 24(\rho, \rho)$. Hence, we have $W_{\mathfrak{g}}(\theta) = 24|\rho|^2$. □

3.2. *Comparing the LMO invariant and the perturbative invariant*

We again assume that $M, L, T, \boldsymbol{f}$ are the same as in Subsection 2.5. Since $\check{Q}^{\mathfrak{g}}(T) \in P_{\hbar}(\mathfrak{g}^{*\ell})^{\mathfrak{g}}$ by Lemma 2.1, we can define $\mathcal{E}_{\mathfrak{g}}^{(\boldsymbol{f})}(\check{Q}^{\mathfrak{g}}(T)) \in \mathbb{R}[[\hbar]]$.

PROPOSITION 3.3. *Assume the above notations. The LMO invariant of M, after the application of the weight map, has the following presentation*

$$\hat{W}_{\mathfrak{g}}(\hat{Z}^{\mathrm{LMO}}(M)) = \frac{I_1(T, \boldsymbol{f})}{\prod_{j=1}^{\ell} I_1(\downarrow, \mathrm{sign}(f_j))}, \tag{3.2}$$

where

$$I_1(T, \boldsymbol{f}) = \left(\prod_{j=1}^{\ell} e^{-f_j|\rho|^2\hbar/2}\right) \mathcal{E}_{\mathfrak{g}}^{(\boldsymbol{f})}(\check{Q}^{\mathfrak{g}}(T)).$$

Proof. Apply the algebra map $\hat{W}_{\mathfrak{g}}$ to (2.8),

$$\hat{W}_{\mathfrak{g}}(\hat{Z}^{\mathrm{LMO}}(M)) = \frac{\hat{W}_{\mathfrak{g}}(\mathcal{I}(T, \boldsymbol{f}))}{\prod_{j=1}^{\ell} \hat{W}_{\mathfrak{g}}(\mathcal{I}(\downarrow, \operatorname{sign}(f_j)))}.$$

Using Lemmas 3.1 and 3.2 and the definition of $\mathcal{I}(T, \boldsymbol{f})$ in (2.7), we get

$$\hat{W}_{\mathfrak{g}}(\mathcal{I}(T, \boldsymbol{f})) = I_1(T, \boldsymbol{f}),$$

which proves the proposition. $\square$

The perturbative invariant has a very similar presentation, as asserted in the following proposition, whose proof will be given in Subsection 6.3, where we present the perturbative invariant.

PROPOSITION 3.4. *The perturbative invariant has the following presentation*

$$\tau^{\mathfrak{g}}(M) = \frac{I_2(T, \boldsymbol{f})}{\prod_{j=1}^{\ell} I_2(\downarrow, \operatorname{sign}(f_j))}, \tag{3.3}$$

where

$$I_2(T, \boldsymbol{f}) = \left(\prod_{j=1}^{\ell} e^{-f_j|\rho|^2\hbar/2}\right) \mathcal{E}_{\mathfrak{h}}^{(\boldsymbol{f})}((\hbar^{\phi_+}\mathcal{D})^{\otimes\ell}\Upsilon_{\mathfrak{g}}(\check{Q}^{\mathfrak{g}}(T))).$$

To prove the main theorem, one needs to understand the relation between $\mathcal{E}_{\mathfrak{g}}^{(\boldsymbol{f})}$ and $\mathcal{E}_{\mathfrak{h}}^{(\boldsymbol{f})}$. We will prove the following proposition for $\ell = 1$ in Section 4 and for a general ℓ in Section 5.

PROPOSITION 3.5. *There is a non-zero constant $c_{\mathfrak{g}}$ such that for $g \in P_\hbar(\mathfrak{g}^{*\ell})^{\mathfrak{g}}$ and any ℓ-tuple $\boldsymbol{f} = (f_1, \ldots, f_\ell)$ of non-zero integers one has*

$$\mathcal{E}_{\mathfrak{g}}^{(\boldsymbol{f})}(g) = \left(\prod_{j=1}^{\ell}(-2f_j)^{\phi_+}c_{\mathfrak{g}}\right) \mathcal{E}_{\mathfrak{h}}^{(\boldsymbol{f})}((\hbar^{\phi_+}\mathcal{D})^{\otimes\ell}\Upsilon_{\mathfrak{g}}(g)), \tag{3.4}$$

where we recall that ϕ_+ is the number of positive roots of $\mathfrak{g}$.

REMARK 1. Recall that if $x \in \mathfrak{g}$, then one can consider x as a polynomial function on $\mathfrak{h}^*$; see Subsection 2.1 and the Appendix. Correspondingly, $\Upsilon_{\mathfrak{g}}(g)$ in the right-hand side of (3.4) belongs to $U(\mathfrak{g})^{\otimes\ell}[[\hbar]]$ and is considered as function on $\mathfrak{h}^{*\ell}$ with values in $\mathbb{R}[[\hbar]]$.

3.3. *Proof of main theorem*

Proof of Theorem 1.1. We prove the theorem in the following two cases.

Case 1 : The manifold M is obtained from S^3 by surgery along an algebraically split link L. We assume that T and $\boldsymbol{f}$ are as in Subsection 2.5. One has

$$\begin{aligned} I_1(T,\boldsymbol{f}) &= \left(\prod_{j=1}^{\ell} e^{-f_j|\rho|^2\hbar/2}\right) \mathcal{E}_{\mathfrak{g}}^{(\boldsymbol{f})}(\check{Q}^{\mathfrak{g}}(T)) \\ &= \left(\prod_{j=1}^{\ell} e^{-f_j|\rho|^2\hbar/2}\right) \left(\prod_{j=1}^{\ell}(-2f_j)^{\phi_+}c_{\mathfrak{g}}\right) \mathcal{E}_{\mathfrak{h}}^{(\boldsymbol{f})}((\hbar^{\phi_+}\mathcal{D})^{\otimes\ell}\Upsilon_{\mathfrak{g}}(\check{Q}^{\mathfrak{g}}(T))) \\ &= \left(\prod_{j=1}^{\ell}(-2f_j)^{\phi_+}c_{\mathfrak{g}}\right) I_2(T,\boldsymbol{f}), \end{aligned} \tag{3.5}$$

where the second equality follows from Proposition 3.5 since $\check{Q}^{\mathfrak{g}}(T) \in P_\hbar(\mathfrak{g}^{*\ell})^{\mathfrak{g}}$. In particular, applying (3.5) to $(T,\boldsymbol{f}) = (\downarrow, \operatorname{sign}(f_j))$, and taking the product over j, one has

$$\prod_{j=1}^{\ell} I_1(\downarrow, \operatorname{sign}(f_j)) = \prod_{j=1}^{\ell}((-2\operatorname{sign}(f_j))^{\phi_+}c_{\mathfrak{g}} I_2(\downarrow, \operatorname{sign}(f_j))). \tag{3.6}$$

Dividing (3.5) by (3.6) and using Propositions 3.3 and 3.4, we have

$$\hat{W}_{\mathfrak{g}}(\hat{Z}^{\mathrm{LMO}}(M)) = \prod_{j=1}^{\ell}|f_j|^{\phi_+}\tau^{\mathfrak{g}}(M).$$

Hence,

$$\hat{W}_{\mathfrak{g}}(\hat{Z}^{\mathrm{LMO}}(M)) = |H_1(M,\mathbb{Z})|^{\phi_+}\tau^{\mathfrak{g}}(M). \tag{3.7}$$

This completes the proof of Theorem 1.1 in this case.

Case 2 : The manifold M is an arbitrary rational homology 3-sphere. This case can be reduced to Case 1 using the well-known trick of diagonalization of Ohtsuki [**29**], as follows.

By Case 1, the theorem holds for the lens space $L(m,1)$, which is the result of surgery on S^3 along the unknot with framing m. Further, since the leading coefficient of the LMO invariant is 1, the formal power series $\hat{W}_{\mathfrak{g}}(\hat{Z}^{\mathrm{LMO}}(M)) \in \mathbb{R}[[\hbar]]$ is invertible. It is known [**29**] that there exist lens spaces $L(m_1,1),\dots,L(m_N,1)$, such that the connected sum $M' := M\#L(m_1,1)\#\dots\#L(m_N,1)$ can be obtained from S^3 by surgery along some algebraically split framed link. Both the LMO invariant and the perturbative invariant are multiplicative with respect to the connected sum. Hence, once we have the theorem, which is the identity (3.7), for M' and all the lens space $L(m_i,1)$, and each $\hat{W}_{\mathfrak{g}}(\hat{Z}^{\mathrm{LMO}}(L(m_i,1)))$ is invertible, we also have the identity (3.7) for M. This completes the proof of Theorem 1.1 in this case. □

4. *The knot case*

In order to complete the proof of the main theorem, we need to prove Propositions 3.4 and 3.5. The aim of this section is to prove Proposition 3.5 in the case $\ell = 1$. We call this case the knot case, since Proposition 3.5 with $\ell = 1$ is enough for the proof of the main theorem when M is obtained from S^3 by surgery along a knot. We present two proofs of Proposition 3.5 with $\ell = 1$: a geometric proof in Subsection 4.1 and an algebraic proof in Subsection 4.2.

