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A TQFT Associated to the LMO Invariant of Three-Dimensional Manifolds ⋆

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Abstract: We construct a Topological Quantum Field Theory associated to the universal finite-type invariant of 3-dimensional manifolds, as a functor from a category of 3-dimensional manifolds with parametrized boundary, satisfying some additional conditions, to an algebraic-combinatorial category. This is built together with its truncations with respect to a natural grading, and we prove that these TQFTs are non-degenerate and anomaly-free. The TQFT(s) induce(s) a (series of) representation(s) of a subgroup $\mathcal{L}_g$ of the Mapping Class Group that contains the Torelli group. The $N = 1$ truncation is a TQFT for the Casson-Walker-Lescop invariant.

A TQFT for the LMO invariant was constructed by Murakami and Ohtsuki [21], but it has complicated anomaly. One of the results of the present paper is to show that their anomaly operator reflects only: 1) the way we define the gluing between two cobordisms, i.e. if we regard $\#_g(S^1 \times S^2)$ or S^3 as the simplest manifold(s), and 2) that they consider the un-normalized invariant. Our construction of TQFT allows us to associate to the LMO invariant an infinite-dimensional *linear* representation of the Torelli group, in fact of a larger *Lagrangian subgroup* of the Mapping Class Group.

The new results of this paper include an isomorphism (Proposition 2.2), reducing the study of the LMO invariant of 3-dimensional manifolds with parametrized boundary to that of finite-type invariants of string-links; the construction (from truncations; Theorem 2.9) of a composition operation on chord diagrams to correspond to the gluing of cobordisms; a limit property (Lemmas 3.7, 3.8); non-degeneracy of the TQFT. The natural truncation induces a TQFT for the Walker-Lescop extension of the Casson invariant, and we can identify Morita's representation as its first non-trivial part. It remains, however, to interpret our TQFT as some sort of "perturbative expansion around 0" of the Reshetikhin-Turaev TQFT.

⋆ The results of this article were obtained when the authors were at the Department of Mathematics, State University of New York at Buffalo, Buffalo, NY 14260-2900, USA.

This paper is organized as follows. In Sect. 1 we recall the topological categories $\mathfrak{Q}$, $\mathfrak{Z}$ introduced in [6], and the pertaining results that we will need subsequently in this paper. The categories $\mathcal{A}$ and $\mathcal{A}^{\leq N}$ of chord diagrams are explained in Sect. 2. We use a simpler (then in [21]) definition of Z on elementary pseudo-quasi-tangles[1], and an even associator instead of the Knizhnik-Zamolodchikov one. In Sect. 3 we formally construct the anomaly-free (by [6]) truncated and full TQFTs. One of the results of that section is showing that the completion of the algebraic image of cobordisms with one boundary component of genus g is precisely the whole space $\mathcal{A}(\uparrow_g)$ of chord diagrams on g vertical lines. This means that the induced representation can (in principle) be used for combinatorial calculations in addressing topological questions about three-dimensional manifolds and the Mapping Class Group. We also finish the proof of Theorem 2.9 there. Functoriality, and other results regarding the TQFT are gathered in Theorem 3.2. In Sect. 4 we restrict to the case $N = 1$ to get a TQFT for the Casson-Walker-Lescop invariant. Also there we describe (algebraically) the map that sends the invariant of a manifold with parametrized boundary to the invariant of the closed manifold obtained from the former by the natural procedure that we call below *filling*.

Let H_g denote a fixed "standard" handlebody of genus g, and let H^g denote $\overline{S^3 - H_g}$. The TQFT construction in [21] and the Reshetikhin-Turaev TQFT for quantum invariants are based on the classical convention under which the identical gluing $H_g \cup_{id} H_g = \#_g(S^1 \times S^2)$. However for invariants that are primarily of homology spheres this convention is not natural. The LMO invariant is very strong for $\mathbb{Z}$- and $\mathbb{Q}$-homology spheres, but is much weaker if the Betti number is higher. In order to address this issue in the present paper we will regard $\Sigma_g = \partial H_g = -\partial H^g \subset \mathbb{R}^3 \subset S^3$ as the standard surface of genus g used to parametrize boundary components of cobordisms, and in order to obtain the composition-cobordism $M_2 \cup_f M_1$ we will always glue the "top" boundary component of M_1 to the "bottom" boundary component of M_2 along an orientation-*preserving* homeomorphism f, as explained in Subsect. 1.1. Hence we shall always have $H^g \cup_{id} H_g = S^3$ for all g, as opposed to Murakami-Ohtsuki's and Reshetikhin-Turaev's $H_g \cup_{id} H_g = \#_g(S^1 \times S^2)$.

Remarks. By fixing a homeomorphism between H^g and H_g, one fixes a Heegaard homeomorphism that Morita [20] calls ι_g. If one then takes, what we call below, the filling $(\widehat{\Sigma_g \times I, \varphi, id})$, one obtains the manifold denoted by Morita W_φ.

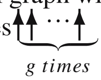
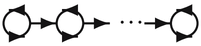
0.1. Chord diagrams. (For details see [2,4].) An *open chord diagram* is a vertex-oriented uni-trivalent graph, i.e. a graph with univalent and trivalent vertices together with a cyclic order of the edges incident to the trivalent vertices. Self-loops and multiple edges are allowed. A univalent vertex is also called a *leg*, and a trivalent vertex is also called an *internal vertex*. In planar pictures, the orientation of the edges incident to a vertex is the counterclockwise orientation, unless otherwise stated; the pictures can not be perfect since not every graph is planar, therefore when reading pictures one should keep in mind that four-valent vertices do not exist. The *degree* of an open chord diagram is half the number of all vertices.

Suppose Γ is a compact oriented 1-manifold (possibly with boundary) and X a finite set of asterisks. A *chord diagram with support* $\Gamma \cup X$ is a vertex-oriented uni-trivalent graph D together with a decomposition $D = \Gamma \cup E$, where E is an open chord diagram

[1] "pseudo" stands for the presence of 3-valent vertices.

with some legs labeled by elements of X, such that D is the result of gluing all non-labeled legs of E to distinct interior points of Γ (the 3-valent vertices resulted from gluing, which will NOT have an associated cyclic order of adjacent edges, are called *external vertices*). Repetition of labels is allowed and not all labels have to be used. The *degree* of D is, by definition, the degree of E. Γ is also called the *skeleton* of D, and in pictures is represented by bold lines. Often the components of Γ, as well as different asterisks in X, are distinguished in pictures by labels.

By *a graph* Γ we will mean a uni-trivalent graph, with all edges oriented, and with a cyclic order of edges incident to trivalent vertices prescribed. Self-loops and multiple edges are allowed. The connected components of the graph have to be always ordered (and this order has to be preserved by a homeomorphism). Additionally we may label (color) some subgraphs within each connected component. One should think of a graph as a generalization of the notion of an oriented compact 1-manifold. One can repeat the definition of the previous paragraph to obtain the notion of a *chord diagram with support a graph*; the graph is then the skeleton of the chord diagram. In the definition of the degree the vertices of the support are NOT counted. Examples of graphs Γ:

- the oriented manifold which is the union of $g \in \mathbb{N}^*$ copies of $[0, 1]$, each copy labeled (colored) by a distinct element of a finite abstract ordered set X of asterisks. This special graph will be denoted $\uparrow_X$, and in planar pictures will be represented by vertical lines  .
- the *chain graph*[2] suggestively denoted  . The order of the edges adjacent to each vertex is everywhere counterclockwise with respect to its standard embedding in $\mathbb{R}^2 \subset \mathbb{R}^3$, the subgraphs  are labeled 1 through g from left to right. Let us denote this special graph by Γ^g. Note that Γ^g standardly embedded in $\mathbb{R}^2 \subset \mathbb{R}^3 \subset S^3$ has a preferred (the blackboard) framing, indicated by a planar surface with Γ^g as core. For $g = 1$, set $\Gamma^1 =$ , one oriented edge (loop), no vertices.
- it is convenient to set $\Gamma^0 =$ *one point* as a chain graph. Chord diagrams on Γ^0 automatically can have only internal vertices.

1. The Topological Category

Definition 1.1 (see also [6]). *1) Two* **triplets** (K, G_1, G_2) *and* (L, H_1, H_2), *consisting each of a framed oriented link and two disjoint embedded* **Framed** *chain graphs in* S^3, *are equivalent (notation* $\cong$*) if there is a PL-homeomorphism* $\phi : S^3 \to S^3$ *which sends* K *to* L, G_1 *to* H_1, *and* G_2 *to* H_2. *Here* $\emptyset$ *is also considered a framed oriented link in* S^3. *We call* G_1 *the* **bottom**, *and* G_2 *the* **top** *of the triplet.*

2) Let M *be a compact oriented* 3*-manifold with boundary* $\partial M = (-S_1) \cup S_2$, *and suppose that parametrizations* $f_i : \Sigma_{g_i} \to S_i$ *are fixed.*[3] *Such* (M, f_1, f_2) *will be called a (parametrized) (2+1)-cobordism,* S_1 *will be called its* **bottom**, *and* S_2 *– its* **top**. *The cobordisms* (M, f_1, f_2) *and* (N, h_1, h_2) *are* **equivalent (homeomorphic)** *if there is a PL-homeomorphism* $F : M \to N$ *such that* $F \circ f_i = h_i$, $i = 1, 2$.

3) Let H_g *be a fixed neighborhood of* Γ^g *in* S^3. $\Sigma_g := \partial H_g \subset S^3$ *is the standard oriented surface of genus g. Denote* $H^g = \overline{S^3 - H_g}$. *Clearly* $\partial H^g = -\Sigma_g$. *Always glue*

[2] We borrow this terminology from [21].

[3] For simplicity restrict to 3-manifolds with ≤ 2 boundary components, one "bottom", and one "top". Note, however, that Propositions 1.2, 1.3 and 1.4 can be easily generalized.

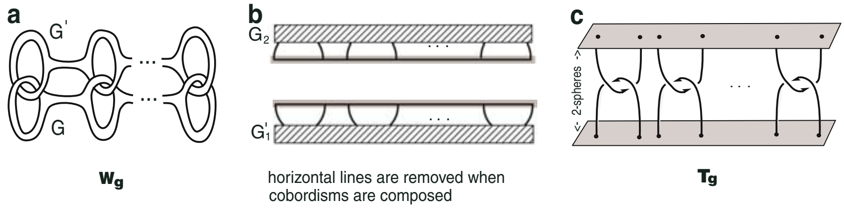


Fig. 1. **a:** The triplet $W_g := (\emptyset, G, G')$ represents the cobordism $(\Sigma_g \times [0,1], p_1, p_2)$; **b:** The horizontal lines (blue) of G_1' and G_2 (The shaded boxes represent symbolically the rest of the triplets); **c:** The framed tangle $T_g \subset \overline{B(0,2) - B(0,1)}$.

*cobordisms along orientation-***preserving** *homeomorphisms of boundary components, as specified by the parametrizations:* $(M_2, f_2, f_2') \circ (M_1, f_1, f_1') = (M_2 \cup_{f_2 \circ (f_1')^{-1}} \cup M_1, f_1, f_2')$. *When* $g = 0$ *we assume that* Γ^g *is a point, and* H_g *is a ball (resp. for* H^g*). Given a parametrized cobordism* (M, f_1, f_2)*, use the maps* f_i *to glue the standard handlebody* H_{g_1} *to the bottom, and the standard* **anti**-*handlebody* H^{g_2} *to the top of* (M, f_1, f_2)*, and call the result* $\widehat{M} := H^{g_2} \cup_{f_2} M \cup_{f_1} H_{g_1}$ **the filling** *of* (M, f_1, f_2).

Remarks. In classical notation our manifold $M_2 \cup_{f_2 \circ (f_1')^{-1}} \cup M_1$ is $M_2 \cup_{-f_2} (\Sigma_g \times [0,1]) \cup_{\iota_g^{-1}} (\Sigma_g \times [0,1]) \cup_{-(f_1')^{-1}} M_1$, where we can fix a Morita's ι_g by fixing a homeomorphism between H^g and H_g, and where the $-$ signs respect the fact that the "bottom" boundary component has orientation opposite to the one induced from the 3-manifold.

1.1. Surgery description of gluing cobordisms. Let $\mathfrak{G}$ denote set of equivalence classes of triplets (L, G_1, G_2) in S^3. Let $\mathfrak{C}$ denote the set of equivalence classes of 3-cobordisms, with connected non-empty bottom and top.

Proposition 1.2. [6] *1) The map* $\kappa : \mathfrak{G} \to \mathfrak{C}$ *that associates to every equivalence class of triplets* (L, G_1, G_2) *the equivalence class of cobordisms* (M, f_1, f_2)*, obtained by doing surgery on* $L \subset S^3$*, removing tubular neighborhoods* N_1, N_2 *of each* G_1, G_2*, and recording the parametrizations of the two obtained boundary components, is well-defined and surjective. If one glues according to these parametrizations a standard handlebody to* $-\partial N_1$ *and a standard anti-handlebody to* ∂N_2*, then one obtains* S^3_L.

2) Let a **first Kirby move** *on a triplet be the cancellation / insertion of a* $\mathbf{O}^{\pm 1}$ *separated by an* S^2 *from anything else, and an* **extended second Kirby move** *be a slide over a link component of an arc, either from another link component or from a chain graph. Then, if one factors* $\mathfrak{G}$ *by the extended Kirby moves and changes of orientations of link components, the induced map* $\overline{\kappa}$ *is a bijection.* □

For example, to represent the *identity cobordism* $(\Sigma_g \times [0,1], p_1, p_2)$, where $p_i : \Sigma_g \to \Sigma_g \times (i-1) \subset S^3$ are two copies of the standard embedding of Σ_g in S^3, $\forall g \geq 1$, we can choose the framed graph $W_g = (G, G')$ shown in Fig. 1a.

Let Γ, respectively Γ', generically denote the bottom, respectively the top, of a triplet. Call the union of the lower half-circles and the horizontal segments of Γ, *the horizontal line* of Γ. Similarly, call the union of the upper half-circles and the horizontal segments of Γ', *the horizontal line* of Γ'. (See Fig. 1b.)