4.1. *First proof: geometric approach*

4.1.1. *Gaussian integral and* $\mathcal{E}_V^{(f)}$. Suppose that V is a Euclidean space and f is a non-zero integer. The following lemma says that the operator $\mathcal{E}_V^{(f)}$ can be expressed by an integral.

LEMMA 4.1. *For $g \in P_\hbar(V^{*\ell})$, considered as a function on V^* with values in $\mathbb{R}[[\hbar]]$, one has*

$$\mathcal{E}_V^{(f)}(g) = \frac{1}{(4\pi)^{\dim V/2}} \int_{V^*} e^{-|x|^2/4} g\left(\frac{x}{\sqrt{-2f\hbar}}\right) dx. \tag{4.1}$$

REMARK 2. Here, $g(x/\sqrt{-2f\hbar})$ is the function on V^* with values in $\mathbb{C}[[\hbar^{1/2}]]$ defined as follows. If g is of the form $g = z^d$, where $z \in V$, then

$$g\left(\frac{x}{\sqrt{-2f\hbar}}\right) := g(x)((\sqrt{-2f\hbar})^{-d}).$$

The square root in the right-hand side does not really appear, since if d is odd, then both sides of (4.1) are 0.

Proof. We can assume that $g \in S(V)$. Every polynomial is a sum of powers of linear polynomials. Since both sides of (4.1) depend linearly on g, we can assume that g is a power of a linear polynomial. By changing coordinates, one can assume that $g = e_1^d$, where e_1 is the first of an orthonormal basis $e_1, \dots, e_n$ of V. The statement now reduces to the case when V is one-dimensional, which follows from a simple Gaussian integral calculation; see, for example, [**8**, Lemma 2.11]. □

4.1.2. *Reduction from $\mathfrak{g}^*$ to $\mathfrak{h}^*$*

PROPOSITION 4.2. *Suppose that g is a G-invariant function on $\mathfrak{g}^*$. Then*

$$\int_{\mathfrak{g}^*} g\, dx = \tilde{c}_{\mathfrak{g}} \int_{\mathfrak{h}^*} \mathcal{D}^2 \mathcal{P}(g)\, dx,$$

provided that both sides converge absolutely. Here, $\tilde{c}_{\mathfrak{g}}$ is a non-zero constant depending only on $\mathfrak{g}$.

Proof. It is clear that if such $\tilde{c}_{\mathfrak{g}}$ exists then it is non-zero, since there are G-invariant functions g, for example, $g(x) = \exp(-|x|^2)$, for which the left-hand side is non-zero.

The co-adjoint action of G on $\mathfrak{g}^*$ is well studied in the literature; see, for example, [**19**]. A point $x \in \mathfrak{g}^*$ is *regular* if its orbit $G \cdot x$ is a submanifold of dimension $\dim \mathfrak{g} - \dim \mathfrak{h} = 2\phi_+$, the maximal possible dimension. It is known that the set of singular points has measure 0. Every orbit has non-empty intersection with $\mathfrak{h}^*$, and if x is regular, then $G \cdot x \cap \mathfrak{h}^*$ has exactly $|W|$ points. Since the function g is constant on each orbit, we have

$$\int_{\mathfrak{g}^*} g(x)\, dx = \frac{1}{|W|} \int_{\mathfrak{h}^*} \mathrm{Vol}(G \cdot x) \mathcal{P}(g)(x)\, dx.$$

The volume function is also well known; it can be calculated, for example, from [**8**, Chapter 7]:

$$\mathrm{Vol}(G \cdot x) = \tilde{c}'_{\mathfrak{g}} \mathcal{D}^2(x), \tag{4.2}$$

where $\tilde{c}'_{\mathfrak{g}}$ is a constant. From (4.2), we can deduce the proposition, with $\tilde{c}_{\mathfrak{g}} = \tilde{c}'_{\mathfrak{g}}/|W|$.

Here is a simple proof of (4.2). (The authors thank A. Kirillov Jr for supplying them the proof.) We will identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the invariant inner product. Let H be the maximal abelian subgroup of G whose Lie algebra is $\mathfrak{h}$. The space G/H is a homogeneous G-space. The tangent space of G/H at H can be identified with $\mathfrak{h}^\perp$, with inner product induced from the invariant; from this we define a Riemannian metric on G/H. When $x \in \mathfrak{h}$ is regular, its stationary group is isomorphic to the torus H. The map $\varphi : G/H \to G \cdot x$, defined by $g \to g \cdot x$ with $g \in G$, is a diffeomorphism. The tangent space of $G\cdot x$ at x can also be identified with the same $\mathfrak{h}^\perp$ with the same inner product. It is easy to see that φ at H has derivative $d\varphi_H = -\operatorname{ad}(x) : \mathfrak{h}^\perp \to \mathfrak{h}^\perp$. Let us calculate the determinant of $d\varphi$. Because G/H is G-homogeneous and φ is G-equivariant, $|\det(d\varphi)|$ is constant on G/H, hence $|\det(d\varphi)| = |\det(\operatorname{ad}(x))|$. To calculate $|\det(\operatorname{ad}(x))|$, it is easier to use the complexification of the adjoint representation, since $\operatorname{ad}(x)$ is diagonal in the complexified representation. The complexified $\mathfrak{h}^\perp_{\mathbb{C}}$ has the standard Chevalley basis $E_\alpha, F_\alpha, \alpha\in\Phi_+$ such that $\operatorname{ad}(x)E_\alpha = i(x,\alpha)E_\alpha$ and $\operatorname{ad}(t)F_\alpha = -i(x,\alpha)F_\alpha$. It follows that $|d\varphi| = \prod_{\alpha\in\Phi_+}|(x,\alpha)|^2$. Hence,

$$\operatorname{Vol}(G\cdot x) = \operatorname{Vol}(G/H)\prod_{\alpha\in\Phi_+^{\mathbb{R}}}|(x,\alpha)|^2 = \tilde{c}'_{\mathfrak{g}}\mathcal{D}^2(x),$$

where $\tilde{c}'_{\mathfrak{g}} = \operatorname{Vol}(G/H)\prod_{\alpha\in\Phi_+}|(\rho,\alpha)|^2$. □

4.1.3. *First proof of Proposition* 3.5 *in the case* $\ell = 1$

Geometric proof of Proposition 3.5 *in the case* $\ell = 1$. We will prove that with $c_{\mathfrak{g}} = \tilde{c}_{\mathfrak{g}}/(4\pi)^{\phi_+}$, for any $g\in P_\hbar(\mathfrak{g}^*)^{\mathfrak{g}}$ and $f\neq 0$,

$$\mathcal{E}^{(f)}_{\mathfrak{g}}(g) = (-2f)^{\phi_+}c_{\mathfrak{g}}\mathcal{E}^{(f)}_{\mathfrak{h}}((\hbar^{\phi_+}\mathcal{D})\Upsilon_{\mathfrak{g}}(g)). \tag{4.3}$$

Assume that $g = \sum_j g_j\hbar^j$, where $g_j\in S(\mathfrak{g})^{\mathfrak{g}}$. It is enough to show (4.3) for each $g = g_j\hbar^j$ since passing to the infinite sum $\sum_j g_j\hbar^j$ is possible due to the tameness of the series g.

Recall that $g_j\in S(\mathfrak{g})^{\mathfrak{g}}$ is considered a function on $\mathfrak{g}^*$; its restriction on $\mathfrak{h}^*$ is denoted by $\mathcal{P}(g_j)$. On the other hand, $\Upsilon_{\mathfrak{g}}(g_j)\in U(\mathfrak{g})^{\mathfrak{g}}$ defines a function on $\mathfrak{h}^*$; see Subsection 2.1. From diagram (2.2) and equation (2.3), we have that, as functions on $\mathfrak{h}^*$,

$$\Upsilon_{\mathfrak{g}}(g_j) = \mathcal{D}\mathcal{P}(g_j). \tag{4.4}$$

By Lemma 4.1, the left-hand side of (4.3) with $g = g_j\hbar^j$ can be expressed as

$$\text{LHS of (4.3)} = \mathcal{E}^{(f)}_{\mathfrak{g}}(g_j\hbar^j) = \frac{\hbar^j}{(4\pi)^{\dim\mathfrak{g}/2}}\int_{\mathfrak{g}^*} e^{-|x|^2/4}g_j\left(\frac{x}{\sqrt{-2f\hbar}}\right)dx.$$

The integrand is invariant under the co-adjoint action. By Proposition 4.2, we have

$$\text{LHS of (4.3)} = \frac{\tilde{c}_{\mathfrak{g}}\hbar^j}{(4\pi)^{\dim\mathfrak{g}/2}}\int_{\mathfrak{h}^*}\mathcal{D}^2(x)\,e^{-|x|^2/4}\mathcal{P}(g_j)\left(\frac{x}{\sqrt{-2f\hbar}}\right)dx. \tag{4.5}$$

We turn to the right-hand side of (4.3). Using (4.4), one has, with $g = g_j\hbar^j$,

$$\text{RHS of (4.3)} = c_{\mathfrak{g}}\hbar^j(-2f\hbar)^{\phi_+}\mathcal{E}^{(f)}_{\mathfrak{h}}(\mathcal{D}^2\mathcal{P}(g)).$$

Again using Lemma 4.1, we have

$$\text{RHS of (4.3)} = \frac{c_{\mathfrak{g}}\hbar^j(-2f\hbar)^{\phi_+}}{(4\pi)^{\dim\mathfrak{h}/2}}\int_{\mathfrak{h}^*} e^{-|x|^2/4}\mathcal{D}^2\left(\frac{x}{\sqrt{-2f\hbar}}\right)g\left(\frac{x}{\sqrt{-2f\hbar}}\right)dx. \tag{4.6}$$

Because $\mathcal{D}^2$ is a homogeneous polynomial of degree $2\phi_+$, one has

$$\mathcal{D}^2(x) = (-2f\hbar)^{\phi_+}\mathcal{D}^2\left(\frac{x}{\sqrt{-2f\hbar}}\right).$$

With $c_{\mathfrak{g}} = \tilde{c}_{\mathfrak{g}}/(4\pi)^{\phi_+}$ and $2\phi_+ = \dim(\mathfrak{g}) - \dim(\mathfrak{h})$, from (4.5) and (4.6) we see that

$$\text{LHS of (4.3)} = \text{RHS of (4.3)}.$$

This completes the first proof of Proposition 3.5 in the case $\ell = 1$. □

4.2. *Second proof: algebraic approach*

In this section, we present another proof of Proposition 3.5 in the case $\ell = 1$.