Proposition 1.3. [6] *Let* (M_1, f_1, f_1') *and* (M_2, f_2, f_2') *be two 3-cobordisms with connected non-empty bottoms and tops, represented by triplets* (L_1, G_1, G_1') *and* (L_2, G_2, G_2')*, and suppose* $genus(G_1') = genus(G_2) = g$*. Remove a 3-ball neighborhood of the horizontal line of* $G_1' \subset S^3$*, and identify the remain with* $\overline{B(0,1)}$*. Remove a 3-ball neighborhood of the horizontal line of* $G_2 \subset S^3$*, and identify the remain with* $\overline{S^3 - B(0,2)}$*. Glue the framed tangle* $T_g \subset \overline{B(0,2) - B(0,1)}$ *shown in Fig. 1c to the ends of the remains of* G_1' *in* $\overline{B(0,1)}$ *and* G_2 *in* $\overline{S^3 - B(0,2)}$*, strictly preserving the order of the points, so that the composition of these framed tangles is a smooth framed oriented link* L_0 *in* $S^3 = (\overline{S^3 - B(0,2)}) \cup (\overline{B(0,2) - B(0,1)}) \cup (\overline{B(0,1)})$*. Then*

$$\kappa(L_1 \cup L_0 \cup L_2, G_1, G_2') = (M_2 \cup_{f_2 \circ (f_1')^{-1}} M_1, f_1, f_2'), \tag{1.1}$$

where the framed graphs G_1, G_2' *in this formula are determined in the obvious way by the original* G_1, G_2' *in the two copies of* S^3*. Hence, any triplet representing* $(M_2 \cup_{f_2 \circ (f_1')^{-1}} M_1, f_1, f_2')$ *is equivalent to* $(L_1 \cup L_0 \cup L_2, G_1, G_2')$ *by extended Kirby moves and changes of orientations of link components.*

Suppose G is an arbitrary embedding of the chain graph Γ^g in S^3, then $H_1(S^3 - G, \mathbb{Z}) \cong \mathbb{Z}^g$, with free generators the meridians of the circle components.

Proposition 1.4. [6] *Suppose* M *is a connected compact oriented 3-manifold with two distinguished boundary components* $\partial M = (-S_1) \cup S_2$*, let* f_1, f_2 *be parametrizations of these surfaces, and let* $i : \partial M \hookrightarrow M$ *be the inclusion. The following conditions are equivalent:*

(1) $H_1(\widehat{M}, \mathbb{F}) = 0$,
(2) $H_1(M, \mathbb{F}) = i_* \left(H_1(\partial M, \mathbb{F})/(f_{1*}H_1(H_{g_1}, \mathbb{F}) + f_{2*}H_1(H^{g_2}, \mathbb{F}))\right)$.

They imply:

(3) $2 \cdot rank\ H_1(M; \mathbb{F}) = rank\ H_1(\partial M; \mathbb{F})$.

□

This proposition holds true for $\mathbb{F} = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{Z}_p$. In condition (2) above, $H_1(H_g, \mathbb{F})$ and $H_1(H^g, \mathbb{F})$ denote (by abuse of notation) the subspaces of $H_1(\Sigma_g, \mathbb{F})$ generated by the longitudes, resp. by the meridians. A 3-cobordism satisfying the equivalent conditions (1), (2) of Proposition 1.4 will be called an $\mathbb{F}$-*cobordism*. Note that in this definition we allow one or both S_i to be empty, although from the point of our TQFT the case of empty top and/or bottom is undistinguished from the case when that component is S^2.

1.2. Description of the categories $\mathfrak{Q} \supset \mathfrak{Z}$. Objects in each of these are *natural numbers*. The morphisms between g_1 and g_2 are equivalence (homeomorphism) classes of connected $\mathbb{F}$-cobordisms with bottom S_1 of genus g_1 and top S_2 of genus g_2, satisfying the $\mathbb{F}$-*doubly-Lagrangian* condition:

$$f_{1*}L^a \supseteq f_{2*}L^a \text{ and } f_{1*}L^b \subseteq f_{2*}L^b, \tag{1.2}$$

where $L^a = \ker(\text{incl}_* : H_1(\Sigma_g, \mathbb{F}) \to H_1(H_g, \mathbb{F}))$, and $L^b = \ker(\text{incl}_* : H_1(\Sigma_g, \mathbb{F}) \to H_1(H^g, \mathbb{F}))$, and $\mathbb{F} = \mathbb{Z}$ (for $\mathfrak{Z}$) or $\mathbb{Q}$ (for $\mathfrak{Q}$). The composition-morphism of two cobordisms (M_2, f_2, f_2') and (M_1, f_1, f_1') is the equivalence class of the 3-cobordism $(M_2 \cup_{f_2 \circ (f_1')^{-1}} M_1, f_1, f_2')$.

In general condition (1.2) over $\mathbb{Z}$ is stronger than (1.2) over $\mathbb{Q}$. It may hold with strict inclusion. [6]

Proposition 1.5. [6] *The composition of two morphisms (say, class of M and class of N) in category $\mathfrak{Q}$ (respectively $\mathfrak{Z}$) is again a morphism in the category $\mathfrak{Q}$ (respectively $\mathfrak{Z}$).* □

Let us restrict to 3-cobordisms M of the form $(\Sigma_g \times [0, 1], f \times 0, f' \times 1)$, with $f, f' \in Aut(\Sigma_g)$, i.e. the parametrization of the top differs by that of the bottom by the automorphism $w = (f')^{-1} \circ f$. The equivalence classes of this cobordism depends only on the isotopy class of w (i.e. we don't need to specify both f, f'). The equivalence class of $M = (\Sigma_g \times [0, 1], f \times 0, f' \times 1)$ is a $\mathbb{Z}$-doubly-Lagrangian cobordism iff it is a $\mathbb{Q}$-doubly-Lagrangian cobordism iff it satisfies $L^a = w_*(L^a)$ and $L^b = w_*(L^b)$. In particular, $\widehat{M}$ is always a $\mathbb{Z}$-homology sphere. [6]

These cobordisms form a category, denoted by $\mathfrak{L}$. The composition of two cobordisms $(\Sigma_g \times I, f_2 \times 0, f_2' \times 1) \cong (\Sigma_g \times I, w_2 \times 0, id \times 1)$ and $(\Sigma_g \times I, f_1 \times 0, f_1' \times 1) \cong (\Sigma_g \times I, w_1 \times 0, id \times 1)$ is the 3-cobordism $(\Sigma_g \times I, f_2 \circ (f_1')^{-1} \circ f_1 \times 0, f_2' \times 1) \cong (\Sigma_g \times I, (f_2')^{-1} \circ f_2 \circ (f_1')^{-1} \circ f_1 \times 0, id \times 1) \cong (\Sigma_g \times I, (w_2 \circ w_1) \times 0, id \times 1)$.

Definition 1.6. [6] *Call* **the Lagrangian subgroup of the Mapping Class Group**, *the subgroup consisting of isotopy classes of elements $w \in Aut(\Sigma_g)$, such that $w_*(L^a) = L^a$ and $w_*(L^b) = L^b$ (over $\mathbb{Q}$ or over $\mathbb{Z}$, is equivalent by the above). Denote it by $\mathcal{L}_g$.*

The TQFT of the LMO invariant (Sect. 3 below) induces a representation of $\mathcal{L}_g$. This subgroup of $MCG(g)$ is big enough to be interesting, it contains the Torelli group. Its image under the action on homology is the group of matrices of the form $\begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix}$, where $A \in GL(g, \mathbb{Z})$.

2. The Algebraic-Combinatorial Category

Murakami and Ohtsuki have extended the Kontsevich integral to an invariant $Z(G)$ of oriented framed trivalent graphs G in S^3 ([21, Theorem 1.4]). A framed graph $G \subset S^3$ is represented as a plane projection (with implicit blackboard framing), then decomposed into elementary pseudo-quasi-tangles, and Z is defined for each piece (see [21], Fig. 2 for the exact definition of Z). It is easy to observe that in order to verify the independence of Z of the decomposition into pseudo-quasi-tangles and the invariance under extended Reidemeister moves for trivalent graphs, one is forced to introduce relations that "move" (in the sense of Proposition 2.1) a box-diagram over a trivalent vertex to a box-diagram. Hence the branching relations (Fig. 4 here, Fig. 1 in [21]). These relations are necessary to impose regardless of the definition of Z for the neighborhood of a trivalent vertex, and obviously regardless of what associator is used.

From the extended Z, Murakami and Ohtsuki [21] derived an invariant of oriented 3-manifolds with boundary, along the same lines the Z^{LMO} is constructed [17] from the Kontsevich integral of framed links.

2.1. The modules of chord diagrams. Let Γ be a graph; we will be mainly interested in the cases Γ = a 1-manifold and Γ = a chain graph. Let $\mathcal{A}(\Gamma)$ be the formal series completion with respect to the degree of the $\mathbb{Q}$-vector space freely generated by the set of homeomorphism classes of chord diagrams with support Γ, without self-loops and

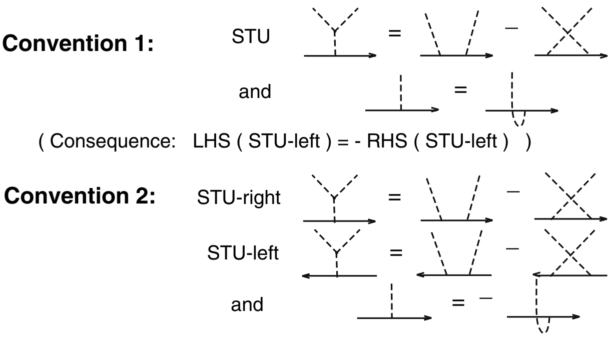


Fig. 2. The two conventions for chord diagrams

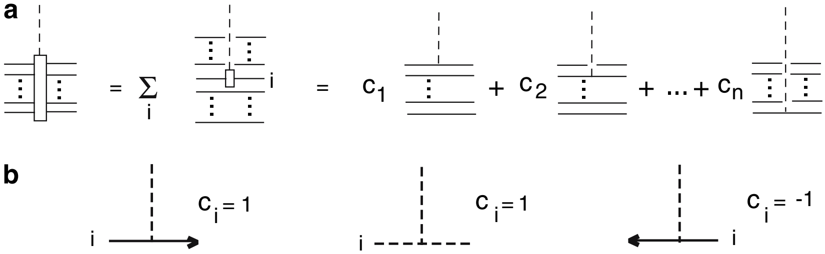


Fig. 3. The box-diagram

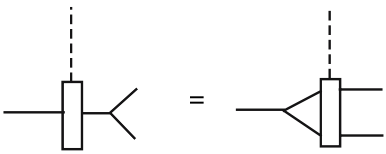


Fig. 4. The 8 branching relations (all but the vertical edge are bold): one for each possible orientation of the 3 bold edges

univalent vertices, modulo AS, IHX, STU and branching relations (which are homogeneous with respect to the degree).[4]

We will use the following *box-diagram* notation for the formal sum of chord diagrams, as shown in Fig. 3a. There outside the drawn part the diagrams are identical, the vertical edge is dashed, the horizontal edges are arbitrary. If the horizontal edge i is dashed, then $c_i = 1$, if it is bold, then c_i is as shown in Fig. 3b.[5] The branching relations, introduced in [21, Fig. 1], are shown in Fig. 4 using this box-notation.

[4] There are essentially two conventions in defining STU and AS relations, and drawing certain elements of $\mathcal{A}(\Gamma)$, as shown in Fig. 2. Note that in Convention 1, which is the one that we use (as well as [17, 21]), AS relations refer only to internal vertices, and no cyclic order of edges adjacent to external trivalent vertices is defined. In this convention, as a consequence, the LHS of STU-left is equal to *minus* the RHS of STU-left. Using the second convention, the definition of a chord diagram has to be changed to account for the cyclic order of edges adjacent to external trivalent vertices. The two $\mathcal{A}(\Gamma)$, from the two conventions, are canonically isomorphic; in fact only the meaning of some diagrams as elements of $\mathcal{A}(\Gamma)$ is changed by adding a $-$ sign.

[5] For Convention 2 all coefficients $c_i = 1$. Then the box-diagrams in the two conventions correspond precisely one to the other via the canonical isomorphism between the conventions.

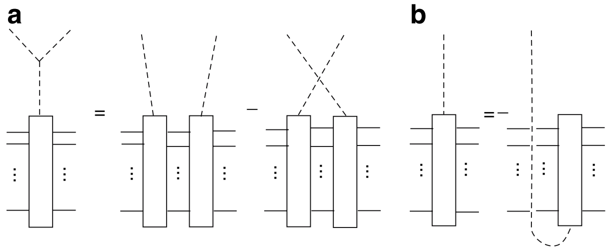


Fig. 5. a: The box-STU relation (each term in the RHS contains a double sum over the horizontal edges), **b:** The box-AS relation

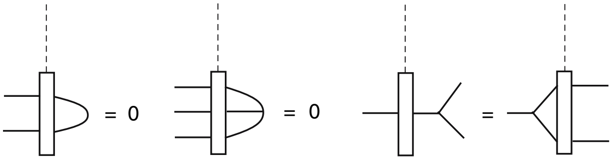


Fig. 6. Invariance over "elementary pseudo-tangles" (the dashed/bold type of horizontal edges is arbitrary, the vertical edge is dashed)

Similarly, let $\mathcal{A}(\emptyset)$ denote the formal series completion with respect to the degree of the $\mathbb{Q}$-module freely generated by the set of homeomorphism classes of open chord diagrams without self-loops and univalent vertices, modulo AS and IHX relations. For a chord diagram D, denote $[D]$ the corresponding element of $\mathcal{A}(\Gamma)$. $\mathcal{A}(\Gamma)$ and $\mathcal{A}(\emptyset)$ are co-algebras with respect to the decomposition of the dashed part of a diagram in connected components (the elements represented by diagrams that have non-empty connected dashed part are defined to be primitive).[6] $\mathcal{A}(\emptyset)$ is an algebra with respect to disjoint union, and together with (completed) comultiplication Δ forms a Hopf algebra. Note that $\mathcal{A}(\Gamma)$ is an $\mathcal{A}(\emptyset)$-module with respect to the disjoint union.

Proposition 2.1. *(a) The box-STU and box-AS relations, schematically shown in Fig. 5 hold in* $\mathcal{A}(\Gamma)$.
(b) The three relations in Fig. 6 hold in $\mathcal{A}(\Gamma)$.
(c) The box-STU and box-AS relations can be "moved" over any trivalent vertex of Γ*, using only branching relations (see Fig. 7 for an example).*[7]

Proof. (a) Let x_i denote the horizontal edges. Let $[D^Y]$, $[D^{II}]$, $[D^X]$ denote the three terms of the box-STU relation. Note that the brackets are also part of the notation, D^Y means the box-diagram, which is not a chord diagram. Let $[D^Y_{x_i}]$ denote the element of $\mathcal{A}(\Gamma)$ corresponding to the chord diagram obtained from D^Y by replacing the box with a prolongation of the vertical edge until the edge x_i. With similar notations $[D^{II}_{x_i x_j}]$ and $[D^X_{x_k x_l}]$, note that for $i \neq j$, $[D^{II}_{x_i x_j}] = [D^X_{x_j x_i}]$. Hence:

$$RHS = \sum_{x_i} \sum_{x_j} c_i c_j [D^{II}_{x_i x_j}] - \sum_{x_j} \sum_{x_i} c_j c_i [D^X_{x_j x_i}]$$

[6] One can check (e.g. by induction on the number of internal vertices of chord diagrams) that this comultiplication is well-defined (remember the presence of STU relations).

[7] One can reformulate this statement: Every IHX (respectively AS) relation on-the-left-of-the-trivalent-vertex is a consequence of branching relations and IHX (respectively AS) relations on-the-right-of-the-trivalent-vertex.

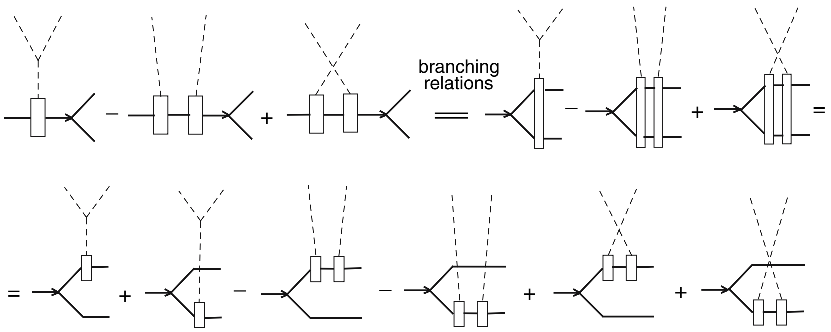


Fig. 7. "Moving" a box-STU relation over a trivalent vertex

$$= \sum_{i=j} c_i^2 [D^{II}_{x_i x_j}] + \sum_{i \neq j} c_i c_j [D^{II}_{x_i x_j}] - \sum_{i=j} c_i^2 [D^{X}_{x_j x_i}] - \sum_{i \neq j} c_j c_i [D^{X}_{x_j x_i}]$$

$$= \sum_{i=j} c_i^2 [D^{II}_{x_i x_j}] - \sum_{i=j} c_i^2 [D^{X}_{x_j x_i}] = \sum_{i} ([D^{II}_{x_i x_i}] - [D^{X}_{x_i x_i}]) = \sum_{i} c_i [D^{Y}_{x_i}] = LHS,$$

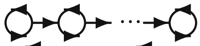
where in the equality before the last we have used an IHX, STU, or Convention-1 form of STU-left for each x_i. The proof of the box-AS relation is elementary, using AS relations and the definition of coefficients c_i.

(b) Consider all dashed/bold possibilities for the edges. The relations then follow from the AS, IHX, STU and branching relations.

(c) Every box-diagram is a sum of box-diagrams with small boxes. For the later follow the calculation shown in Fig. 7, for the box-STU case. The box-AS case is obvious. □

Note that this proposition for the case of Γ being a 1-manifold is part of [24, Prop. 1.4].

The "formal series completion" (i.e. the topology is given by[8] $distance(p, q) \leq \frac{1}{2^n} \Leftrightarrow p - q$ has no terms of degree $< n$) is algebraically nothing else but the direct product over $i \in \mathbb{N}$ of the vector spaces generated by diagrams of a fixed order i. AS, IXH, STU and branching relations are homogeneous with respect to the degree. For every i, the degree i part $\mathcal{A}_i$ is defined as $\mathcal{D}_i/\mathfrak{R}_i$, where $\mathcal{D}_i$ is the $\mathbb{Q}$-module freely generated by the chord diagrams of degree i (without factoring through relations), and $\mathfrak{R}_i$ is the $\mathbb{Q}$-module freely generated by the relations involving only diagrams of order i. By the universal property of the direct product $\mathcal{A} = \prod_{i \in \mathbb{N}} \mathcal{A}_i \cong \prod_{i \in \mathbb{N}} \mathcal{D}_i / \prod_{i \in \mathbb{N}} \mathfrak{R}_i = \mathcal{D}/\mathfrak{R}$, i.e. factoring and taking completion commute. We will not use anywhere below the next proposition that $\mathcal{A} \cong \mathcal{D}/\mathfrak{R}$, our object is always $\mathcal{A}$.

Proposition 2.2. *Denote* $[g] \overset{\text{not}}{=} \{1, \ldots, g\}$. *Let* $\phi : \uparrow_g \rightarrow$  $_{[g]} = \Gamma^g$ *be the embedding of* $\uparrow_g$ *onto the upper half-circles of*  *, sending the arrow labeled i to the* i^{th} *upper half-circle of* Γ^g*, preserving orientation. Then it extends to an isomorphism of* $\mathbb{Q}$*-vector spaces* $\phi_* : \mathcal{A}(\uparrow_g) \rightarrow \mathcal{A}(\Gamma^g)$.

Proof. Fix an arbitrary degree i of chord diagrams. Then ϕ induces a homomorphism of vector spaces $\phi_* : \mathcal{D}_i(\uparrow_g) \rightarrow \mathcal{D}_i(\Gamma^g)$, under which $\mathfrak{R}_i(\uparrow_g)$ is sent exactly to the set of AS, IHX and STU relations in Γ^g that involve only diagrams with support in $\phi(\uparrow_g)$. For simplicity of notation, let us denote $\phi_* \mathcal{D}_i(\uparrow_g)$ by $\mathcal{D}_i(\uparrow_g)$, and $\phi_* \mathfrak{R}_i(\uparrow_g)$ by $\mathfrak{R}_i(\uparrow_g)$.

[8] The only reason we choose $\frac{1}{2^n}$ instead of $\frac{1}{n}$ is Lemma 3.7. See the remark after it.

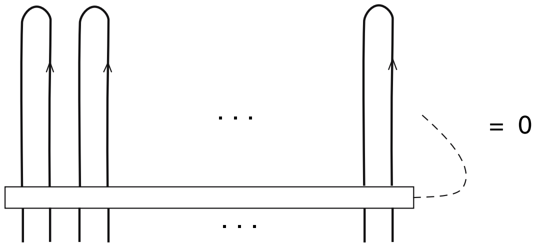


Fig. 8. Relation 9: the ambiguity of "moving" a dashed end off the horizontal line.

Replace each external trivalent vertex in $\Gamma^g - \phi(\uparrow_g)$ of a chord diagram by a small box (and add a sign to it, the coefficient c_i), then "move", using the branching relations, one by one all boxes off $\Gamma^g - \phi(\uparrow_g)$. This assigns to an arbitrary chord diagram with support in Γ^g a diagram with boxes (with a $\pm$ sign) with support in $\phi(\uparrow_g)$. It depends on the choice of the sequence of trivalent vertices over which boxes are "moved" in Γ^g. Observe, however, that different such choices result in diagrams with boxes, representing elements of $\mathcal{D}_i(\uparrow_g)$ that differ one from the other by a sum (with coefficients ± 1) of relations depicted in Fig. 8. Let us call them *Relations 9* as reference to Fig. 9 in [21]. By linearity, this defines a homomorphism of $\mathbb{Q}$-vector spaces $\alpha : \mathcal{D}_i(\Gamma^g) \to \mathcal{D}_i(\uparrow_g)/R9$, which when restricted to $\mathcal{D}_i(\uparrow_g) \to \mathcal{D}_i(\uparrow_g)/R9$ is the canonical quotient map. Here $R9$ is the $\mathbb{Q}$-vector subspace of $\mathcal{D}_i(\uparrow_g)$ generated by the set of Relations 9.

Proposition 2.1(b) implies that Relations 9 are true in $\mathcal{A}_i(\uparrow_g)$, i.e. $R9 \subset \mathfrak{R}_i(\uparrow_g)$. Let $\beta : \mathcal{D}_i(\uparrow_g)/R9 \to \mathcal{D}_i(\uparrow_g)/\mathfrak{R}_i(\uparrow_g)$ be the canonical projection. Let us observe that for every branching relation R, $\alpha(R) = 0$. Therefore $(\beta \circ \alpha)(R) = 0$, so if we denote by $\mathfrak{B}_i$ the $\mathbb{Q}$-vector subspace of $\mathcal{D}_i(\Gamma^g)$ generated by the set of branching relations, then $\mathfrak{B}_i \cap \mathcal{D}_i(\uparrow_g) \subset \mathfrak{R}_i(\uparrow_g)$.

On the other hand, any IHX, AS, STU or branching relation on Γ^g is, by Proposition 2.1, a sum of IHX, AS and STU relations on $\phi(\uparrow_g)$, plus a sum of branching relations. Indeed, an IHX or AS relation refers only to a neighborhood outside $\Gamma^g - \phi(\uparrow_g)$, hence the "moving" procedure can be applied simultaneously to all terms of the relation; while a STU relation is, up to sign, a box-STU relation, therefore using Proposition 2.1(c) can be "moved" to a box-STU relation with support in $\Gamma^g - \phi(\uparrow_g)$, the later being a consequence of $\mathfrak{R}_i(\uparrow_g)$ by Proposition 2.1(a). The difference between the start and the end of each step of a "moving" procedure is, of cause, an element of $\mathfrak{B}_i$. Hence $\mathfrak{R}_i(\Gamma^g) = \mathfrak{R}_i(\uparrow_g) + \mathfrak{B}_i$.

The two established relations imply $\mathfrak{R}_i(\Gamma^g) \cap \mathcal{D}_i(\uparrow_g) \subset \mathfrak{R}_i(\uparrow_g)$. Since the opposite inclusion is obvious, $\mathfrak{R}_i(\Gamma^g) \cap \mathcal{D}_i(\uparrow_g) = \mathfrak{R}_i(\uparrow_g)$. Then, by the second isomorphism theorem for vector spaces, $\mathcal{D}_i(\uparrow_g)/\mathfrak{R}_i(\uparrow_g) \cong \mathcal{D}_i(\Gamma^g)/\mathfrak{R}_i(\Gamma^g)$. Composing with ϕ_* from the first paragraph, we obtain $\phi_* : \mathcal{A}_i(\uparrow_g) \to \mathcal{A}_i(\Gamma^g)$, for every $i \geq 0$. Moreover the induced $\phi_* : \mathcal{A}(\uparrow_g) \to \mathcal{A}(\Gamma^g)$ preserves the topology. □

Remarks. This proposition still holds if Γ has two or more connected components, but we can "eliminate" the horizontal line of only ONE component. If we "eliminate" more than one horizontal line, the corresponding ϕ_* is still well-defined and surjective.

2.2. The algebra structure of $\mathcal{A}(\uparrow_g)$. Let $\mathcal{A}_c(\uparrow_g)$ be the $\mathbb{Q}$-vector subspace of $\mathcal{A}(\uparrow_g)$ generated by formal series of diagrams on $\uparrow_g$ with no components of the dashed graph

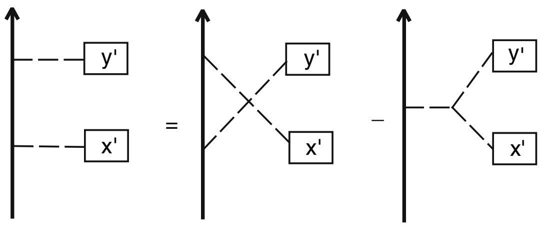


Fig. 9. Two consecutive external vertices from connected components x' and y' can be interchanged up to $\pm$ a diagram with dashed graph having one component less

disconnected from the support. Viewing each chord diagram as a union of the connected components of the dashed graph that do not meet the support with the part that meets the support, we get $\mathcal{A}(\uparrow_g) = \mathcal{A}(\emptyset) \otimes_{\mathbb{Q}} \mathcal{A}_c(\uparrow_g)$. Let $\mathfrak{a}(\uparrow_g)$ be the $\mathbb{Q}$-vector subspace of $\mathcal{A}_c(\uparrow_g)$ generated by formal series of diagrams on $\uparrow_g$ with *non-empty and connected dashed graph* (and connected to the support). $\mathfrak{a}(\uparrow_g)$ is precisely the set of primitive elements of $\mathcal{A}_c(\uparrow_g)$. A similar notation $\mathfrak{a}(\Gamma)$ for any abstract graph Γ is self-evident. $\mathcal{A}_c(\uparrow_g)$ is an algebra with respect to justaposition of the bold vertical arrows. Denote this associative, generally (if $g > 1$) non-commutative operation $\bullet$. In fact $\mathcal{A}_c(\uparrow_g)$ is a co-commutative Hopf algebra [24, Prop. 1.5]. Recall the following "common knowledge":

Proposition 2.3. *1)* $\mathfrak{a}(\uparrow_g)$ *is a Lie algebra over* $\mathbb{Q}$ *with respect to the operation* $(x, y) \mapsto x \bullet y - y \bullet x$.

2) Let $\widehat{I}$ *be the topological ideal of* $\mathcal{A}_c(\uparrow_g)$ *generated by* $\mathfrak{a}(\uparrow_g)$. *Then* $\exp : \widehat{I} \to 1+\widehat{I}$ *and* $\log : 1+\widehat{I} \to \widehat{I}$, *defined by* $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ *and* $\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$, *where the product is the operation* $\bullet$, *satisfy* $\exp \circ \log = id_{1+\widehat{I}}$ *and* $\log \circ \exp = id_{\widehat{I}}$. *In particular,* $\exp$ *and* $\log$ *are bijections.*

3) $\exp$ *is a bijection from* $\mathfrak{a}(\uparrow_g) \subset \widehat{I}$ *to the set of group-like elements in* $1 + \widehat{I}$.

4) If $\alpha, \beta \in \mathfrak{a}(\uparrow_g)$, *then* $\exp(\alpha) \bullet \exp(\beta) = \exp(\gamma)$ *for some* $\gamma \in \mathfrak{a}(\uparrow_g)$. *Moreover,* γ *is given by the Campbell-Hausdorff formula.*

5) $\widehat{I}$ *coincides with the set of formal series of chord diagrams of degree* ≥ 1.

Proof. 1) The statement is sufficient to prove for $x, y =$ diagrams with connected dashed graph. Using STU relations, as shown in Fig. 9, we can interchange two consecutive external vertices, one from x, the other from y, on any bold arrow, up to $\pm$ a diagram with connected dashed graph. Therefore, iteratively we can interchange all external vertices of x, with all external vertices of y, obtaining $x \bullet y - y \bullet x =$ a sum of diagrams with connected dashed graph.

2), 3) and 4) are classical statements. The proofs in [22, Theorem 7.2, Corollary 7.3 and Theorem 7.4] apply môt-a-môt. For 3) and 4) note that if γ is primitive, then $\gamma \in \mathfrak{a}(\uparrow_g)$.

5) Since the set of formal series of chord diagrams of degree ≥ 1 is an ideal containing $\mathfrak{a}(\uparrow_g)$, and is closed topologically, $\widehat{I}$ certainly belongs to it. Conversely, pick an arbitrary connected component y' of the dashed graph of a chord diagram. Observe that using the "'trick"' in Fig. 9, up to $\pm$ a sum of diagrams with the number of connected components of the dashed graph less by 1, y' can be assumed to have all external vertices below all the other external vertices of the diagram. Hence an induction on the number of connected

components of the dashed graph shows that any chord diagram of degree ≥ 1 is a sum of terms of type $\pm z_1 \bullet z_2 \bullet \ldots \bullet z_k$, $k \geq 1$, with z_i a diagram in $\mathfrak{a}(\uparrow_g)$. We conclude that the set of finite sums of chord diagrams of degree ≥ 1 is contained in $\widehat{I}$. Hence so is its completion. □

This proposition also holds if we replace $\mathcal{A}_c(\uparrow_g)$ by $\mathcal{A}(\uparrow_g)$, $\mathbb{Q}$ by $\mathcal{A}(\emptyset)$, and $\mathfrak{a}(\uparrow_g)$ by $\mathfrak{a}(\uparrow_g) + \mathfrak{a}(\emptyset) \cdot 1$, where $\mathfrak{a}(\emptyset)$ is the set of primitive elements of $\mathcal{A}(\emptyset)$.