4.2.1. *Harish-Chandra's radial component formula and its applications.* An invariant function $g \in P(\mathfrak{g}^*)^{\mathfrak{g}}$ is totally determined by its restriction $\mathcal{P}(g) \in P(\mathfrak{h}^*)^W$. If $\mathcal{O} : P(\mathfrak{g}^*) \to P(\mathfrak{g}^*)$ is a differential operator with polynomial coefficients, then Harish-Chandra showed that the map $P(\mathfrak{h}^*)^W \to P(\mathfrak{h}^*)^W$ defined by $\mathcal{P}(g) \to \mathcal{P}(D(g))$ is a differential operator, called the *radial component* of $\mathcal{O}$, and gave a description of the radial component for the case when $\mathcal{O}$ is a $\mathfrak{g}$-invariant differential operator with constant coefficients; see [**15**, Chapter II]. This description is called the *radial component formula*. Let us briefly recall this formula, when $\mathcal{O}$ is the Laplacian.

The Laplacian $\Delta_{\mathfrak{g}^*}$, originally acting on $S(\mathfrak{g})$, can be naturally extended to an action on $S(\mathfrak{g}) \otimes \mathbb{C} = S(\mathfrak{g}_{\mathbb{C}})$, and we also denote this extended action by $\Delta_{\mathfrak{g}^*}$. Similarly, we extend the action of $\Delta_{\mathfrak{h}^*}$ to $S(\mathfrak{h}_{\mathbb{C}})$. Let $\mathcal{P}$ be the restriction from $\mathfrak{g}^*_{\mathbb{C}}$ to $\mathfrak{h}^*_{\mathbb{C}}$. Denote by $\pi = \prod_{\alpha \in \Phi_+} \alpha$. Then Harish-Chandra's *radial component formula*, also known as Harish-Chandra's *restriction formula*, says that for any $g \in S(\mathfrak{g}_{\mathbb{C}})^{\mathfrak{g}_{\mathbb{C}}}$,

$$\pi \mathcal{P}(\Delta_{\mathfrak{g}^*}(g)) = \Delta_{\mathfrak{h}^*}(\pi \mathcal{P}(g)).$$

Actually, the above formula is obtained from Proposition II.3.14 of Helgason's book [**15**] by identifying $\mathfrak{g}_{\mathbb{C}}$ with ${\mathfrak{g}_{\mathbb{C}}}^*$ via the Killing form. Note that π is a constant times $\mathcal{D}$, namely $\pi = \mathcal{D} \prod_{\alpha \in \Phi_+} (\mathcal{J}^{-1}\alpha, \rho)$. Hence, by restricting $\Delta_{\mathfrak{g}^*}$ to the real part, we get the following proposition.

Proposition 4.3. *For any $g \in S(\mathfrak{g})^{\mathfrak{g}}$,*

$$\mathcal{D}\mathcal{P}(\Delta_{\mathfrak{g}^*}(g)) = \Delta_{\mathfrak{h}^*}(\mathcal{D}\mathcal{P}(g)).$$

Remark 3. We got Proposition 4.3 through the Harish-Chandra theorem, which is usually formulated for the complexified $\mathfrak{g}_{\mathbb{C}}$. Actually, the real version, that is, Proposition 4.3, is simply [**15**, Corollary II.3.13], if one properly translates our notations to the ones in [**15**]. For another discussion (and proof) of the real case, the reader is referred to [**32**]; see also [**16**, Theorem 2.1.8] for the $U(n)$ case.

Lemma 4.4. *One has $\Delta_{\mathfrak{h}^*}(\mathcal{D}) = 0$.*

Proof. If $g = 1$, then the left-hand side of the formula of Proposition 4.3 is 0, while the right-hand side is $\Delta_{\mathfrak{h}^*}(\mathcal{D})$. Hence, $\Delta_{\mathfrak{h}^*}(\mathcal{D}) = 0$. □

Proposition 4.5. *For any homogeneous polynomial $g \in S(\mathfrak{g})^{\mathfrak{g}}$ of degree $2d$,*

$$\frac{c_{\mathfrak{g}}}{d!} \Delta^d_{\mathfrak{g}^*}(g) = \frac{1}{(d+\phi_+)!} \Delta^{d+\phi_+}_{\mathfrak{h}^*}(\mathcal{D}^2 \mathcal{P}(g)),$$

where $c_{\mathfrak{g}} = \Delta^{\phi_+}_{\mathfrak{h}^*}(\mathcal{D}^2)/(\phi_+)!$ is a non-zero constant depending on $\mathfrak{g}$ only.

Proof. Since $\Delta^d_{\mathfrak{g}^*}(g)$ is a scalar, we have that

$$\mathcal{D}\Delta^d_{\mathfrak{g}^*}(g) = \mathcal{D}\mathcal{P}(\Delta^d_{\mathfrak{g}^*}(g)) = \Delta^d_{\mathfrak{h}^*}(\mathcal{D}\mathcal{P}(g)),$$

where we obtain the second equality by applying Proposition 4.3 repeatedly. Hence,

$$c_{\mathfrak{g}} \cdot \phi_+!\, \Delta^d_{\mathfrak{g}^*}(g) = \Delta^{\phi_+}_{\mathfrak{h}^*}(\mathcal{D}^2)\Delta^d_{\mathfrak{g}^*}(g) = \Delta^{\phi_+}_{\mathfrak{h}^*}(\mathcal{D}^2\Delta^d_{\mathfrak{g}^*}(g)) = \Delta^{\phi_+}_{\mathfrak{h}^*}(\mathcal{D}\Delta^d_{\mathfrak{h}^*}(\mathcal{D}\mathcal{P}(g))),$$

and the identity of Proposition 4.5 is reduced to

$$\Delta_{\mathfrak{h}^*}^{d+\phi_+}(\mathcal{D}^2\mathcal{P}(g)) = \binom{d+\phi_+}{\phi_+}\Delta_{\mathfrak{h}^*}^{\phi_+}(\mathcal{D}\Delta_{\mathfrak{h}^*}^{d}(\mathcal{D}\mathcal{P}(g))).$$

Furthermore, by putting $g' = \mathcal{D}\mathcal{P}(g)$, the above equality is rewritten as

$$\Delta_{\mathfrak{h}^*}^{d+\phi_+}(\mathcal{D}g') = \binom{d+\phi_+}{\phi_+}\Delta_{\mathfrak{h}^*}^{\phi_+}(\mathcal{D}\Delta_{\mathfrak{h}^*}^{d}(g')). \tag{4.7}$$

It is sufficient to show (4.7).

By definition, $\Delta_{\mathfrak{h}^*} = \sum_i \partial_{e_i}^2$ for an orthonormal basis $\{e_i\}$ of $\mathfrak{h}$. Let $Z \subset P(\mathfrak{h}^*)$ be the set of all elements of the form $\mathcal{O}(\mathcal{D})$, where $\mathcal{O} \in \mathbb{R}[\partial_{e_1}, \dots, \partial_{e_n}]$. Since $\mathcal{O}$ commutes with $\Delta_{\mathfrak{h}^*}$, by Lemma 4.4 we have $\Delta_{\mathfrak{h}^*}(x) = 0$ for every $x \in Z$. Hence, for $x \in Z$ and $y \in P(\mathfrak{h}^*)$,

$$\Delta_{\mathfrak{h}^*}(xy) = 2\sum_i \partial_{e_i}(x)\partial_{e_i}(y) + x\Delta_{\mathfrak{h}^*}(y).$$