2.3. The LMO invariant for closed manifolds and extending the maps ι_n. In [17] from the Kontsevich integral an invariant of oriented framed links L was constructed, which does not change under Kirby-1,2 moves and change of orientation of components of L. We recall it here, together with the maps $\tilde{\iota}_n$, necessary to extend it to an invariant of unions of embedded framed *chain* graphs in S^3.

Let $\overset{\circ}{\mathcal{A}}(\emptyset)$ be the formal series completion of the $\mathbb{Q}$-vector space generated by the homeomorphism classes of open chord diagrams without univalent vertices (but allowing dashed self-loops - these are set of degree 0) modulo AS and IHX relations. Let $\mathcal{B}(X)$ be the formal power series completion of the $\mathbb{Q}$-vector space generated by the homeomorphism classes of open chord diagrams without dashed self-loops, with the univalent vertices colored by elements of X, modulo AS and IHX relations.

Denote $[m] \overset{\text{not}}{=} \{1, \ldots, m\}$, and let $\Gamma = \sqcup_m S^1$, where each component is colored by a different element of $[m]$. Let $\widetilde{\mathcal{B}([m])}$ be the subspace of $\mathcal{B}([m] \cup \{*\})$, generated by the diagrams with one $*$-colored vertex, and $f_i : \widetilde{\mathcal{B}([m])} \to \mathcal{B}([m])$, $f_i :=$ *average of the diagrams obtained by attaching the $*$-vertex near all i-vertices*. We can define a map $\varphi : \mathcal{C}(m) := \frac{\mathcal{B}([m])}{<im\ f_i|\forall i>} \to \mathcal{A}(\Gamma)$, $\varphi :=$ *average of the diagrams obtained by attaching i-colored vertices to the i^{th} copy of S^1 in Γ, $\forall i$.* (One checks that the definition on diagrams extends over relations to a map between formal series completions.) This map is in fact an isomorphism of $\mathbb{Q}$-modules (and co-algebras). For details, please consult [17, 24]. (If $\Gamma = S^1$, $\mathcal{C}(1)$ and $\mathcal{A}(S^1)$ are algebras, but φ is not an algebra homomorphism.)

For every $n \geq 0$ define a map $\kappa_n : \mathcal{C}(m) \to \overset{\circ}{\mathcal{A}}(\emptyset)$; $\kappa_n(K) = 0$, if $\exists i$ such that the number of i-colored vertices is not $2n$, $\kappa_n(K) =$ *sum of all ways of attaching i-colored vertices in pairs*, $\forall i$, otherwise. Let $\mathcal{O}_n$ be the ideal of $\overset{\circ}{\mathcal{A}}(\emptyset)$ generated by $\bigcirc + 2n$. ($\overset{\circ}{\mathcal{A}}(\emptyset)$ is an algebra with respect to disjoint union.) It can be shown that as modules (and even as algebras) $\overset{\circ}{\mathcal{A}}(\emptyset)\ /\mathcal{O}_n \cong \mathcal{A}(\emptyset)$. Now, let $\iota_n = q_n \circ \kappa_n \circ \varphi^{-1}$: $\mathcal{A}(\Gamma) \to \mathcal{C}(m) \to \overset{\circ}{\mathcal{A}}(\emptyset) \to \overset{\circ}{\mathcal{A}}(\emptyset)\ /\mathcal{O}_n \cong \mathcal{A}(\emptyset)$, where q_n is the quotient map.

Let $\kappa_n^* : \mathcal{B}(X \sqcup \{*\}) \to \mathcal{B}(X)$ be defined as κ_n, but only involving $*$-colored vertices (see [17, 24] for details). Let $P_n = \mathrm{Im}(\kappa_n^*)$. The map κ_n^* passes to the quotient from the definition on $\mathcal{C}(m)$, and hence we get a submodule P_n of $\mathcal{C}(m)$. The relations P_n also commute with φ. Define the quotient map $j_n : \overset{\circ}{\mathcal{A}}(\emptyset)\ /\mathcal{O}_n \to \overset{\circ}{\mathcal{A}}(\emptyset)\ /\mathcal{O}_n, P_{n+1}$ of graded modules. It is isomorphism in degree $\leq n$, and is the main ingredient in showing that $deg_{\leq n}\iota_n(\check{Z}(L))$ is invariant under the second Kirby move [17, 24].

This construction can be extended for $\Gamma = \Gamma^{g_1} \sqcup (\sqcup_m S^1) \sqcup \Gamma^{g_2}$, disjoint union of two chain graphs and m copies of S^1, i.e. $\iota_n : \mathcal{A}(\sqcup_m S^1) \to \overset{\circ}{\mathcal{A}}(\emptyset)\ /\mathcal{O}_n \cong \mathcal{A}(\emptyset)$ can be extended (meaning that for $g_1 = g_2 = 0$, $\tilde{\iota}_n$ acts exactly as ι_n) to a map:

$$\tilde{\iota}_n = \tilde{q}_n \circ \tilde{\kappa}_n \circ \tilde{\varphi}^{-1} : \mathcal{A}(\Gamma^{g_1} \sqcup (\sqcup_m S^1) \sqcup \Gamma^{g_2}) \to \mathcal{A}(\Gamma^{g_1} \overset{\circ}{\sqcup} \Gamma^{g_2}) \,/\mathcal{O}_n \cong \mathcal{A}(\Gamma^{g_1} \sqcup \Gamma^{g_2}), \tag{2.1}$$

where the corresponding homomorphism $\tilde{\varphi}^{-1}$ refers only to *all present circle components* of Γ. Here, to define the preimage of $\tilde{\varphi} : \mathcal{C}(\Gamma^{g_1}, [m], \Gamma^{g_2}) \to \mathcal{A}(\Gamma^{g_1}, \sqcup_m S^1, \Gamma^{g_2})$ we consider absolutely analogous chord diagrams with support the disjoint union of two chain graphs Γ^{g_1}, Γ^{g_2}, and points indexed by elements of $[m]$ (it is convenient NOT to call these points vertices), $\tilde{\kappa}_n$ is extended in the same manner, and $\tilde{q}_n$ is just the quotient map. $\tilde{\varphi}$ is an isomorphism with the proof of Sect. 2 of [17]. Moreover, the similarly constructed map $\tilde{j}_n$ is an isomorphism in degree $\leq n$. Namely, and this is exactly the statement of Lemma 2.3 of [21], if $\Gamma = \Gamma^{g_1} \sqcup (\sqcup_m S^1) \sqcup \Gamma^{g_2}$, then $\tilde{j}_n : deg_{\leq n}(\mathcal{A}(\Gamma^{g_1} \sqcup \Gamma^{g_2}) \cong \mathcal{A}(\Gamma^{g_1} \overset{\circ}{\sqcup} \Gamma^{g_2}) \,/\mathcal{O}_n) \xrightarrow{\cong} deg_{\leq n}(\mathcal{A}(\Gamma^{g_1} \overset{\circ}{\sqcup} \Gamma^{g_2}) \,/\mathcal{O}_n, P_{n+1})$. To check this fact it is enough to follow the proof of Lemma 3.3 in [17] or Proposition 4.4 in [24].

Let $Z(L)$ be the usual Kontsevich integral of the (oriented) framed link L, $\nu = Z(zero\!-\!framed\,unknot)$. Denote $\check{Z}(L) := Z(L) \otimes \nu^{|L|}$, meaning we take the "connected sum" of $Z(L)$ on each of its component with ν. Like $Z(L)$, $\check{Z}(L)$ is also group-like of the form $1 + (terms\ of\ degree \geq 1)$.[9] Let $\sigma_\pm$ be the number of positive, resp. negative, eigenvalues of the linking matrix of L. Denote $\mathbf{O}^{+1}$, resp. $\mathbf{O}^{-1}$, the unknot with +1, resp. -1, framing, and S^3_L the 3-manifold obtained by surgery on the framed link L in S^3. Recall the definition of the LMO invariant for oriented closed 3-manifolds $M \equiv S^3_L$:

$$\Omega_n(S^3_L) := deg_{\leq n}\left(\frac{\iota_n(\check{Z}(L))}{\iota_n(\check{Z}(\mathbf{O}^{+1}))^{\sigma_+} \cdot \iota_n(\check{Z}(\mathbf{O}^{-1}))^{\sigma_-}}\right) \tag{2.2}$$

and

$$Z^{lmo}(M) := \sum_{n \geq 0} deg_n \Omega_n(M), \tag{2.3}$$

and for $\mathbb{Q}$-homology spheres also:

$$Z^{LMO}(M) := \sum_{n \geq 0} d(M)^{-n} deg_n \Omega_n(M), \tag{2.4}$$

where $d(M) = |det(lk(L))|$, which is 0 if $H_1(S^3_L, \mathbb{Q}) \neq 0$ and $|H_1(M, \mathbb{Z})|$ otherwise. We use the convention $|det(lk(\emptyset))| = 1$. Then we have $deg_{\leq n}\Omega_{n+1}(S^3_L) = d(M) \cdot \Omega_n(S^3_L)$, hence we can write $deg_{\leq n} Z^{LMO}(M) = d(M)^{-n}\Omega_n(M)$. More precisely, the following holds [24, Prop. 4.5]:

$$deg_{\leq n}[\iota_{n+1}\check{Z}(L)] = (-1)^{|L|} det(lk(L)) deg_{\leq n}[\iota_n \check{Z}(L)], \tag{2.5}$$

and therefore we can define:

[9] It can be shown by induction that then for $|L| = 1$ the formal graded series $log(element)$ is a primitive element of $\mathcal{A}(\sqcup S^1)$, and has no part of degree 0, hence it is a formal power series of chord diagrams with connected dashed part. More precisely, a statement similar to Proposition 2.3 holds.

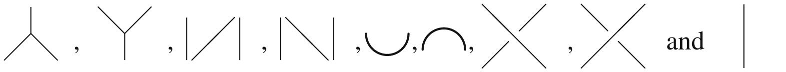
$$c_+ = \lim_{n\to\infty} (-1)^n deg_{\le n}[\iota_n \check{Z}(\mathbf{O}^{+1})], \tag{2.6}$$

$$c_- = \lim_{n\to\infty} deg_{\le n}[\iota_n \check{Z}(\mathbf{O}^{-1})]. \tag{2.7}$$

These elements of $\mathcal{A}(\emptyset)$ are canonical constants in the theory of LMO invariant. Equation (2.5) implies

$$deg_{\le N} Z^{LMO}(M) = \frac{(-1)^{N\sigma_+}}{d(M)^N} \cdot deg_{\le N}\left(\iota_N\left(\frac{\check{Z}(L)}{c_+^{\sigma_+} c_-^{\sigma_-}}\right)\right). \tag{2.8}$$

2.4. The definition of Z on elementary pseudo-quasi-tangles. To extend $\Omega_n(S^3_L) \in \mathcal{A}(\emptyset)$ to invariants $\Omega_n(L, G) \in \mathcal{A}(\Gamma)$, where Γ is G as an abstract graph, we will extend now $Z(L)$ to $Z(L \cup G)$. However we shall do this differently from Murakami and Ohtsuki [21], see Fig. 1 there, who use the Knizhnik-Zamolodchikov associator. We will use the even associator.

Let G be an embedded framed graph in S^3. Fix a plane projection such that G is given the blackboard framing. This projection of G can be decomposed into elementary tangles[10]:  .

We need only to specify the definition of Z on the first two, since on the others we know it from the link case.

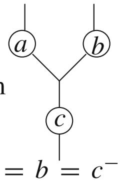
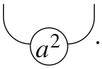
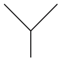
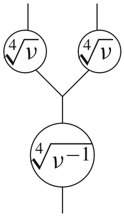
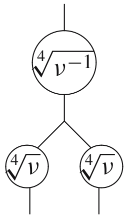
Let Γ be an abstract (disjoint union of) chain graph(s), and $\epsilon_e\Gamma$ be Γ with edge e erased. Suppose $\epsilon_e\Gamma$ is also a chain graph. A similar notation $\epsilon_e G$ for a framed graph G is self-evident. Define the map $\epsilon_{(e)} : \mathcal{A}(\Gamma) \to \mathcal{A}(\epsilon_e\Gamma)$, $\epsilon_{(e)}(D) = 0$, if D has an external vertex on the removed edge, and $\epsilon_{(e)}(D) = D$, otherwise. To verify well-definedness of $\epsilon_{(e)}$ it is enough to check its invariance under branching relations of diagrams on Γ. There are 3 diagrams involved in a branching relation. Suppose v is a trivalent vertex of Γ, and e_1, e_2, e_3 the edges adjacent to v. Edge e cannot be repeated twice among e_1, e_2, e_3, since then $\epsilon_e\Gamma$ would not be a (union of) chain graph(s). Therefore we can assume $e = e_1$, $e \neq e_2$, $e \neq e_3$. It is easy to check that then one of the three diagrams in the relation is sent to 0 by $\epsilon_{(e)}$, while the other two are sent to diagrams that form an AS relation in $\mathcal{A}(\epsilon_e\Gamma)$.

If e is an edge of G, denote by S_eG the graph obtained from G by reversing the orientation of the edge e (without changing the framing). If Γ is the underlying abstract graph of G, denote by $S_e\Gamma$ the underlying abstract graph of S_eG. Let $S_{(e)} : \mathcal{A}(\Gamma) \to \mathcal{A}(S_e\Gamma)$ be the linear map which sends every diagram D in $\mathcal{A}(\Gamma)$ to the diagram obtained from D by reversing the orientation of e, multiplied by $(-1)^m$, where m is the number of vertices of D on the edge e.

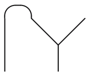
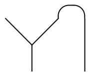
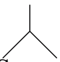
We define Z for the elementary tangles  and  to satisfy the following two conditions (compare with [21, Prop. 1.5]):

(1) $Z(S_eG) = S_{(e)}Z(G)$, for any embedded framed graph G and edge e.
(2) $Z(\epsilon_eG) = \epsilon_{(e)}Z(G)$, for any (disjoint union of) embedded chain graph(s) G and edge e, such that ϵ_eG is still a (disjoint union of) embedded chain graph(s).

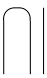
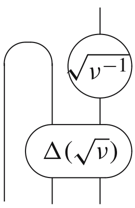
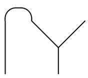
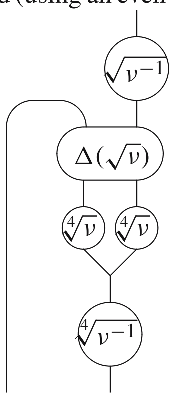
[10] The words *quasi* and *pseudo* are left out for simplicity of language.

Moreover, we seek to define $Z($  $)$ of the form  (for all possible 8 orientations). By Condition (2) above we must have $a = b = c^{-1}$, and hence also $Z($$) =$ . But $Z($$) = \nu^{1/2}$, therefore we must require $a = b = c^{-1} = \nu^{1/4}$, i.e. $Z($  $) =$ . Similarly $Z($  $) =$ . (These formulas are each for the 8 possible orientations.)