Let $\mu : Z \otimes P(\mathfrak{h}^*) \to P(\mathfrak{h}^*)$ be the multiplication, $\mu(x \otimes y) = xy$. Then, the above formula becomes

$$\Delta_{\mathfrak{h}^*}(\mu(\beta)) = \mu[(\mathcal{O}_1 + \mathcal{O}_2)(\beta)]$$

for every $\beta \in Z \otimes P(\mathfrak{h}^*)$, where $\mathcal{O}_1 = 2\sum_i \partial_{e_i} \otimes \partial_{e_i}$ and $\mathcal{O}_2 = \mathrm{id} \otimes \Delta_{\mathfrak{h}^*}$. Note that $\mathcal{O}_1$ and $\mathcal{O}_2$ commute, and $(\mathcal{O}_1 + \mathcal{O}_2)(\beta) \in Z \otimes P(\mathfrak{h}^*)$. Applying the above formula repeatedly m times, we get

$$\Delta_{\mathfrak{h}^*}^{m}(\mu(\beta)) = \mu[(\mathcal{O}_1 + \mathcal{O}_2)^m(\beta)] = \sum_k \binom{m}{k}\mu[\mathcal{O}_1^k\mathcal{O}_2^{m-k}(\beta)]. \tag{4.8}$$

Let $\beta = \mathcal{D} \otimes g'$. From (4.8) we have

$$\Delta_{\mathfrak{h}^*}^{d+\phi_+}(\mathcal{D}g') = \sum_k \binom{d+\phi_+}{k}\mu[\mathcal{O}_1^k(\mathcal{D} \otimes \Delta_{\mathfrak{h}^*}^{d+\phi_+-k}(g'))].$$

Since $\deg(\mathcal{D}) = \phi_+$ and $\deg(g') = 2d + \phi_+$, the only non-zero term in the right-hand side is the one with $k = \phi_+$. Hence,

$$\Delta_{\mathfrak{h}^*}^{d+\phi_+}(\mathcal{D}g') = \binom{d+\phi_+}{d}\mu[\mathcal{O}_1^{\phi_+}(\mathcal{D} \otimes \Delta_{\mathfrak{h}^*}^{d}(g'))]. \tag{4.9}$$

Let $g'' = \Delta_{\mathfrak{h}^*}^{d}(g')$, which has degree ϕ_+. We have

$$(\mathcal{O}_1 + \mathcal{O}_2)^{\phi_+}(\mathcal{D} \otimes g'') = \sum_k \binom{\phi_+}{k}\mathcal{O}_1^{\phi_+-k}(\mathcal{D} \otimes \Delta_{\mathfrak{h}^*}^{k}(g'')).$$

Again the degree restriction shows that the only non-zero term in the right-hand side is the one with $k = 0$. Hence,

$$(\mathcal{O}_1 + \mathcal{O}_2)^{\phi_+}(\mathcal{D} \otimes g'') = \mathcal{O}_1^{\phi_+}(\mathcal{D} \otimes g''). \tag{4.10}$$

Combining (4.9) and (4.10), we have

$$\begin{aligned}\Delta_{\mathfrak{h}^*}^{d+\phi_+}(\mathcal{D}g') &= \binom{d+\phi_+}{d}\mu[(\mathcal{O}_1 + \mathcal{O}_2)^{\phi_+}(\mathcal{D} \otimes g'')] \\ &= \binom{d+\phi_+}{d}\Delta_{\mathfrak{h}^*}^{\phi_+}(\mathcal{D} \otimes g''),\end{aligned}$$

where the second equality follows from (4.8). This completes the proof of (4.7). □

4.2.2. *Second proof of Proposition* 3.5 *in the case* $\ell = 1$

Algebraic proof of Proposition 3.5 *in the case* $\ell = 1$. Again we can assume that $g \in S(\mathfrak{g})^{\mathfrak{g}}$. We can further assume that g is homogeneous. If the degree of g is odd, then both sides of (3.4) are 0, and we are done. Assume now g has degree $2d$.

By definition, the left-hand side of (3.4) is

$$\mathcal{E}_{\mathfrak{g}}^{(f)}(g) = \exp\left(-\frac{1}{2f\hbar}\Delta_{\mathfrak{g}^*}\right)(g)\Big|_{x=0}.$$

By expanding the exponential,

$$\text{LHS of (3.4)} = \left(-\frac{1}{2f\hbar}\right)^d \frac{1}{d!}\Delta_{\mathfrak{g}^*}^d(g). \tag{4.11}$$

Let us turn to the right-hand side of (3.4). Recall that $\mathcal{D}$ has degree ϕ_+. By (4.4)

$$\begin{aligned}
\text{RHS of (3.4)} &= c_{\mathfrak{g}}(-2f\hbar)^{\phi_+}\mathcal{E}_{\mathfrak{h}}^{(f)}(\mathcal{D}^2\mathcal{P}(g)) \\
&= c_{\mathfrak{g}}(-2f\hbar)^{\phi_+}\exp\left(-\frac{1}{2f\hbar}\Delta_{\mathfrak{h}^*}\right)(\mathcal{D}^2\mathcal{P}(g))\Big|_{x=0} \\
&= c_{\mathfrak{g}}(-2f\hbar)^{\phi_+}\left(-\frac{1}{2f\hbar}\right)^{d+\phi_+}\frac{1}{(d+\phi_+)!}\Delta_{\mathfrak{h}^*}^{d+\phi_+}(\mathcal{D}^2\mathcal{P}(g_d)) \\
&= c_{\mathfrak{g}}\left(-\frac{1}{2f\hbar}\right)^{d}\frac{1}{(d+\phi_+)!}\Delta_{\mathfrak{h}^*}^{d+\phi_+}(\mathcal{D}^2\mathcal{P}(g_d)).
\end{aligned} \tag{4.12}$$

Comparing (4.11) and (4.12) by using Proposition 4.5, we have immediately

$$\text{LHS of (3.4)} = \text{RHS of (3.4)}.$$

This completes the algebraic proof of Proposition 3.5 in the knot case. □

5. *The link case*

In Section 4, we discussed the proofs of Proposition 3.5 in the knot case. Here, in Subsection 5.1, we discuss a proof of Proposition 3.5 in the general case. In Subsection 5.2, we also show that, without Proposition 3.5 for the case $\ell > 1$, one can still prove the main theorem using general results on finite-type invariants.

5.1. *Proof of Proposition* 3.5 *for arbitrary* ℓ

Proof of Proposition 3.5. Using the tameness, we can assume that $g \in P(\mathfrak{g}^{*\ell})^{\mathfrak{g}} = (S(\mathfrak{g})^{\otimes \ell})^{\mathfrak{g}}$. The left-hand side of (3.4) is

$$\text{LHS of (3.4)} = \mathcal{E}_{\mathfrak{g}}^{(\boldsymbol{f})}(g) = \left(\bigotimes_{j=1}^{\ell}\mathcal{E}_{\mathfrak{g}}^{(f_j)}\right)(g).$$

Note that $\mathcal{E}_{\mathfrak{h}}^{(f)}$ acts on $P(\mathfrak{h}^*)$. We define a modification of $\mathcal{E}_{\mathfrak{h}}^{(f)}$, which acts on the bigger space $P(\mathfrak{g}^*) = S(\mathfrak{g})$, as follows:

$$\tilde{\mathcal{E}}_{\mathfrak{h}}^{(f)}(g) := ((-2f\hbar)^{\phi_+}c_{\mathfrak{g}})\mathcal{E}_{\mathfrak{h}}^{(f)}(\mathcal{D}\Upsilon_{\mathfrak{g}}(g)). \tag{5.1}$$

Then the right-hand side of (3.4) can be rewritten as

$$\text{RHS of (3.4)} = \left(\bigotimes_{j=1}^{\ell}\tilde{\mathcal{E}}_{\mathfrak{h}}^{(f_j)}\right)(g).$$

Identity (3.4) is then equivalent to

$$\left(\bigotimes_{j=1}^{\ell} \mathcal{E}_{\mathfrak{g}}^{(f_j)}\right)(g) = \left(\bigotimes_{j=1}^{\ell} \tilde{\mathcal{E}}_{\mathfrak{h}}^{(f_j)}\right)(g),$$

which is the case $m = \ell$ of the following identity,

$$\left(\bigotimes_{1 \leqslant j \leqslant m} \mathcal{E}_{\mathfrak{g}}^{(f_j)} \otimes \bigotimes_{m < j \leqslant \ell} \tilde{\mathcal{E}}_{\mathfrak{h}}^{(f_j)}\right)(g) = \bigotimes_{j=1}^{\ell} \tilde{\mathcal{E}}_{\mathfrak{h}}^{(f_j)}(g). \tag{5.2}$$

We will prove (5.2) by induction on m. The case $m = 0$ is a tautology. Note also that when $\ell = 1$, the identity holds since we proved it in Section 4. We put

$$g' = \left(\bigotimes_{1 \leqslant j < m} \mathcal{E}_{\mathfrak{g}}^{(f_j)} \otimes \bigotimes_{m < j \leqslant \ell} \tilde{\mathcal{E}}_{\mathfrak{h}}^{(f_j)}\right)(g) \in S(\mathfrak{g}).$$

Then equality (5.2) becomes

$$\mathcal{E}_{\mathfrak{g}}^{(f_m)}(g') = \tilde{\mathcal{E}}_{\mathfrak{h}}^{(f_m)}(g'). \tag{5.3}$$

Since $\mathcal{E}_{\mathfrak{g}}^{(f_j)}$ and $\tilde{\mathcal{E}}_{\mathfrak{h}}^{(f_j)}$ are intertwiners by Lemmas 5.1 and 5.2, $g' \in S(\mathfrak{g})^{\mathfrak{g}}$. Hence, (5.3) follows from the case $\ell = 1$, completing the induction. □

LEMMA 5.1. *The map $\mathcal{E}_{\mathfrak{g}}^{(f)} : S(\mathfrak{g}) \to \mathbb{R}[1/\hbar]$ is an intertwiner with respect to the action of $\mathfrak{g}$.*

Proof. By definition, $\mathcal{E}_{\mathfrak{g}}^{(f)}$ takes a monomial of odd degree in $S(\mathfrak{g})$ to 0. It is enough to consider the case $g = Y_1 Y_2 \dots Y_{2d}$, where each Y_j is a linear form. Then $\mathcal{E}_{\mathfrak{g}}^{(f)}$ takes $Y_1 Y_2 \dots Y_{2d}$ to a constant multiple of

$$\sum_{\tau} B(Y_{\tau(1)}, Y_{\tau(2)}) \dots B(Y_{\tau(2d-1)}, Y_{\tau(2d)}),$$

where the sum runs over all permutations on $\{1, 2, \dots, 2d\}$ and B is the invariant inner product. Since the invariant form B is an intertwiner, $\mathcal{E}_{\mathfrak{g}}^{(f)}$ is also an intertwiner.