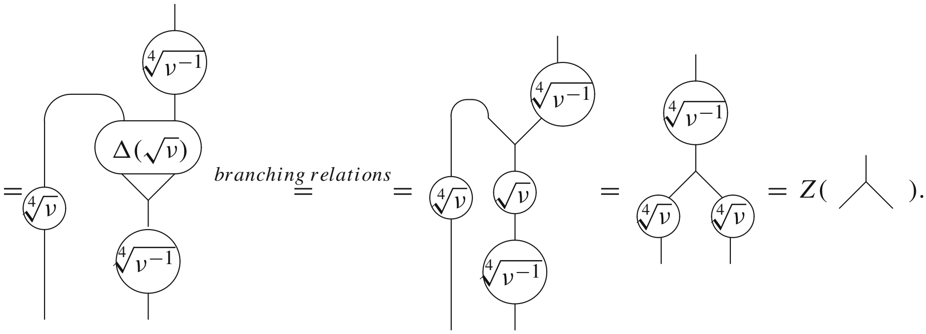
Theorem 2.4. [11] *$Z(G)$ is an isotopy invariant of embedded framed chain graphs.*

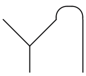
Proof. In large part, this is mostly a repetition of the proofs of the statements in [21, Sect. 1], hence we only sketch here the details that are not identical. First, one shows that $Z(G)$ is invariant under isotopies of the plane. If such isotopies fix a neighborhood of each trivalent vertex, the result is known from the link case. If isotopies move such neighborhoods "as a whole", the result follows using branching relations [21, Lemma 1.2.] Finally, it is sufficient to show $Z($$) = Z($  $)$ and $Z($$) = Z($  $)$.

Secondly, one shows that $Z(G)$ is invariant under extended Reidemeister moves. This is also easily achieved from results known from the link case and the branching relations [21, Lemma 1.4.]

To prove the two remaining relations, note that in [16, p. 8] it is proved (using an even associator) that $Z($$) =$ . Therefore $Z($$) =$ 

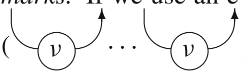
[11] This statement is considered known, but a complete proof was missing from the literature.



Similarly $Z($  $) = Z($  $)$. □

The above properties (1) and (2) we have now for granted (compare to [21, Prop. 1.5]). It then follows directly from their definitions in Sect. 3.1 that τ^N and τ also enjoy properties (1) and (2).

Conjecture 2.5. [12] *If G is a chain graph, then this definition of $Z(G)$ using an even associator coincides with the definition in [21], which uses a KZ associator.*

Remarks. If we use an even associator it is easy to see that $Z($  $) =$ $\phi_*($  $)$, where ϕ_* is the isomorphism from Proposition 2.2.

Definition 2.6 [21]. *For any $L \cup G \hookrightarrow S^3$, let $\check{Z}(L \cup G) = Z(L \cup G) \otimes (\nu^{\otimes |L|})$.*

2.5. The composition $*$ of chord diagrams. Denote $\mathcal{A}(\Gamma^{g_1}, \Gamma^{g_2}) := \mathcal{A}(\Gamma^{g_1} \sqcup \Gamma^{g_2})$, where the order $(\Gamma^{g_1}, \Gamma^{g_2})$ is specified. Γ^{g_1} is the union of its horizontal line and the upper half-circles, Γ^{g_2} is the union of its horizontal line and lower half-circles. As remarked after Proposition 2.2, every element of $\mathcal{A}(\Gamma^{g_1}, \Gamma^{g_2})$ can be represented as a formal series (with rational coefficients) of chord diagrams whose external vertices don't meet one horizontal line. For $\alpha \in \mathcal{A}(\Gamma^{g_1}, \Gamma^{g_2})$, $\beta \in \mathcal{A}(\Gamma^{g_2}, \Gamma^{g_3})$ represented by single chord diagrams x, respectively y, let $\alpha * \beta$ denote the element of $\mathcal{A}(\Gamma^{g_1}, \sqcup_{g_2} S^1, \Gamma^{g_3})$, represented by the diagram obtained by attaching $\phi_*^{-1} y$ (the horizontal line of the first graph removed) on top of $\phi_*^{-1} x$ (the horizontal line of the second graph removed). For $g = 0$ set $*$ to be the disjoint union. Extend $*$ by linearity to formal power series of chord diagrams. Note that $*$ is associative.

Every chord diagram D is a disjoint union $D_1 \sqcup D_2$, where every connected component of D_1 intersects the support, and every connected component of D_2 does not. Define the **vacuum degree** $vdeg(D) := deg(D_2)$. Since all relations are $vdeg$-homogeneous, the vacuum-degree N part $vdeg_N(\alpha)$ is well-defined $\forall N \in \mathbb{N}$, $\forall \alpha \in \mathcal{A}(\Gamma^{g_1}, \Gamma^{g_2})$,

[12] The results of this paper are equally true for any associator for which Theorem 2.4 holds. We have been able to obtain only partial results toward the proof of this conjecture with direct methods. It follows, however, from results in [7].

and is in general a series. For any $N \in \mathbb{N}$ define $\mathcal{A}^{\leq N}(\emptyset) = \mathcal{A}(\emptyset)/(vdeg > N)$. Let $\mathcal{A}^{\leq N+1}(\emptyset) \to \mathcal{A}^{\leq N}(\emptyset)$ be the forgetful map. The sequence

$$\ldots \to \mathcal{A}^{\leq N+1}(\emptyset) \to \mathcal{A}^{\leq N}(\emptyset) \to \ldots \to \mathcal{A}^{\leq 1}(\emptyset) \to \mathcal{A}^{\leq 0}(\emptyset) = \mathbb{Q}$$

has inverse limit $\mathcal{A}(\emptyset)$. Similarly, let $\mathcal{A}^{\leq N}(\Gamma^{g_1}, \Gamma^{g_2}) = \mathcal{A}(\Gamma^{g_1}, \Gamma^{g_2})/(vdeg > N)$. Then for any Γ, and $\forall N$, $\mathcal{A}^{\leq N}(\Gamma) = \mathcal{A}^{\leq N}(\emptyset) \otimes \mathcal{A}_c(\Gamma)$, hence $\mathcal{A}(\Gamma^{g_1}, \Gamma^{g_2})$ is the inverse limit of the sequence:

$$\ldots \to \mathcal{A}^{\leq N+1}(\Gamma^{g_1}, \Gamma^{g_2}) \to \mathcal{A}^{\leq N}(\Gamma^{g_1}, \Gamma^{g_2}) \to \ldots \to \mathcal{A}^{\leq 1}(\Gamma^{g_1}, \Gamma^{g_2}) \to \mathcal{A}^{\leq 0}(\Gamma^{g_1}, \Gamma^{g_2}) = \mathcal{A}_c(\Gamma^{g_1}, \Gamma^{g_2}).$$

$\mathcal{A}^{\leq N}(\emptyset)$ is an algebra, and $\mathcal{A}^{\leq N}(\Gamma)$ is a $\mathcal{A}^{\leq N}(\emptyset)$-module: $\forall \alpha \in \mathcal{A}^{\leq N}(\emptyset)$, $\forall \beta \in \mathcal{A}^{\leq N}(\Gamma)$ or $\mathcal{A}^{\leq N}(\emptyset)$, $\alpha \cdot \beta := vdeg_{\leq N}(\alpha \sqcup \beta)$, where $\sqcup$ is the disjoint union of chord diagrams. The multiplication $\bullet$ in $\mathcal{A}(\uparrow_g)$ induces one on $\mathcal{A}^{\leq N}(\uparrow_g)$, making it an algebra. $\Delta : \mathcal{A}(\Gamma) \to \mathcal{A}(\Gamma)\widehat{\otimes}\mathcal{A}(\Gamma)$, induced by summing over all ways of splitting chord diagrams into connected components, preserves vacuum degree parts, hence induces a co-multiplication $\Delta : \mathcal{A}^{\leq N}(\Gamma) \to vdeg_{\leq N}(\mathcal{A}^{\leq N}(\Gamma)\widehat{\otimes}\mathcal{A}^{\leq N}(\Gamma)) \subset \mathcal{A}^{\leq N}(\Gamma)\widehat{\otimes}\mathcal{A}^{\leq N}(\Gamma)$. But Δ is not an algebra homomorphism when $\Gamma =\uparrow_g$. Call an element of $\beta \in \mathcal{A}^{\leq N}(\Gamma)$ *primitive* if $\Delta\beta = \beta \otimes 1 + 1 \otimes \beta$. Call $\alpha \in \mathcal{A}^{\leq N}(\Gamma)$ *group-like* if $\Delta\alpha = vdeg_{\leq N}(\alpha\widehat{\otimes}\alpha)$. Since $\mathcal{A}^{\leq N}(\uparrow_g) = \mathcal{A}^{\leq N}(\emptyset) \otimes_{\mathbb{Q}} \mathcal{A}_c(\uparrow_g)$, Proposition 2.3 holds if we replace $\mathcal{A}(\uparrow_g)$, $\mathfrak{a}(\uparrow_g)$ and $\widehat{I}$ by their vacuum degree $\leq N$ truncations $\mathcal{A}^{\leq N}(\uparrow_g)$, $\mathfrak{a}^{\leq N}(\uparrow_g)$ and $\widehat{I}^{\leq N}$, provided we use the above notions of primitive and group-like. By Proposition 2.2, it then also holds for $\mathcal{A}^{\leq N}(\Gamma^g)$. Let:

$$z_g = \frac{Z(T_g) \otimes (\nu^{1/2})^{\otimes 2g}}{c_+^g \cdot c_-^g} \in \mathcal{A}(\uparrow_g, \uparrow_g), \tag{2.9}$$

$$z_g^N = vdeg_{\leq N}\left(\frac{Z(T_g) \otimes (\nu^{1/2})^{\otimes 2g}}{c_+^g \cdot c_-^g}\right) \in \mathcal{A}^{\leq N}(\uparrow_g, \uparrow_g), \tag{2.10}$$

where T_g is the q-tangle from Fig. 1c with the non-associative structure $(\ldots(((\bullet\bullet)(\bullet\bullet))(\bullet\bullet))\ldots)$, $\nu = Z(\mathbf{O}) \in \mathcal{A}(\bigcirc)$ is the Kontsevich integral of the zero-framed unknot, and $\otimes$ means taking the connected sum of chord diagrams on each of the $2g$ components, c_+, c_- have been defined in Sect. 2.3. Note that $Z(T_g) \otimes (\nu^{1/2})^{\otimes 2g} = vdeg_0(Z(T_g) \otimes (\nu^{1/2})^{\otimes 2g})$. For $g = 0$ define $z_0 = z_0^N = 1$.

Proposition 2.7. *1) Let $*$ be the gluing operation defined above, let $\tilde{\iota}_N : \mathcal{A}(\Gamma^{g_1} \sqcup (\sqcup_{2g_2} S^1) \sqcup \Gamma^{g_3}) \to \mathcal{A}(\Gamma^{g_1} \sqcup \Gamma^{g_3})$ be the $\mathcal{A}(\emptyset)$-linear map defined by (2.1), which refers exactly to all present circle components (in this case $2g_2$). Then*[13]

$$\ell_N(\alpha, \beta) := vdeg_{\leq N}((-1)^{gN}\tilde{\iota}_N(\alpha * z_{g_2}^N * \beta)) \tag{2.11}$$

defines a $\mathcal{A}^{\leq N}(\emptyset)$-bilinear form $\mathcal{A}^{\leq N}(\Gamma^{g_1}, \Gamma^{g_2}) \otimes \mathcal{A}^{\leq N}(\Gamma^{g_2}, \Gamma^{g_3}) \to \mathcal{A}^{\leq N}(\Gamma^{g_1}, \Gamma^{g_3})$.

2) For any N and any respective elements α, β, γ we have $\ell_N(\ell_N(\alpha, \beta), \gamma) = \ell_N(\alpha, \ell_N(\beta, \gamma))$.

[13] For elements belonging to the space of chord diagrams on specific $2g$ arrows which connect $2g$ points on a "bottom line" with $2g$ points on a "top line", as is z_g, one can similarly define $*$.

Proof. 1) Let $\alpha \in \mathcal{A}^{\leq N}(\Gamma^{g_1}, \Gamma^{g_2}), \beta \in \mathcal{A}^{\leq N}(\Gamma^{g_2}, \Gamma^{g_3}), z_{g_2}^N \in \mathcal{A}^{\leq N}(\Gamma^{g_2}, \Gamma^{g_2})$. $\ell_N(\alpha, \beta)$ is well-defined. Indeed, it can be calculated in two steps: first $\tilde{\varphi}^{-1}(\alpha * z_{g_2}^N * \beta)$, then apply $\tilde{q}_N \circ \tilde{\kappa}_N$. Suppose $\alpha = 0 \in \mathcal{A}^{\leq N}(\Gamma^{g_1}, \Gamma^{g_2})$. Then, since $\tilde{\varphi}$ is an isomorphism, $vdeg_{\leq N}\tilde{\varphi}^{-1}(\alpha * z_{g_2}^N * \beta) = 0$ (i.e. if we factor in $\mathcal{C}(\Gamma^{g_1}, [m], \Gamma^{g_2})$ by the subspace spanned by diagrams with vacuum degree $> N$). Hence $vdeg_{\leq N}\tilde{\iota}_N \left(\alpha * z_{g_2}^N * \beta\right) = 0 \in \mathcal{A}^{\leq N}(\Gamma^{g_1}, \Gamma^{g_2})$, and similarly for β. $\mathcal{A}^{\leq N}(\emptyset)$-bilinearity of $(\alpha, \beta) \mapsto vdeg_{\leq N}\tilde{\iota}_N \left(\alpha * z_{g_2}^N * \beta\right)$ is obvious.

2) follows from the fact that $*$ is associative, and $\tilde{\iota}_N(\tilde{\iota}_N(\alpha * z_{g_1}^N * \beta) * z_{g_2}^N * \gamma) = \tilde{\iota}_N(\alpha * z_{g_1}^N * \tilde{\iota}_N(\beta * z_{g_2}^N * \gamma)) = \tilde{\iota}_N(\alpha * z_{g_1}^N * \beta * z_{g_2}^N * \gamma)$. $\square$

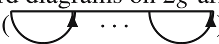
Note that when $g_1 = 0$, ℓ_N becomes a $\mathcal{A}^{\leq N}(\emptyset)$-linear map:

$$\ell_N : \mathcal{A}^{\leq N}(\Gamma^{g_2}) \otimes \mathcal{A}^{\leq N}(\Gamma^{g_2}, \Gamma^{g_3}) \to \mathcal{A}^{\leq N}(\Gamma^{g_3}).$$

Hence every element in $\mathcal{A}^{\leq N}(\Gamma^{g_2}, \Gamma^{g_3})$ defines a $\mathcal{A}^{\leq N}(\emptyset)$-linear map from $\mathcal{A}^{\leq N}(\Gamma^{g_2})$ to $\mathcal{A}^{\leq N}(\Gamma^{g_3})$, and the induced map $\tilde{\ell}_N : \mathcal{A}^{\leq N}(\Gamma^{g_2}, \Gamma^{g_3}) \to \mathcal{A}^{\leq N}(\Gamma^{g_2})^* \otimes \mathcal{A}^{\leq N}(\Gamma^{g_3})$ is $\mathcal{A}^{\leq N}(\emptyset)$-linear. Using the isomorphism $\phi_* : \mathcal{A}^{\leq N}(\uparrow_{g_i}) \to \mathcal{A}(\Gamma^{g_i})$ we obtain an $\mathcal{A}^{\leq N}(\emptyset)$-linear map also denoted $\tilde{\ell}_N : \mathcal{A}^{\leq N}(\Gamma^{g_1}, \Gamma^{g_2}) \to \mathcal{A}^{\leq N}(\uparrow_{g_1})^* \otimes \mathcal{A}^{\leq N}(\uparrow_{g_2})$. In fact the image of $\tilde{\ell}_N$ lies in the space of continuous operators, since we will show in Sect. 3 that the pairing ℓ_N is continuous.