Another proof is to use Lemma 4.1 to present $\mathcal{E}_{\mathfrak{g}}^{(f)}$ by an integral:

$$\mathcal{E}_{\mathfrak{g}}^{(f)}(g) = \frac{1}{(4\pi)^{\dim \mathfrak{g}/2}} \int_{\mathfrak{g}^*} e^{-|x|^2/2} g\left(\frac{x}{\sqrt{-2f\hbar}}\right) dx.$$

Since $|x|^2$ and dx are G-invariant, the right-hand side is G-invariant. □

LEMMA 5.2. *The map $\tilde{\mathcal{E}}_{\mathfrak{h}}^{(f)} : S(\mathfrak{g}) \to \mathbb{R}[1/\hbar]$ is an intertwiner with respect to the action of $\mathfrak{g}$.*

Proof. Since $\mathfrak{g}$ acts trivially on $\mathbb{R}$, it is sufficient to show that

$$\tilde{\mathcal{E}}_{\mathfrak{h}}^{(f)}(\mathrm{ad}_X(g)) = 0$$

for $X \in \mathfrak{g}$ and $g \in S(\mathfrak{g})$. Using the definition of $\tilde{\mathcal{E}}_{\mathfrak{h}}^{(f)}$ in (5.1), this is equivalent to

$$\mathcal{E}_{\mathfrak{h}}^{(f)}(\mathcal{D}\Upsilon_{\mathfrak{g}}(\mathrm{ad}_X(g))) = 0.$$

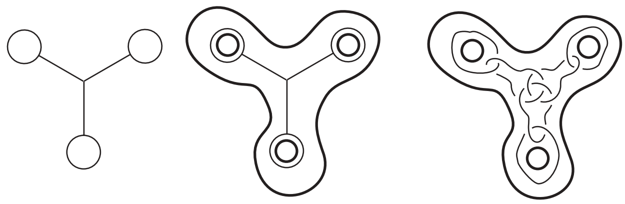


FIGURE 4. *Y-graph, its neighborhood and surgery link.*

It is enough to show that $\Upsilon_{\mathfrak{g}}(\mathrm{ad}_X(g)) = 0$ as a function on $\mathfrak{h}^*$. Evaluating $\Upsilon_{\mathfrak{g}}(\mathrm{ad}_X(g))$ on $\lambda \in \mathfrak{h}^*$ such that $\lambda - \rho$ is a real dominant weight, one has

$$\begin{aligned}
\Upsilon_{\mathfrak{g}}(\mathrm{ad}_X(g))(\lambda) &= \mathrm{Tr}_{V_{\lambda-\rho}} \Upsilon_{\mathfrak{g}}(\mathrm{ad}_X(g)) \quad \text{by definition} \\
&= \mathrm{Tr}_{V_{\lambda-\rho}} \mathrm{ad}_X(\Upsilon_{\mathfrak{g}}(g)) \quad \text{since } \Upsilon_{\mathfrak{g}} \text{ is an intertwiner} \\
&= \mathrm{Tr}_{V_{\lambda-\rho}} (X\Upsilon_{\mathfrak{g}}(g) - \Upsilon_{\mathfrak{g}}(g)X) \quad \text{by definition of } \mathrm{ad}_X \text{ on } U(\mathfrak{g}) \\
&= 0.
\end{aligned}$$

□

5.2. *The link case through the knot case*

Here we discuss another approach to the link case using general results on finite-type invariants. We will prove that if two multiplicative finite-type invariants of rational homology 3-spheres coincide on the set of rational homology 3-spheres obtained from S^3 by surgery along knots, then they are equal.

Let $\mathcal{H}_1$ be the set of all integral homology 3-spheres that can be obtained from S^3 by surgery along knots with framing ± 1, and $\mathcal{H}_1^{\oplus}$ the set of all finite connected sums of elements in $\mathcal{H}_1$.

5.2.1. *Finite-type invariants of rational homology 3-spheres.* We summarize here some basic facts about finite-type invariants of *rational* homology 3-spheres (Ohtsuki, Goussarov–Habiro, for details see [**10**, **11**]).

Consider the *standard Y-graph* in $\mathbb{R}^3$; see Figure 4. A Y-graph C in M is the image of an embedding of a small neighborhood of the standard Y-graph into M. Let L be the six-component link in a small neighborhood of the standard Y-graph as shown in Figure 4, each component having framing 0. The surgery of M along the image of the six-component link is called a Y-surgery along C, denoted by M_C.

Matveev [**28**] proved that M and M' are related by a finite sequence of Y-surgeries if and only if there is an isomorphism from $H_1(M, \mathbb{Z})$ onto $H_1(M', \mathbb{Z})$ preserving the linking form on the torsion group. For a 3-manifold M, let $\mathcal{C}(M)$ be the free R-module with basis all 3-manifolds that have the same H_1 and linking form as M. Here R is a commutative ring with unit. For example, $\mathcal{C}(S^3)$ is the free R-module spanned by all integral homology 3-spheres. We will always assume that 2 is invertible in R. Actually, for the application in this paper, it is enough to consider the case when R is a field of characteristic 0.

Let E be a finite collection of disjoint Y-graphs in a 3-manifold N. Define

$$[N, E] = \sum_{E' \subset E} (-1)^{|E'|} N_{E'}.$$

Define $\mathcal{F}_n\mathcal{C}(M)$ as R-submodule of $\mathcal{C}(M)$ spanned by all $[N, E]$ such that N is in $\mathcal{C}(M)$ and $|E| = n$. Any invariant I of 3-manifolds in $\mathcal{C}(M)$ with values in an R-module A can be extended linearly to an R-linear function $I : \mathcal{C}(M) \to A$. Such an invariant I is a *finite-type invariant*

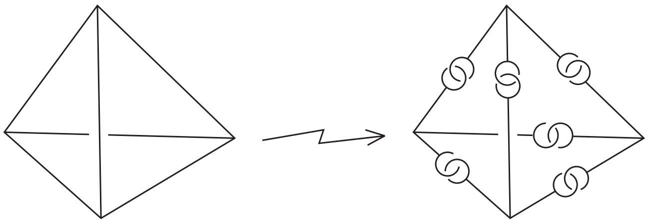


FIGURE 5. *From Jacobi graph to Y-graphs.*

of degree at most n, if $I|_{\mathcal{F}_{n+1}} = 0$. Matveev's result shows that an invariant of degree 0 is a constant invariant in each class $\mathcal{C}(M)$.

Goussarov and Habiro [11] showed that $\mathcal{F}_{2n-1} = \mathcal{F}_{2n}$. There is a *surjective* map

$$W : \mathrm{Gr}_n\mathcal{A}(\emptyset) \to \mathcal{F}_{2n}\mathcal{C}(M)/\mathcal{F}_{2n+1}\mathcal{C}(M),$$

known as the universal weight map, defined as follows. Suppose that $D \in \mathrm{Gr}_n\mathcal{A}(\emptyset)$ is a Jacobi graph of degree n. Embed D into S^3 arbitrarily. Then from the image of D construct a set E of Y-graphs as in Figure 5. By definition, $[M\#S^3, E] \in \mathcal{F}_{2n}\mathcal{C}(M)$. A priori, $[M\#S^3, E]$ depends on the way D is embedded in S^3. However,

$$W(D) := [M\#S^3, E] \pmod{\mathcal{F}_{2n+1}\mathcal{C}(M)}$$

depends only on D as an element in $\mathcal{A}(\emptyset)$. Moreover, the map $W : \mathrm{Gr}_n\mathcal{A}(\emptyset) \to \mathcal{F}_{2n}\mathcal{C}(M)/\mathcal{F}_{2n+1}\mathcal{C}(M)$, known as the universal weight, is *surjective*.