Extend $\Gamma^g \to (\Gamma^g, \emptyset)$ to $\mathcal{A}^{\leq N}(\Gamma^g) \to \mathcal{A}^{\leq N}(\Gamma^g, \emptyset)$, and compose with $\tilde{\ell}_N$ to obtain a $\mathcal{A}^{\leq N}(\emptyset)$-linear map $(\ell_N)^* : \mathcal{A}^{\leq N}(\Gamma^g) \to \mathcal{A}^{\leq N}(\Gamma^g)^*$. Namely $(\ell_N)^*(\beta)(\alpha) = \ell_N(\alpha, \beta), \forall \alpha, \beta \in \mathcal{A}^{\leq N}(\Gamma^g)$. Similarly, there is a map $(\ell_N)^* : \mathcal{A}^{\leq N}(\uparrow_g) \to \mathcal{A}^{\leq N}(\uparrow_g)^*$. The second part of the above proposition shows that $\tilde{\ell}_N(\ell_N(\beta, \gamma)) = \tilde{\ell}_N(\gamma) \circ \tilde{\ell}_N(\beta)$ for any corresponding β, γ, i.e. the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{A}^{\leq N}(\Gamma^{g_1}, \Gamma^{g_2}) \otimes \mathcal{A}^{\leq N}(\Gamma^{g_2}, \Gamma^{g_3}) & \xrightarrow{\ell_N} & \mathcal{A}^{\leq N}(\Gamma^{g_1}, \Gamma^{g_3}) \\ \downarrow \tilde{\ell}_N \otimes \tilde{\ell}_N & & \downarrow \tilde{\ell}_N \\ \mathcal{A}^{\leq N}(\Gamma^{g_1})^* \otimes \mathcal{A}^{\leq N}(\Gamma^{g_2}) \otimes \mathcal{A}^{\leq N}(\Gamma^{g_2})^* \otimes \mathcal{A}^{\leq N}(\Gamma^{g_3}) & \xrightarrow{evaluation} & \mathcal{A}^{\leq N}(\Gamma^{g_1})^* \otimes \mathcal{A}^{\leq N}(\Gamma^{g_3}) \end{array}$$

Remarks. If we were to use Knizhnik-Zamolodchikov or any other associator, in the definition of ℓ, between α, z_g and β, we would have to insert an element A (and its horizontal reflection) from the space of chord diagrams on $2g$ arrows alternatively oriented downward and upward, such that $\left(\phi_*^{-1} Z(\right.$  $\left.)\right) * A = \nu^{\otimes g}$; and similarly for β. For the even associator, A can be taken 1, i.e. it can be omitted. Conjecture 2.5 above claims that one can take $A = 1$ for any associator.

2.6. The categories $\mathcal{A}^{\leq N}$ and $\mathcal{A}$. Let $\mathcal{A}^{\leq N}$ be the category with objects $\mathcal{A}^{\leq N}(\uparrow_g) \equiv \mathcal{A}^{\leq N}(\Gamma^0, \uparrow_g)$, $g \geq 0$, and morphisms the set of $\mathcal{A}^{\leq N}(\emptyset)$-homomorphisms between these modules. Similarly define the category $\mathcal{A}$. The isomorphism ϕ_* of Proposition 2.2 identifies $\mathcal{A}(\Gamma_g)$ and $\mathcal{A}(\uparrow_g)$. The following two statements are proved in Sect. 3.

Proposition 2.8. *Let ℓ_N be the bilinear form defined by the previous proposition, and $\tilde{\ell}_N$ be the induced map $\mathcal{A}^{\leq N}(\Gamma^{g_1}, \Gamma^{g_2}) \to \mathcal{A}^{\leq N}(\uparrow_{g_1})^* \otimes \mathcal{A}^{\leq N}(\uparrow_{g_2}) = Hom(\mathcal{A}^{\leq N}(\uparrow_{g_1}), \mathcal{A}^{\leq N}(\uparrow_{g_2}))$. Denote $w_g := Z(W_g) = vdeg_{\leq N} Z(W_g) \in \mathcal{A}^{\leq N}(\Gamma^g, \Gamma^g)$, where W_g is the* *embedded framed graph in Fig. 1a, with the first Γ^g corresponding to the lower of the two chain graphs in the picture, and the second - to the upper. For $g = 0$, set $w_g = 1$. Then $\tilde{\ell}_N(w_g)$ is the identity operator on $\mathcal{A}^{\leq N}(\uparrow_g)$.*

Theorem 2.9. *1) There is a (unique) $\mathcal{A}(\emptyset)$-bilinear form $\ell : \mathcal{A}(\Gamma^{g_1}, \Gamma^{g_2}) \otimes \mathcal{A}(\Gamma^{g_2}, \Gamma^{g_3}) \to \mathcal{A}(\Gamma^{g_1}, \Gamma^{g_3})$, such that $(vdeg_{\leq N}) \circ \ell = \ell_N$.*

2) Let $\tilde{\ell}$ be the induced map $\mathcal{A}(\Gamma^{g_1}, \Gamma^{g_2}) \to \mathcal{A}(\uparrow_{g_1})^* \otimes \mathcal{A}(\uparrow_{g_2}) = Hom_{\mathcal{A}(\emptyset)}(\mathcal{A}(\uparrow_{g_1}), \mathcal{A}(\uparrow_{g_2}))$, and denote as before $w_g = Z(W_g) \in \mathcal{A}(\Gamma^g, \Gamma^g)$, where W_g is shown in Fig. 1a. Then $\tilde{\ell}(w_g)$ is the identity operator on $\mathcal{A}(\uparrow_g)$.

3. The TQFT

Now we construct the truncated (with respect to the vacuum degree[14]) and full TQFTs. We will show that they are non-degenerate and anomaly-free. All cobordisms in the category $\mathfrak{Q}$ are connected, hence it does not make sense to require multiplicativity or self-duality. A TQFT based on $\mathfrak{Q}$ has to satisfy four axioms [6], similar to those of Atiyah [1]:

Naturality: Axiom (A1) in [6], or (III.1.4.1) in [23].
Functoriality: Axiom (A2) in [6], or (A4) in [21], or (III.1.4.3) in [23].
Normalization: Axiom (A3) in [6], or (III.1.4.4) in [23].
pseudo-Hermitianity: Axiom (A4) in [6], or (A2) in [21], or (III.5.2.2) in [23].

Assign to every oriented closed surface of genus $g \geq 0$ the $\mathcal{A}(\emptyset)$-vector space $\mathcal{A}(\Gamma^g) \cong \mathcal{A}(\uparrow_g)$, and, in the case of truncations, $\forall N \geq 0$, the $\mathcal{A}^{\leq N}(\emptyset)$-vector space $\mathcal{A}^{\leq N}(\Gamma^g) \cong \mathcal{A}^{\leq N}(\uparrow_g)$. Naturality is therefore trivial. The other axioms, together with the non-degeneracy property[15] are stated in Theorem 3.2 below.

Let $L \cup G \hookrightarrow S^3$ be an arbitrary embedding of a link and a (union of) chain graph(s) in S^3. Let σ_+, σ_- be the number of positive, respectively negative eigenvalues of $lk(L)$, the linking matrix of L, let g be the number of circle components of G, and let Γ be G as an abstract graph. For every $n \in \mathbb{N}$ define:

$$\Omega_n(L, G) := vdeg_{\leq n}\left(\frac{\tilde{\iota}_n(\check{Z}(L \cup G))}{\iota_n(\check{Z}(\mathbf{O}^{+1}))^{\sigma_+} \cdot \iota_n(\check{Z}(\mathbf{O}^{-1}))^{\sigma_-}}\right) \in \mathcal{A}^{\leq n}(\Gamma) \subset \mathcal{A}(\Gamma).$$

Here $\check{Z}(L \cup G)$ is an isotopy invariant (see [21, Theorem 1.4], where KZ associator is used, or Sect. 2.4, where an even associator is used) of $L \cup G \subset S^3$. Note that once $Z(L \cup G)$ has been shown well-defined, it does not matter which associator we use.

Proposition 3.1. *1) The following relation holds in $\mathcal{A}(\Gamma)$ for any[16] chain graph G and link L:*

$$vdeg_{\leq n}\left(\tilde{\iota}_{n+1}\check{Z}(L, G)\right) = (-1)^{|L|} det(lk(L)) vdeg_{\leq n}\left(\tilde{\iota}_n \check{Z}(L, G)\right), \tag{3.1}$$

[14] Since the map $\tilde{\iota}_N$, which we had to introduce if we want to have invariance under Kirby moves for chain graphs, decreases the total degree of a diagram by $2gN$, and since $\tilde{\iota}_N$ must be applied every time we glue two cobordisms, one does not expect the theory to truncate with respect to the total degree of chord diagrams.

[15] We show ($\forall N$ and non-truncated, and $\forall g$) that the *closure* of the subspace spanned by the images of cobordisms is *all* the respective space of chord diagrams.

[16] possibly with several connected components

and therefore:

$$vdeg_{\leq n}\Omega_{n+1}(L, G) = d(S^3_L) \cdot \Omega_n(L, G).$$

2) $\frac{1}{d(S^3_L)}\Omega_n(L, G)$ *is a group-like element of* $\mathcal{A}^{\leq n}(\Gamma) \subset \mathcal{A}(\Gamma)$ *of the form* $1+$ *higher order terms.*

Proof. 1) Follow the proof of Proposition 4.5 in [24].

2) First note that Z of any elementary pseudo-quasi-tangle is group-like of the desired form. Indeed, if one uses a KZ associator, the elements a, b in [21, p. 503] used in the definition of Z for the vicinity of trivalent vertices are clearly so. If one uses an even associator, then this follows from the fact that $\Delta\nu = \nu \otimes \nu$ and $\nu = 1 + h.o.t.$ Hence $\check{Z}(L \cup G)$ is group-like of the form $1 + h.o.t.$ for any $L \cup G \hookrightarrow S^3$ (compare with [17, Subsect. 1.4]). That Δ commutes with $\tilde{\iota}_n$ follows from the fact that Δ commutes with $\tilde{\varphi}$, and an explicit calculation of $\Delta \circ (\tilde{q}_n \circ \tilde{\kappa}_n)$ and $(\tilde{q}_n \circ \tilde{\kappa}_n) \otimes (\tilde{q}_n \circ \tilde{\kappa}_n) \circ \Delta$ for any diagram with $2n$ legs of each color $1, \dots, |L|$, just as in the case $G = \emptyset$ [24, 17]. Similarly it then follows that $\frac{1}{d(S^3_L)}\Omega_n(L, G)$ has the form $1 + h.o.t.$ (compare with [17, Lemma 4.7]). □

Let (M, f_1, f_2) be an arbitrary $\mathbb{Q}$-cobordism (in particular, a morphism in the category $\mathfrak{Q}$ between g_1 and g_2). Let (L, G_1, G_2) be such that $\kappa(L, G_1, G_2) = (M, f_1, f_2)$. By Proposition 2.1 in [21], the ambiguity in this choice is a finite sequence of KI and extended KII moves, and change of orientation of a link component. Define:

$$\begin{aligned} \tau(M, f_1, f_2) &= \sum_{n\geq 0} \frac{1}{|det(lk(L))|^n} \cdot vdeg_n\left(\frac{\tilde{\iota}_n(\check{Z}(L \cup G_1 \cup G_2))}{\iota_n(\check{Z}(\mathbf{O}^{+1}))^{\sigma_+} \cdot \iota_n(\check{Z}(\mathbf{O}^{-1}))^{\sigma_-}}\right) \\ &\overset{(2.5)}{=} \sum_{n\geq 0} \frac{(-1)^{\sigma_+ n}}{|det(lk(L))|^n} \cdot vdeg_n\left(\frac{\tilde{\iota}_n(\check{Z}(L \cup G_1 \cup G_2))}{c_+^{\sigma_+} \cdot c_-^{\sigma_-}}\right) \in \mathcal{A}(\Gamma^{g_1}, \Gamma^{g_2}), \end{aligned} \tag{3.2}$$

where $\tilde{\iota}_n$ refers to the circle components of chord diagrams, all coming here from the components of the link L, c_+, c_- have been defined in Sect. 2.3, $vdeg_n$ means taking the vacuum degree n part, and for simplicity we suppose that G_1, G_2 as abstract graphs are $\Gamma^{g_1}, \Gamma^{g_2}$. We use the convention $det(lk(\emptyset)) = 1$. Also, let:

$$\tau^N(M, f_1, f_2) = \frac{(-1)^{\sigma_+ N}}{d(\widehat{M})^N} \cdot vdeg_{\leq N}\left(\frac{\tilde{\iota}_N(\check{Z}(L \cup G_1 \cup G_2))}{c_+^{\sigma_+} \cdot c_-^{\sigma_-}}\right) \in \mathcal{A}^{\leq N}(\Gamma^{g_1}, \Gamma^{g_2}), \tag{3.3}$$

where $d(\widehat{M}) = |H_1(\widehat{M}, \mathbb{Z})|$. Note that $\tau^N(M, f_1, f_2) = vdeg_{\leq N}\tau(M, f_1, f_2)$. The same proof as in the case of closed manifolds (Sect. 3 in [17], or Sect. 4 in [24], or [21, Prop. 2.4]), shows[17] that $\tau(M, f_1, f_2)$, being invariant under KI and extended KII moves, and change of orientation of a link component, is independent of the choice of the triplet (L, G_1, G_2) for (M, f_1, f_2). Hence $\tau(M, f_1, f_2) \in \mathcal{A}(\Gamma^{g_1}, \Gamma^{g_2})$ and $\tau^N(M, f_1, f_2) \in \mathcal{A}^{\leq N}(\Gamma^{g_1}, \Gamma^{g_2})$ are invariants of $\mathbb{Q}$-cobordisms (in particular of

[17] looking at every (total) degree part

morphisms in the category $\mathfrak{Q}$). As we have seen in Sect. 2.5, $\tilde{\ell}_N(\tau^N(M, f_1, f_2))$ is then a $\mathcal{A}^{\leq N}(\emptyset)$-homomorphism from $\mathcal{A}^{\leq N}(\uparrow_{g_1})$ to $\mathcal{A}^{\leq N}(\uparrow_{g_2})$.

It is obvious to similarly define (3.2) and (3.3) for cobordisms $(M, \emptyset, f)$ with only one connected and parametrized boundary component, as long as $\widehat{M}$ is a $\mathbb{Q}$-homology sphere[18]. Note that we think of the boundary as the *top* of the 3-cobordism. We associate to $(M, \emptyset, f)$ an element $\tau(M, \emptyset, f) \in \mathcal{A}(\Gamma^g) \overset{\phi_*^{-1}}{\cong} \mathcal{A}(\uparrow_g)$, and an element $\tau^N(M, \emptyset, f) \in \mathcal{A}^{\leq N}(\Gamma^g) \overset{\phi_*^{-1}}{\cong} \mathcal{A}^{\leq N}(\uparrow_g), \forall N$.

Theorem 3.2. *1) Let $\mathcal{A}_\tau^{\leq N}(\uparrow_g)$, respectively $\mathcal{A}_\tau(\uparrow_g)$, be the $\mathbb{Q}$-vector subspace of $\mathcal{A}^{\leq N}(\uparrow_g)$, respectively $\mathcal{A}(\uparrow_g)$, generated by all $\phi_*^{-1}\tau^N(M, \emptyset, f)$, respectively by all $\phi_*^{-1}\tau(M, \emptyset, f)$, such that $\widehat{M}$ is a $\mathbb{Z}$-homology sphere. Then the completion of $\mathcal{A}_\tau^{\leq N}(\uparrow_g)$ is $\mathcal{A}^{\leq N}(\uparrow_g)$, and the completion of $\mathcal{A}_\tau(\uparrow_g)$ is $\mathcal{A}(\uparrow_g)$.*

2) Let $g, g' \in \mathbb{N}$, and let $\mathcal{A}_\tau^{\leq N}(\Gamma^g, \Gamma^{g'})$, respectively $\mathcal{A}_\tau(\Gamma^g, \Gamma^{g'})$, be the $\mathbb{Q}$-vector subspace of $\mathcal{A}^{\leq N}(\Gamma^g, \Gamma^{g'})$, respectively $\mathcal{A}(\Gamma^g, \Gamma^{g'})$, generated by all $\tau^N(M, f, f')$, respectively by all $\tau(M, f, f')$, which belong to the category $\mathfrak{Z}$. Then the completion of $\mathcal{A}_\tau^{\leq N}(\Gamma^g, \Gamma^{g'})$ is $\mathcal{A}^{\leq N}(\Gamma^g, \Gamma^{g'})$, and the completion of $\mathcal{A}_\tau(\Gamma^g, \Gamma^{g'})$ is $\mathcal{A}(\Gamma^g, \Gamma^{g'})$.

3) Let (M_1, f_1, f_1') and (M_2, f_2, f_2') be two $\mathbb{Q}$-cobordisms, such that their composition is also a $\mathbb{Q}$-cobordism. Then, for any $N \in \mathbb{N}$, the following gluing formula without anomaly *holds:*

$$\tau^N(M_2 \cup_{f_2 \circ (f_1')^{-1}} M_1, f_1, f_2') = \ell_N(\tau^N(M_1, f_1, f_1'), \tau^N(M_2, f_2, f_2')). \tag{3.4}$$

4) Let (M_1, f_1, f_1') and (M_2, f_2, f_2') be two $\mathbb{Q}$-cobordisms, such that their composition is also a $\mathbb{Q}$-cobordism. Then the following gluing formula without anomaly *holds:*

$$\tau(M_2 \cup_{f_2 \circ (f_1')^{-1}} M_1, f_1, f_2') = \ell(\tau(M_1, f_1, f_1'), \tau(M_2, f_2, f_2')). \tag{3.5}$$

5) Let $(\Sigma \times [0, 1], (\Sigma \times 0, p_1), (\Sigma \times 1, p_2))$ be the identity 3-cobordism of genus g (see Sect. 1.1), then the following holds:

$$\tau^N(\Sigma_g \times [0, 1], (\Sigma_g \times 0, p_1), (\Sigma_g \times 1, p_2)) = id_{\mathcal{A}^{\leq N}(\Gamma^g, \Gamma^g)}, \tag{3.6}$$
$$\tau(\Sigma_g \times [0, 1], (\Sigma_g \times 0, p_1), (\Sigma_g \times 1, p_2)) = id_{\mathcal{A}(\Gamma^g, \Gamma^g)}. \tag{3.7}$$

6) For every $g, g' \in \mathbb{N}$, there exist antimorphisms (maps linear in 0-supergrading and antilinear in 1-supergrading) $\bar{\cdot} : \mathcal{A}(\Gamma^g) \to \mathcal{A}(\Gamma^g)$ and $\bar{\cdot} : \mathcal{A}(\Gamma^{g'}, \Gamma^g) \to \mathcal{A}(\Gamma^g, \Gamma^{g'})$, such that for every cobordism M between g and g':

$$\tau(-M) = \overline{\tau(M)} \quad \text{and} \quad \tau^N(-M) = \overline{\tau^N(M)}. \tag{3.8}$$

It is known [11, Prop. 13.1], that for every chord diagram $\xi \in \mathcal{A}(\uparrow_g)$ of degree m, with connected dashed graph, there exist string links $L^\pm$, with $lk(L^\pm) = (\mathbf{0})$, such that $Z(L^\pm) = 1 \pm \xi + o(m + 1)$. There is a very intuitive topological realization of $L^\pm$ from Habiro's calculus of claspers [14, 16].

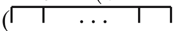
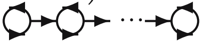
[18] which is equivalent to the equivalence class of the cobordism $(M - \{a\,ball\}, S^2 \to \partial(the\,ball), f)$ being in the category $\mathfrak{Q}$

Lemma 3.3. *For every $n \geq 0$ and every chord diagram ξ on $\uparrow_g$, $deg(\xi) \leq n$, there exist string links $L_1, \ldots, L_k$, with zero linking matrix, and positive integers $a_1, \ldots, a_k$, such that $\sum_{i=1}^{k} a_i Z(L_i) = \xi + o(n+1)$.*

Proof. Induction on n. For $n = 0$, $Z(\text{trivial string link}) = 1 \in \mathcal{A}(\uparrow_g)$. For $n = 1$, ξ must have connected dashed graph, hence the claim follows from the mentioned result of Habegger and Masbaum, because $Z(\text{trivial string link}) = 1$. For general n, suppose ξ has degree m. We prove the statement first for $m = n$, then for $m = n-1, \ldots, 1$ ($m = 0$ is obvious). For arbitrary m, by the same argument as in the proof of Proposition 2.3.5) we can assume $\xi = \sum \pm \xi_1 \bullet \ldots \bullet \xi_k$, ξ_i have connected dashed graph and degree ≥ 1. If $k > 1$, by the induction hypothesis there exist $\alpha^i_j \in \mathbb{Z}$ and string links L^i_j, such that $\sum_j a^i_j Z(L^i_j) = \xi_i + o(n)$, $\forall i$. Therefore $\sum_{i_1,\ldots,i_k} a^{i_1}_1 \cdots a^{i_k}_k \cdot Z(L^{i_1}_1 \bullet \ldots \bullet L^{i_k}_k) = \sum_{i_1,\ldots,i_k} a^{i_1}_1 Z(L^{i_1}_1) \bullet \ldots \bullet a^{i_k}_k Z(L^{i_k}_k) = (\xi_1 + o(n)) \bullet \ldots \bullet (\xi_k + o(n)) = \xi_1 \bullet \ldots \bullet \xi_k + o(n+1)$. Moreover, since all $L^{i_j}_j$ have zero linking matrix, so does $L^{i_1}_1 \bullet \ldots \bullet L^{i_k}_k$. If $k = 1$, by the Habegger-Masbaum result, there is a string link L, such that $Z(L) - Z(\text{trivial string link}) = \xi_i + o(m+1)$. Therefore the statement for m follows from the fact that it holds for $m+1$ (express in the later formula the degree $m+1$ terms of $o(m+1)$). If $k = 1$ and $m = n$, it is precisely the Habegger-Masbaum result. Note that all coefficients a_i appearing throughout the proof can be arranged positive or negative as we wish [14, 16], hence the ones in the statement can be ensured positive. □

It is known [16 Theorem 4.5; see also 12] that for any connected trivalent graph D of degree n there exist $\mathbb{Z}$-homology 3-spheres $M^{\pm}$ such that $Z^{LMO}(M^{\pm}) = 1 \pm D + o(n+1) \in \mathcal{A}(\emptyset)$. (This is proved there for Z^{lmo}, but it is obviously then true for Z^{LMO}.) Since $Z^{LMO}(S^3) = 1 \in \mathcal{A}(\emptyset)$, with a proof absolutely similar to the one above, we have:

Lemma 3.4. *For any $n \geq 0$ and any chord diagram $\xi \in \mathcal{A}(\emptyset)$, $deg(\xi) \leq n$, there exist $\mathbb{Z}$-homology spheres $M_1, \ldots, M_k$ and positive integers $b_1, \ldots, b_k$ such that $\sum_{i=1}^{k} b_i Z^{LMO}(M_i) = \xi + o(n+1)$. In particular the set $\{\sum_i b_i Z^{LMO} | M_i\ \mathbb{Z}\text{-homology sphere}, b_i \in \mathbb{N}^*\}$ is dense in $\mathcal{A}(\emptyset)$.*

Proof of Theorem 3.2.1). Let $\downarrow\uparrow_{2g}$ denote the graph with $2g$ edges oriented alternatively down- and upward. Lemma 3.3 is clearly true for $\xi \in \mathcal{A}(\downarrow\uparrow_{2g})$ as well. Therefore for every $n \geq 0$ and every $\beta \in \mathcal{A}(\downarrow\uparrow_{2g})$ there exist string links L_i and $a_i \in \mathbb{Q}$ such that $\sum a_i \cdot Z(L_i) = \beta \bullet \left(\downarrow \otimes \nu^{-1}\right)^{\otimes g} + o(n+1)$. Using the operation $*$ defined in Sect. 2.5, attach $Z($$)$ on top of, and $Z($$)$ below each side of this equality, to obtain the existence of embedded framed graphs G_i and $a_i \in \mathbb{Q}$, such that $\sum a_i \cdot Z(G_i) = Z($$) * \left(\beta \bullet \left(\downarrow \otimes \nu^{-1}\right)^{\otimes g}\right) * Z($$) + o(n+1) = \widehat{\beta} + o(n+1)$, where $\left(\beta \mapsto \widehat{\beta}\right) : \mathcal{A}(\downarrow\uparrow_{2g}) \to \mathcal{A}($$)$ is the map induced by inclusion. The later is well-defined, since at the level of $\mathcal{D}(\downarrow\uparrow_{2g})$ AS, IHX and STU relations are sent to the same type relations on . It is clearly surjective by Proposition 2.2. Therefore for every $n \geq 0$ and $\alpha \in \mathcal{A}(\uparrow_g)$ there exist G_i and $a_i \in \mathbb{Q}$ such that $\sum a_i \cdot Z(G_i) = \phi_*(\alpha) + o(n+1)$.

Let $N = 0$. Then by (3.3) we have $\tau^0(M, \emptyset, f) = \tilde{\iota}_0 \check{Z}(L \cup G)$ whenever $\kappa(L, G) = (M, \emptyset, f)$. Since $\tilde{\iota}_N$ refer only to link components, for every embedded framed graph G we obtain $\tau^N(\kappa(\emptyset, G)) = \tau^0(\kappa(\emptyset, G)) = \tilde{\iota}_0 \check{Z}(G) = \check{Z}(G) = Z(G)$. Together with the conclusion of the previous paragraph this shows that, for every $n \geq 0$, and every $\alpha \in \mathcal{A}^{\leq 0}(\uparrow_g) = \mathcal{A}_c(\uparrow_g)$, there exist cobordisms $(M_i, \emptyset, f_i)$ having $\widehat{M_i} = S^3$, and $a_i \in \mathbb{Q}$, such that

$$\sum_i a_i \cdot \tau^0(M_i, \emptyset, f_i) = \phi_*(\alpha) + o(n+1). \tag{3.9}$$

This proves the theorem for $N = 0$.

Let $N > 1$. Recall that $\mathcal{A}^{\leq N}(\uparrow_g) = \mathcal{A}^{\leq N}(\emptyset) \otimes_{\mathbb{Q}} \mathcal{A}_c(\uparrow_g)$. Therefore the statement is enough to prove for $\xi \cdot \alpha$, $\xi \in \mathcal{A}^{\leq N}(\emptyset)$ and $\alpha \in \mathcal{A}_c(\uparrow_g)$. By Lemma 3.4 there exist $\mathbb{Z}$-homology spheres M_i and $b_i \in \mathbb{N}^*$, such that $\sum b_i Z^{LMO}(M_i) = \xi + o(N+1)$. Then, by the previous paragraph, for every $n \geq 0$, there exist cobordisms $(M_j, \emptyset, f_j)$ having $\widehat{M_j} = S^3$, and $a_i \in \mathbb{Q}$, such that $vdeg_{\leq N}\left(\sum_i b_i Z^{LMO}(M_i)\right) \cdot \left(\sum_j a_j \tau^0(M_j, \emptyset, f_j)\right) = \xi \cdot \phi_*(\alpha) + o(n+1)$. Therefore:

$$\begin{aligned}
\phi_*(\xi \cdot \alpha) = \xi \cdot \phi_*(\alpha) &= \sum_{i,j} b_i a_j vdeg_{\leq N} Z^{LMO}(M_i) \cdot \tau^0(M_j, \emptyset, f_j) + o(n+1) \\
&= \sum_{i,j} b_i a_j \tau^N(M_i) \tau^0(M_j, \emptyset, f_j) \\
&\quad + o(n+1) = \sum_{i,j} b_i a_j \left(\tau^N(S^3_{L_i}) \tau^0(\kappa(\emptyset, G_j))\right) + o(n+1) \\
&= \sum_{i,j} b_i a_j \tau^N\left(S^3_{L_i} \# \kappa(\emptyset, G_j)\right) + o(n+1),
\end{aligned}$$

where $b_i a_j \in \mathbb{Q}$, $M_i = S^3_{L_i}$, $(M_j, \emptyset, f_j) = \kappa(\emptyset, \Gamma_j)$ and $S^3_{L_i} \# \kappa(\emptyset, G_j) = \kappa(L_i \sqcup G_j)$, whose filling is $S^3_{L_i}$, a $\mathbb{Z}$-homology sphere. This proves the theorem for arbitrary N. Since for any cobordism $(M, \emptyset, f)$ we have $vdeg_{\leq N} \tau(M, \emptyset, f) = \tau^N(M, \emptyset, f)$, by taking $N = n$, we can see that, for every $n \geq 0$ and any $\alpha \in \mathcal{A}(\uparrow_g)$, there exist $c_i \in \mathbb{Q}$, and $(M_i, \emptyset, f_i)$ with $\widehat{M_i}$ $\mathbb{Z}$-homology spheres, such that $\phi_*(\alpha) = \sum_i c_i vdeg_{\leq n} \tau(M_i, \emptyset, f_i) + o(n+1) = \sum_i c_i \tau(M_i, \emptyset, f_i) + o(n+1)$. □