LEMMA 5.3. *Suppose that D is connected. Then S^3_E can be obtained from S^3 by surgery along a knot with framing ± 1, and $S^3_E \in \mathcal{H}_1$.*

Proof. Choose a sublink E' of E consisting of all components of E except for one component K, and do surgery along this sublink. Using repeatedly, the move that removes a 0-framing trivial knot together with another knot piercing the trivial knot, it is easy to see that the resulting manifold is still S^3. Let K' be the image of K in the resulting S^3. Now one has $S^3_E = S^3_{K'}$, an integral homology 3-sphere. The framing of K' must be ± 1 because the result is an integral homology 3-sphere. □

If I is a finite-type invariant of degree at most $2n$, then its nth weight is defined as the composition

$$w_I^{(n)} = I \circ W : \mathrm{Gr}_n\mathcal{A}(\emptyset) \to V.$$

It is clear that if $w_I^{(n)} = 0$, then I has degree at most $2n - 2$.

5.2.2. *Multiplicative finite-type invariants and surgery on knots.* The following result shows that finite invariants are determined by their values on a smaller subset of the set of all applicable 3-manifolds. Besides application to the proof of the LMO conjecture, the result is also interesting by itself.

THEOREM 5.4. (a) *Suppose that I is a finite-type invariant of* integral *homology 3-spheres with values in an R-module A such that $I(M) = 1$ for every $M \in \mathcal{H}_1^{\oplus}$. Then $I(M) = 1$ for every integral homology 3-sphere.*

(b) *Suppose that I is a* multiplicative *finite-type invariant of* rational *homology 3-spheres with values in an R-algebra A. If $I(M) = 1$ for every $M \in \mathcal{H}_1$ and every lens space $M = L(p, 1)$, then $I(M) = 1$ for every rational homology 3-sphere. In particular, if $I(M) = 1$ for any rational homology 3-sphere obtained from S^3 by surgery along knots, then $I(M) = 1$ for any rational homology 3-sphere.*

Proof. (a) Suppose that I has degree at most $2n$. Let D be a Jacobi diagram of degree n. Suppose that $D = \prod_{j=1}^{s} D_j$. Let E_j be the Y-graphs corresponding to D_j as constructed in Paragraph 5.2.1, and put $E = \bigsqcup_{j=1}^{s} E_j$. Since each of $S^3_{E_j}$ is in $\mathcal{H}_1$ by Lemma 5.3, $S^3_E = \#_{j=1}^{s} S^3_{E_j}$ is in $\mathcal{H}_1^{\oplus}$.

Then

$$\begin{aligned} w_I^{(n)}(D) &= I([S^3, E]) \\ &= I(S^3) - I(S^3_E) \\ &= 0 \text{ because } S^3_E \in \mathcal{H}_1^{\oplus}. \end{aligned}$$

It follows that I is an invariant of degree at most $2n - 2$. Induction then shows that I is an invariant of degree 0, or just a constant invariant. Hence, $I(M) = I(S^3) = 1$ for every integral homology 3-sphere M.

(b) Suppose that I is a finite-type invariant of degree at most $2n$, and D is a Jacobi diagram of degree n. Let us restrict I on the class $\mathcal{C}(M)$. One has

$$\begin{aligned} w_I^{(n)}(D) &= I([M\#S^3, E]) \\ &= I(M) - I(M\#S^3_E) \\ &= I(M) - I(M)I(S^3_E) \text{ because } I \text{ is multiplicative} \\ &= 0. \end{aligned}$$

Hence, again I is an invariant of degree 0, or I is a constant invariant on every class $\mathcal{C}(M)$.

Since $I(M) = 1$ for every lens space of the form $L(p, 1)$, it follows that if a rational homology sphere M belongs to $\mathcal{C}(N)$, where N is the connected sum of a finite number of lens spaces of the form $L(p, 1)$, then $I(M) = 1$.

Ohtsuki's lemma [**29**] says that, for every rational homology sphere M, there are lens spaces $L(p_1, 1), \ldots, L(p_s, 1)$ such that the linking form of $N = M\#(\#_{j=1}^{s} L(p_j, 1))$ is the sum of the linking forms of a finite number of lens spaces of the form $L(p, 1)$. Since I is multiplicative

$$I(N) = I(M) \prod_{j=1}^{s} I(L(p_j, 1)).$$

With $I(N) = 1 = I(L(p_j, 1))$, it follows that $I(M) = 1$. □

5.2.3. *Another proof of Theorem* 1.1 *in the link case*

Proof of Theorem 1.1 *in the link case.* When R is a field of characteristic 0, the LMO invariant is universal among finite-type invariants. This fact can be reformulated as $W : \mathrm{Gr}_n\mathcal{A}(\emptyset) \to \mathcal{F}_{2n}\mathcal{C}(M)/\mathcal{F}_{2n+1}\mathcal{C}(M)$ is a bijection. This was proved for integral homology 3-spheres by Le [**23**] and for general rational homology spheres by Habiro [**11**]. In particular, this result says that the part of degree at most n of $\hat{Z}^{\mathrm{LMO}}$ is a (universal) finite-type invariant of degree at most $2n$.

Note that $\hat{W}_{\mathfrak{g}}(\hat{Z}^{\mathrm{LMO}})$ and τ^g are multiplicative invariant with values in $\mathbb{R}[[\hbar]]$. By Proposition 6.1, the degree at most n part $\tau^{\mathfrak{g}}_{\leqslant n}$ of $\tau^{\mathfrak{g}}$ is a finite-type invariant of degree at most $2n$. Let $I = |H_1|^{\phi_+}\tau^g/\hat{W}_{\mathfrak{g}}(\hat{Z}^{\mathrm{LMO}})$. Then the part $I_{\leqslant n}$ of degree at most n is an invariant of degree at most $2n$. Clearly, $I_{\leqslant n}$ is multiplicative. Moreover, $I_{\leqslant n}(M) = 1$ if M is obtained from S^3 by surgery along knots by the knot case. Hence, by Theorem 5.4, $I_{\leqslant n} = 1$. Since this holds true for every n, one has $I = 1$, and hence, $\hat{W}_{\mathfrak{g}}(\hat{Z}^{\mathrm{LMO}}) = \tau^{\mathfrak{g}}$. □

6. *Presentations of the perturbative invariants*

In this section, we discuss the perturbative invariant $\tau^{\mathfrak{g}}(M)$. In particular, we prove Proposition 3.4 and show that the degree n part of the perturbative invariant is a finite-type invariant of degree at most $2n$. We also give an informal way to explain how one can arrive at the formula of the perturbative invariant given by Proposition 3.4.

6.1. *Perturbative expansion of a Gaussian integral*

In this section, we explain how a Gaussian integral with a formal parameter in the exponent can be understood in perturbative expansions. For the perturbative expansion of a Gaussian integral, see also [**3**, Appendix].

Suppose that V is a finite-dimensional Euclidean space, f is a non-zero integer, and $R \in S(V) = P(V^*)$. The Gaussian integral

$$I = \int_{V^*} e^{f\hbar|x|^2/2} R(x)\, dx$$

does not make sense if $\hbar$ is a formal parameter. If $\hbar$ is a real number such that $f\hbar < 0$, then the integral converges absolutely, and one can calculate the integral as follows. A substitution $x = u/\sqrt{-2f\hbar}$ leads to

$$\begin{aligned} I &= \frac{1}{(-2f\hbar)^{\dim V/2}} \int_{V^*} e^{-|u|^2/4} R\left(\frac{u}{\sqrt{-2f\hbar}}\right) du \\ &= \left(\frac{2\pi}{-f\hbar}\right)^{\dim V/2} \mathcal{E}_{\mathfrak{h}}^{(f)}(R) \quad \text{by Lemma 4.1.} \end{aligned}$$

If $\hbar$ is a formal parameter, then the right-hand side still makes sense as an element in $\mathbb{R}[1/\hbar]$. Thus, we should declare

$$\int_{V^*} e^{f\hbar|x|^2/2} R(x)\, dx = \left(\frac{2\pi}{-f\hbar}\right)^{\dim V/2} \mathcal{E}_{\mathfrak{h}}^{(f)}(R) \tag{6.1}$$

for a formal parameter $\hbar$. Note that if $R \in S(V)[[\hbar]]$ is tame, then the right-hand side is in $\mathbb{R}[[\hbar]]$.

6.2. *Derivation of the perturbative invariants from the WRT invariant*

First we review the 3-manifold WRT invariant; for details, see, for example, [**25**]. We again assume that M is obtained from S^3 by surgery along an algebraically split link L with framing $\boldsymbol{f} = (f_1, \ldots, f_\ell)$. Let L_0 be the link L with all framings 0, and let T be an algebraically string link (with 0-framing on each component) such that its closure is L_0.

For an ℓ-tuple $(V_{\lambda_1-\rho}, \ldots, V_{\lambda_\ell-\rho})$ of $\mathfrak{g}$-modules one can define the quantum link invariant $Q^{\mathfrak{g};V_{\lambda_1-\rho},\ldots,V_{\lambda_\ell-\rho}}(L_0)$ of the link L_0 (see [**33**], we use here notation from the book [**31**]). This invariant can be calculated through the Kontsevich invariant by results of Kassel [**18**] and Le and Murakami [**26**]:

$$Q^{\mathfrak{g};V_{\lambda_1-\rho},\ldots,V_{\lambda_\ell-\rho}}(L_0) = (Z(T)\Delta^{(\ell)}(\nu))(\lambda_1, \ldots, \lambda_\ell). \tag{6.2}$$

In particular, when $L_0 = U$, the unknot with framing 0, $Q^{\mathfrak{g};V_{\lambda-\rho}}(U)$ is called the quantum dimension of $V_{\lambda-\rho}$, denoted by q-$\mathcal{D}(\lambda)$; its value is well known:

$$\text{q-}\mathcal{D}(\lambda) = \prod_{\alpha\in\Phi_+} \frac{[(\lambda,\alpha)]}{[(\rho,\alpha)]}, \tag{6.3}$$

where $[n] := (e^{n\hbar/2} - e^{-n\hbar/2})/(e^{\hbar/2} - e^{-\hbar/2})$.

The quantum invariant of L differs from that of L_0 by the framing factors, which will play the role of the exponential function in the Gaussian integral:

$$Q^{\mathfrak{g};V_{\lambda_1-\rho},\ldots,V_{\lambda_\ell-\rho}}(L) = \left(\prod_{j=1}^{\ell} e^{f_j(|\lambda_j|^2-|\rho|^2)\hbar/2}\right) Q^{\mathfrak{g};V_{\lambda_1-\rho},\ldots,V_{\lambda_\ell-\rho}}(L_0). \tag{6.4}$$

The normalization used in the definition of the WRT invariant is

$$F_L(\lambda_1,\ldots,\lambda_\ell) := \left(\prod_{j=1}^{\ell} \text{q-}\mathcal{D}(\lambda_j)\right) Q^{\mathfrak{g};V_{\lambda_1-\rho},\ldots,V_{\lambda_\ell-\rho}}(L).$$

Using (6.2) and (6.4), one can show that

$$F_L(\lambda_1,\ldots,\lambda_\ell) = (e^{-\sum_j f_j|\rho^2|\hbar/2})(e^{\sum_j f_j\hbar|\lambda_j|^2/2} R(\lambda_1,\ldots,\lambda_\ell)), \tag{6.5}$$

where $R = \mathcal{D}^{\otimes\ell}\Upsilon_{\mathfrak{g}}(\check{Q}^{\mathfrak{g}}(T)) = F_{L_0}$.

Suppose that $e^{\hbar}$ is a complex root of unity of order r. Then, $F_L(\lambda_1,\ldots,\lambda_\ell)$ is a polynomial in $e^{\hbar}$ and is component-wise invariant under the translation by $r\alpha$ for any α in the root lattice; see [**24**]. Let $D_r \subset \mathfrak{h}^*$ be any fundamental domain of the translations by $r\alpha$ with α in the root lattice. Then, with $e^{\hbar}$ an rth root of 1,

$$I(L) := \sum_{\lambda_j\in D_r} F_L(\lambda_1,\ldots,\lambda_\ell) \tag{6.6}$$

is invariant under the handle slide move. A standard normalization of $I(L)$ gives us an invariant of 3-manifolds, which is the WRT invariant.

Because of the translational invariance of F_L, we could define the WRT invariant if we replace D_r by ND_r in (6.6), where N is any positive integer. When we let $N \to \infty$, we should sum over all the weight lattice in (6.6) which does not converge. Instead, we use integral over $\mathfrak{h}^*$, that is, instead of $I(L)$ we consider the integral

$$\int_{(\mathfrak{h}^*)^\ell} F_L(\lambda_1,\ldots,\lambda_\ell)\, d\lambda_1\ldots d\lambda_\ell,$$

which might not make sense in a usual sense. However, using $F_L(\lambda_1,\ldots,\lambda_\ell)$ in (6.5), the integral has the form of a Gaussian integral discussed in the previous section. According to (6.1), the above integral should be a constant multiple of the following modification of $I(L)$:

$$I_2(T,\boldsymbol{f}) := \left(\prod_{j=1}^{\ell} e^{-f_j|\rho|^2\hbar/2}\right) \mathcal{E}_{\mathfrak{h}}^{(\boldsymbol{f})}((\hbar^{\phi_+}\mathcal{D})^{\otimes\ell}\Upsilon_{\mathfrak{g}}(\check{Q}^{\mathfrak{g}}(T))),$$

which leads to the formula in Proposition 3.5.

6.3. *Proof of Proposition* 3.4

First we review Le's formula of $\tau^{\mathfrak{g}}$; for details, see [**25**]. As noted in the previous section, as functions on $\mathfrak{h}^{*\ell}$,

$$F_{L_0} = \mathcal{D}^{\otimes\ell}\Upsilon_{\mathfrak{g}}(\check{Q}^{\mathfrak{g}}(T)).$$

Let $\mathcal{O}^{(f)}: P(\mathfrak{h}^*) = S(\mathfrak{h}) \to \mathbb{R}[1/\hbar]$ be the unique linear operator defined by

$$\mathcal{O}^{(f)}(\beta^k) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ e^{-f|\rho|^2\hbar/2}(2d-1)!! \left(-\frac{|\beta|^2}{f}\right)^d \hbar^{-d} & \text{if } k = 2d, \end{cases} \tag{6.7}$$

for $\beta \in \mathfrak{h}$. We also define its multi-linear extension

$$\mathcal{O}^{(\boldsymbol{f})}: P_\hbar(\mathfrak{h}^{*\ell}) \to \mathbb{R}[[\hbar]], \quad \mathcal{O}^{(\boldsymbol{f})} := \bigotimes_{j=1}^{\ell} \mathcal{O}^{(f_j)}.$$

Let

$$I_2'(T, \boldsymbol{f}) := \mathcal{O}^{(\boldsymbol{f})}(\hbar^{\ell\phi_+} F_{L_0}) = \mathcal{O}^{(\boldsymbol{f})}((\hbar^{\phi_+}\mathcal{D})^{\otimes \ell} \Upsilon_{\mathfrak{g}}(\check{Q}^{\mathfrak{g}}(T))).$$

Then, as in [**25**], the perturbative invariant $\tau^{\mathfrak{g}}(M)$ is given by

$$\tau^{\mathfrak{g}}(M) = \frac{I_2'(T, \boldsymbol{f})}{\prod_{j=1}^{\ell} I_2'(\downarrow, \operatorname{sign}(f_j))}.$$

To prove Proposition 3.4, one needs only to show that $I_2(T; \boldsymbol{f}) = I_2'(T; \boldsymbol{f})$. It is enough to show that

$$\mathcal{O}^{(f)}(g) = e^{-f|\rho|^2\hbar/2}\mathcal{E}_{\mathfrak{h}}^{(f)}(g) \tag{6.8}$$

for every $g \in S(\mathfrak{h})$. Since both operators $\mathcal{O}^{(f)}$ and $\mathcal{E}_{\mathfrak{h}}^{(f)}$ are linear and W-invariant, it is sufficient to consider the case when $g = x_1^k$, where x_1 is the first vector of an orthonormal basis $x_1, \ldots, x_n$ of $\mathfrak{h}$. In this case $\Delta_{\mathfrak{h}^*} = \sum \partial_{x_j}^2$, and one can easily calculate $\mathcal{E}_{\mathfrak{g}}^{(f)}(x_1^k)$:

$$\begin{aligned} \mathcal{E}_{\mathfrak{g}}^{(f)}(x_1^k) &= \exp\left(\frac{\Delta_{\mathfrak{h}}}{-2f\hbar}\right)(x_1^k)|_{x_j=0} = \sum_d \frac{\Delta^d}{d!(-2f\hbar)^d}(x_1^k) \\ &= \begin{cases} 0 & \text{if } k \text{ is odd,} \\ (2d-1)!! \left(-\frac{1}{f}\right)^d \hbar^{-d} & \text{if } k = 2d, \end{cases} \end{aligned}$$

which is precisely the right-hand side of (6.7) without the factor $e^{-f|\rho|^2\hbar/2}$ (with $\beta = x_1$). This proves (6.8).

6.4. *The coefficients of $\tau^{\mathfrak{g}}$ are of finite type*

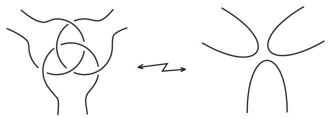
PROPOSITION 6.1. *The degree n part of the perturbative invariant $\tau^{\mathfrak{g}}$ is a finite-type invariant of degree at most $2n$.*

REMARK 4. The proposition is a consequence of the main theorem. However, we used this proposition in the alternative proof of the main theorem in Subsection 5.2. This is why we give here a proof of the proposition independently of the main theorem.

Proof. Let M be a rational homology 3-sphere and E a collection of $2n+1$ disjoint Y-graphs in M. We only need to prove that $\tau^{\mathfrak{g}}([M, E]) \in \hbar^{n+1}\mathbb{Q}[[\hbar]]$.

By taking the connecting sum with lens spaces, we assume that the pair (M, E) can be obtained from (S^3, E) by surgery along an algebraically split link $L \subset S^3$. By adding trivial knots with framing ± 1 (which are unlinked with L) to L if needed, we can assume that the leaves of $E \in S^3$ form a trivial link. Let L_0 be the link L with 0-framing, and choose a string link T in a cube such that L_0 is the closure of T. We can assume that E is also in the cube.

For a sub-collection $E' \subset E$, let $L_{E'}$ be the link obtained by surgery of S^3 along E' (see [**10**, **11**]). We define similarly $(L_0)_{E'}$ and $T_{E'}$. Clearly, $(L_0)_{E'}$ is the closure of $T_{E'}$. For every link L and every Y-graph C whose leaves form a 0-framing trivial link, the move from L to L_C is a repetition of the Borromeo move (see [**10**, **11**]):



Hence, by Le [**23**, Lemma 5.3], $Z(T - T_C)$ has i-degree at least 1. Here $x \in \mathcal{A}(\sqcup^\ell \downarrow)$ has i-degree at least k if it is a linear combination of Jacobi diagrams with at least k trivalent vertices. It follows that $\check{Z}([T, E])$ has i-degree at least $2n+1$, where $[T, E] := \sum_{E' \subset E} (-1)^{|E'|} T_{E'}$. Note that all the links $L_{E'}, E' \subset E$ are algebraically split, having the same number of components, and having the same framings $\boldsymbol{f} = (f_1, \dots, f_\ell)$. By definition, one has

$$[M, E] = \sum_{E' \subset E} (-1)^{|E'|} (S^3)_{L_{E'}}.$$

Hence,

$$\begin{aligned}
\tau^{\mathfrak{g}}([M, E]) &= \sum_{E' \subset E} (-1)^{|E'|} \tau^{\mathfrak{g}}((S^3)_{L_{E'}}) \\
&= \sum_{E' \subset E} (-1)^{|E'|} \frac{I_2(T_{E'}, \boldsymbol{f})}{\prod_{j=1}^{\ell} I_2(\downarrow, \operatorname{sign}(f_j))} \\
&= \frac{I_2([T, E], \boldsymbol{f})}{\prod_{j=1}^{\ell} I_2(\downarrow, \operatorname{sign}(f_j))} \\
&= \frac{(\prod_{j=1}^{\ell} e^{-f_j |\rho|^2 \hbar/2}) \mathcal{E}_{\mathfrak{h}}^{(\boldsymbol{f})}((\hbar^{\phi_+} \mathcal{D})^{\otimes \ell} \check{Z}([T, E]))}{\prod_{j=1}^{\ell} I_2(\downarrow, \operatorname{sign}(f_j))}.
\end{aligned} \tag{6.9}$$

By Lemma 6.2, since $\check{Z}([T, E])$ has i-degree at least $2n+1$, the numerator of (6.9) belongs to $\hbar^{n+1}\mathbb{R}[[\hbar]]$, while the denominator is invertible in $\mathbb{R}[[\hbar]]$. It follows that the right-hand side of (6.9) belongs to $\hbar^{n+1}\mathbb{R}[[\hbar]]$. □

LEMMA 6.2. (a) *If $D \in \mathcal{A}(\sqcup^\ell \downarrow)$ is a Jacobi diagram having at least $2n+1$ trivalent vertices, then $\mathcal{E}_{\mathfrak{h}}^{(\boldsymbol{f})}((\hbar^{\phi_+} \mathcal{D})^{\otimes \ell} \hat{W}_{\mathfrak{g}}(D)) \in \hbar^{n+1}\mathbb{R}[[\hbar]]$.*

(b) *The lowest degree of $\hbar$ in $I_2(\downarrow, \pm 1) \in \mathbb{R}[[\hbar]]$ is 0, that is, $I_2(\downarrow, \pm 1)$ is invertible.*

Proof. (a) Suppose that D has degree d. Then D has $2d$ vertices, among which $2d - 2n - 1$ are univalent. It follows that $W_{\mathfrak{g}}(D)$, as an element of $U(\mathfrak{g})^{\otimes \ell}$, has degree at most $(2d - 2n - 1)$ and, as a function on $(\mathfrak{h}^*)^\ell$, is a polynomial of degree at most $\ell\phi_+ + (2d - 2n - 1)$; see [**25**]. Hence, the degree of $\mathcal{D}^{\otimes \ell} W_{\mathfrak{g}}(D)$ is at most $2\ell\phi_+ + 2d - 2n - 1$. Recall that $\mathcal{E}_{\mathfrak{h}}^{(\boldsymbol{f})}(g)$ decreases the degree of $\hbar$ by at most half the degree of g. The degree of $\hbar$ in $\mathcal{E}_{\mathfrak{h}}^{(\boldsymbol{f})}(\mathcal{D}^{\otimes \ell} \hat{W}_{\mathfrak{g}}(D)) = \hbar^d \mathcal{E}_{\mathfrak{h}}^{(\boldsymbol{f})}(\mathcal{D}^{\otimes \ell} W_{\mathfrak{g}}(D))$ is at least $d - \frac{1}{2}(2\ell\phi_+ + 2d - 2n - 1) = \frac{1}{2} + n - \ell\phi_+$. Hence, $\mathcal{E}_{\mathfrak{h}}^{(\boldsymbol{f})}((\hbar^{\phi_+} \mathcal{D})^{\otimes \ell} \hat{W}_{\mathfrak{g}}(D)) \in \hbar^{n+1}\mathbb{R}[[\hbar]]$.

(b) By definition

$$I_2(\downarrow, \pm 1) = e^{\mp |\rho|^2 \hbar/2} \mathcal{E}_{\mathfrak{h}}^{(\pm 1)}(\hbar^{\phi_+} \mathcal{D} \hat{W}_{\mathfrak{g}}(\check{Z}(\downarrow))).$$

For the trivial knot, everything can be calculated explicitly. One has $\mathcal{D}\hat{W}_{\mathfrak{g}}(\check{Z}(\downarrow)) = (\text{q-}\mathcal{D})^2$, and using (6.3) one can easily show that

$$\mathcal{D}\hat{W}_{\mathfrak{g}}(\check{Z}(\downarrow)) = \mathcal{D}^2 \left(1 + \sum_{k=1}^{\infty} g_k \hbar^{2k}\right),$$

where g_k has degree exactly $2k$. Thus,

$$e^{\pm|\rho|^2\hbar/2} I_2(\downarrow, \pm 1) = \hbar^{\phi_+}\mathcal{E}_{\mathfrak{h}}^{(\pm 1)}(\mathcal{D}^2) + \hbar^{\phi_+}\mathcal{E}_{\mathfrak{h}}^{(\pm 1)}\left(\sum_{k=1}^{\infty} g_k \mathcal{D}^2 \hbar^{2k}\right).$$

Since $\deg(g_k) = 2k$ and $\deg(\mathcal{D}^2) = 2\phi_+$, the second term belongs to $\hbar\mathbb{R}[[\hbar]]$, while the first term is

$$\hbar^{\phi_+}\mathcal{E}_{\mathfrak{h}}^{(\pm 1)}(\mathcal{D})^2 = \frac{\Delta_{\mathfrak{h}}^{\phi_+}(\mathcal{D}^2)}{(\phi_+)!(\mp 2)^{\phi_+}} = \frac{c_{\mathfrak{g}}}{(\mp 2)^{\phi_+}}.$$

Since $c_{\mathfrak{g}} \neq 0$, we conclude that $I_2(\downarrow, \pm 1)$ is invertible in $\mathbb{R}[[\hbar]]$. □

Appendix. Elements of $U(g)$ as polynomial functions on $\mathfrak{h}^*$

Let $\mathrm{Tr}(g, V_{\lambda-\rho})$ be the trace of the action of g on the $\mathfrak{g}_{\mathbb{C}}$-module $V_{\lambda-\rho}$.

PROPOSITION A.1. (a) *For every $g \in U(\mathfrak{g}_{\mathbb{C}})$, there exists a unique polynomial function p_g on $\mathfrak{h}_{\mathbb{C}}^*$, such that for every dominant real weight $\lambda - \rho$,*

$$\mathrm{Tr}(g, V_{\lambda-\rho}) = p_g(\lambda).$$

Moreover, p_g is divisible by $\mathcal{D}$, and the polynomial function $\psi_{\mathfrak{g}}(g) := p_g/\mathcal{D}$ is W-invariant. If g is central, then $\psi_{\mathfrak{g}}(g)$ coincides with the one defined in diagram (2.2).

(b) *If $g \in U(\mathfrak{g})$, then p_g is real, that is, $p_g \in P(\mathfrak{h}^*)$.*

Proof. (a) If g is central, then the statement is [**36**, Theorem XVII.7]. For the general case, we use the decomposition of $U(\mathfrak{g}_{\mathbb{C}})$ into $\mathfrak{g}$-module:

$$U(\mathfrak{g}_{\mathbb{C}}) = U(\mathfrak{g}_{\mathbb{C}})^{\mathfrak{g}_{\mathbb{C}}} \oplus U', \tag{A.1}$$

where $U' = \mathrm{ad}_{\mathfrak{g}_{\mathbb{C}}} U(\mathfrak{g}_{\mathbb{C}}) = \{xy - yx \mid x, y \in U(\mathfrak{g}_{\mathbb{C}})\}$ (see [**17**, Exercise 23.7]). For $g \in U(\mathfrak{g}_{\mathbb{C}})$, let g' and g'' be the projections of g onto the first and the second components of (A.1). Since g'' is a commutator, its trace on any module is 0. Hence, we have $\mathrm{Tr}(g, V) = \mathrm{Tr}(g', V)$ for any $\mathfrak{g}_{\mathbb{C}}$-module V, and we can define $p_g = p_{g'}$.

(b) has been proved at the end of Subsection 2.3, using the commutativity of the complexified version of diagram (2.2). □

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