Let (L, G, G') be a triplet and $(M, f, f') = \kappa(L, G, G')$. We can talk about linking number between a link component K and a circle U of a chain graph, as well as between two circles U and V of chain graphs: $lk(K, U) = lk(U, K)$ is defined to be the linking number between K, and the knot obtained from the graph by deleting all but the circle component U, and similarly for $lk(U, V)$. *The linking matrix of a triplet* is then:

$$lk(L, G, G') = \begin{pmatrix} lk(L) & lk(L, G) & lk(L, G') \\ lk(G, L) & lk(G, G) & lk(G, G') \\ lk(G', L) & lk(G', G) & lk(G', G') \end{pmatrix} = \begin{pmatrix} A & B^T & C^T \\ B & D & E^T \\ C & E & F \end{pmatrix}, \tag{3.10}$$

where A, D, F are symmetric matrices. In [6] it has been shown that the doubly-Lagrangian condition can be expressed:

$$\begin{aligned} D &= BA^{-1}B^T, \\ F &= CA^{-1}C^T \end{aligned} \tag{3.11}$$

(for $\mathbb{Q}$-cobordisms this in particular means that the entries on the left-hand side, a priori in $\mathbb{Z}\left[\frac{1}{\det A}\right]$, must be in $\mathbb{Z}$), and for the case $\mathbb{F}=\mathbb{Q}$, additionally $BA^{-1}C^T \in \mathcal{M}_{g_1\times g_2}(\mathbb{Z})$. In the case $L=\emptyset$, the doubly-Lagrangian condition can be expressed:

$$D = F = 0. \tag{3.12}$$

Proof of Theorem 3.2.2). The proof of part 1) does not use the fact that in Lemma 3.3 the links L_i have zero linking matrix, because for connected chain graphs the Lagrangian condition is trivial. Since a $\mathbb{Z}$-cobordism (M, f, f') is doubly-Lagrangian iff any representing triplet satisfies condition (3.12), the same proof as in part 1) shows that, for every $n \geq 0$, and every $\alpha \in \mathcal{A}_c(\Gamma^g, \Gamma^{g'})$, there exist $a_i \in \mathbb{Q}$, and doubly-Lagrangian $\mathbb{Z}$-cobordisms (M_i, f_i, f_i') having $\widehat{M_i} = S^3$, such that $\sum_i a_i \cdot \tau^0(M_i, f_i, f_i') = \alpha + o(n+1)$, proving the theorem for $N=0$. This identity replaces (3.9) in the proof of part 1), and the argument can be continued to prove part 2) for any N, and for non-truncated τ. □

Lemma 3.5. [6, Lemma 5] *Let (M_1, f_1, f_1') and (M_2, f_2, f_2') be two 3-cobordisms. Suppose $(M_1, f_1, f_1') = \kappa(L_1, G_1, G_1')$, $(M_2, f_2, f_2') = \kappa(L_2, G_2, G_2')$, and $(M_2 \cup_{f_2\circ(f_1')^{-1}} M_1, f_1, f_2') = \kappa(L_1 \cup L_0 \cup L_2, G_1, G_2')$, the later triplet obtained from the previous two by the construction described in Proposition 1.3. Denote $\sigma_+^1 = sign_+(lk(L_1))$, $\sigma_+^2 = sign_+(lk(L_2))$, $\sigma_+ = sign_+(lk(L_1 \cup L_0 \cup L_2))$, and let g be the genus of the connected closed surface along which is this splitting. Then the integer $s(M, M_1, M_2) = \sigma_+^1 + \sigma_+^2 + g - \sigma_+$ is an invariant of the decomposition $M = M_2 \cup_{f_2\circ(f_1')^{-1}} M_1$), i.e. it does not depend on the choice of triplets representing the 3-cobordisms M_1 and M_2.* □

Lemma 3.6. *Let (M_1, f_1, f_1') and (M_2, f_2, f_2') be two $\mathbb{Q}$-cobordisms, such that their composition is also a $\mathbb{Q}-cobordism$. Denote $d = |H_1(\widehat{M_2 \cup_{f_2\circ(f_1')^{-1}} M_1}, \mathbb{Z})|$, $d_1 = |H_1(\widehat{M_1}, \mathbb{Z})|$, $d_2 = |H_1(\widehat{M_2}, \mathbb{Z})|$. Suppose that these cobordisms are glued along a surface of genus g. Then:*

$$\tau^N(M_2 \bigcup_{f_2\circ(f_1')^{-1}} M_1, f_1, f_2') = \left((-1)^N vdeg_{\leq N}\left(\frac{c_+}{c_-}\right)\right)^{\sigma_+^1+\sigma_+^2+g-\sigma_+} \left(\frac{d_1 d_2}{d}\right)^N \cdot \ell_N(\tau^N(M_1, f_1, f_1'), \tau^N(M_2, f_2, f_2')), \tag{3.13}$$

where $(-1)^N \cdot vdeg_{\leq N}(c_+/c_-) \in \mathcal{A}^{\leq N}(\emptyset)$, the multiplication by scalars is thought in the category $\mathcal{A}^{\leq N}$, and $\sigma_+^1 + \sigma_+^2 + g - \sigma_+$ is an integer.

Proof. Let (L_1, G_1, G_1'), $(L_2, G_2, G_2',)$, and $(L_1 \cup L_0 \cup L_2, G_1, G_2')$ represent the 3-cobordisms M_1, M_2, and $M_2 \circ M_1$, as in the previous lemma, and let (σ_+, σ_-), (σ_+^1, σ_-^1), respectively (σ_+^2, σ_-^2) be the signatures of $lk(L_1 \cup L_0 \cup L_2)$, $lk(L_1)$, resp. $lk(L_2)$. Then, locally abbreviating (when space requires) $vdeg_{\leq N}(c_+)$ and $vdeg_{\leq N}(c_-)$ to c_+ and c_-:

$$\tau^N\left(M_2 \bigcup_{f_2\circ(f_1')^{-1}} M_1, f_1, f_2'\right) = \frac{(-1)^{\sigma_+ N}}{d^N} \cdot vdeg_{\leq N}\left(\frac{\check{\iota}_N \check{Z}(L_1 \cup L_0 \cup L_2 \cup G_1 \cup G_2')}{c_+^{\sigma_+} \cdot c_-^{\sigma_-}}\right)$$

$$= \left((-1)^N \frac{c_+}{c_-}\right)^{\sigma_+^1+\sigma_+^2+g-\sigma_+} \cdot \frac{d_1^N d_2^N}{d^N} \cdot$$

$$\times\, vdeg_{\leq N}\left(\frac{\tilde{\iota}_N(\tilde{\iota}_N \check{Z}(L_1, G_1, G_1') * (Z(T_g) \otimes (\nu^{1/2})^{\otimes 2g}) * \tilde{\iota}_N \check{Z}(L_2, G_2, G_2'))}{(-1)^{\sigma_+^1 N}(-1)^{\sigma_+^2 N}(-1)^{\sigma_+ N} d_1^N d_2^N \cdot c_+^{\sigma_+^1} c_-^{\sigma_-^1} \cdot c_+^g c_-^g \cdot c_+^{\sigma_+^2} c_-^{\sigma_-^2}}\right)$$

$$= \left((-1)^N \frac{c_+}{c_-}\right)^{\sigma_+^1+\sigma_+^2+g-\sigma_+} \cdot \frac{d_1^N d_2^N}{d^N} \cdot$$

$$\times\, vdeg_{\leq N}\tilde{\iota}_N\left(\frac{\tilde{\iota}_N \check{Z}(L_1, G_1, G_1')}{(-1)^{\sigma_+^1 N} d_1^N c_+^{\sigma_+^1} c_-^{\sigma_-^1}} * \frac{Z(T_g) \otimes (\nu^{1/2})^{\otimes 2g}}{(-1)^{gN} c_+^g c_-^g} * \frac{\tilde{\iota}_N \check{Z}(L_2, G_2, G_2')}{(-1)^{\sigma_+^2 N} d_2^N c_+^{\sigma_+^2} c_-^{\sigma_-^2}}\right)$$

$$= \left((-1)^N vdeg_{\leq N}\left(\frac{c_+}{c_-}\right)\right)^{\sigma_+^1+\sigma_+^2+g-\sigma_+} \cdot \left(\frac{d_1^N d_2^N}{d^N}\right) \cdot \ell_N(\tau^N(M_1), \tau^N(M_2)),$$

where we have used that $\sigma_+ + \sigma_- = \sigma_1^+ + \sigma_1^- + \sigma_2^+ + \sigma_2^- + 2 \cdot g$. Observe that in the second equality, when "braking" $\check{Z}$ into three, on each component of L_0 a $\nu^{1/2}$ "goes" to Z of G_1' or G_2, and another $\nu^{1/2}$ goes to z_g. In fact, the two middle expressions are written for the even associator. For any other associator we would insert between the $*$'s the element A mentioned in the remark at the end of Sect. 2.5. □

Proof of Theorem 3.2.3). We use the notations of the previous lemma. Observe that (3.11) implies:

$$lk(L_1 \cup L_0 \cup L_2) = \begin{pmatrix} A & B^T & \mathbf{0} & \mathbf{0} \\ B & BA^{-1}B^T & -I & \mathbf{0} \\ \mathbf{0} & -I & DC^{-1}D^T & D \\ \mathbf{0} & \mathbf{0} & D^T & C \end{pmatrix}, \tag{3.14}$$

where $A = lk(L_1) \in \mathcal{M}_{|L_1|\times|L_1|}(\mathbb{Z})$, $C = lk(L_2) \in \mathcal{M}_{|L_2|\times|L_2|}(\mathbb{Z})$, $B = lk(G_1', L_1) \in \mathcal{M}_{g\times|L_1|}(\mathbb{Z})$, $D = lk(G_2, L_2) \in \mathcal{M}_{g\times|L_2|}(\mathbb{Z})$, $BA^{-1}B^T, DC^{-1}D^T \in \mathcal{M}_{g\times g}(\mathbb{Z})$. In [6] it has been shown that the signature of this linking matrix is $(\sigma_+^1+\sigma_+^2+g, \sigma_-^1+\sigma_-^2+g)$, where (σ_+^1, σ_-^1), respectively (σ_+^2, σ_-^2) is the signature of $lk(L_1)$, respectively $lk(L_2)$, and that the following holds:

$$\det(lk(L_1 \cup L_0 \cup L_2)) = (-1)^g \cdot \det(lk(L_1)) \cdot \det(lk(L_2)). \tag{3.15}$$

Substituting into (3.13), we obtain the functoriality of τ^N. □

Lemma 3.7. *Let $\beta \in \mathcal{A}(\Gamma^{g_2}, \Gamma^{g_3})$, and let $\alpha_n, n \in \mathbb{N}$ be a sequence of elements of $\mathcal{A}(\Gamma^{g_1}, \Gamma^{g_2})$, such that for every n, $deg_{\leq n}(\alpha_n) = deg_{\leq n}(\alpha_m)$, $\forall m > n$. Then, both sides of the following equality are well-defined, and the equality holds:*

$$\ell_N(\lim_n \alpha_n, \beta) = \lim_n \ell_N(\alpha_n, \beta). \tag{3.16}$$

A similar property holds for the rôle of two arguments of ℓ_N reversed.

Proof. The existence of $\alpha := \lim_n \alpha_n \in \mathcal{A}(\Gamma^{g_1}, \Gamma^{g_2})$ follows directly from the fact that we defined the topology on $\mathcal{A}(\Gamma^{g_1}, \Gamma^{g_2})$ such that $dist(p, q) < \frac{1}{2^n}$ if and only if $p - q$ has degree $> n$. Then, since $deg_{>n+gN}(\alpha)$ does not contribute to $deg_{\leq n}\ell_N(\alpha, \beta)$, we have:

$$\lim_n deg_{\leq n}\ell_N(\alpha, \beta) = \lim_n deg_{\leq n}\ell_N(deg_{\leq n+gN}\alpha, \beta) = \lim_n \ell_N(deg_{\leq n+gN}\alpha, \beta)$$
$$= \lim_m \ell_N(deg_{\leq m}\alpha, \beta) = \ell_N(\lim_m deg_{\leq m}\alpha, \beta) = \ell_N(\alpha, \beta).$$

The existence of the third limit and the second equality follow from a standard Cauchy-sequences argument. The fourth equality is true since $\lim_m$ commutes with $*$, $\widetilde{\iota_N}$ and $vdeg_{\leq N}$. On the other hand $\ell_N(\alpha_n, \beta)$ and $\ell_N(\alpha, \beta)$ agree in degree $\leq n - 2gN$. Hence $\lim \ell_N(\alpha_n, \beta) = \lim deg_{\leq n}\ell_N(\alpha, \beta)$. Putting the two together we obtain (3.16). □

Remark. If in the statement of this lemma we assume that $\lim_n \alpha_n$ exists, which is the case whenever the result is used in this paper, then we can relax the topology: $distance(p, q) \leq \frac{1}{n} \Leftrightarrow p - q$ has no terms of degree $< n$.

Elementary remark. The signature of a symmetric $2g \times 2g$-matrix $\begin{pmatrix} A & -I \\ -I & \mathbf{0} \end{pmatrix}$ with integer, respectively real entries is (g, g). The determinant of such a matrix is $(-1)^g$.

Proof of Proposition 2.8. Note that $w_g = \tau(\Sigma_g \times [0, 1], (\Sigma_g \times 0, p_1), (\Sigma_g \times 1, p_2))$. Using the gluing formula (3.13), for any $\mathbb{Q}$-cobordism (M, f_1, f_2), $\tau^N((\Sigma_g \times [0, 1]) \cup_{p_1 \circ (f_2)^{-1}} M, f_1, p_2) = \left((-1)^N vdeg_{\leq N}\left(\frac{c_+}{c_-}\right)\right)^{\sigma_+^1+\sigma_+^2+g-\sigma_+} \cdot \left(\frac{d_1 d_2}{d}\right)^N \cdot \ell_N(\tau^N(M, f_1, f_2), w_g)$. If L is a link such that $\widehat{M} = S^3_L$, and the linking matrix of L is $lk(L)$, then the linking matrix of the link $L \cup L_0$ is $\begin{pmatrix} lk(L) & * & \mathbf{0} \\ * & * & -I \\ \mathbf{0} & -I & \mathbf{0} \end{pmatrix} \sim \begin{pmatrix} lk(L) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & * & -I \\ \mathbf{0} & -I & \mathbf{0} \end{pmatrix}$. Using the above remark, $\sigma_+ = \sigma_+^1 + g$, $\sigma_- = \sigma_-^1 + g$, $\sigma_+^2 = \sigma_-^2 = 0$, $d_2 = lk(\emptyset) = 1$, $d_1 = d$. Observe that $((\Sigma_g \times [0, 1]) \cup_{p_1 \circ (f_2)^{-1}} M, f_1, p_2) \cong (M, f_1, f_2)$. Hence $\ell_N(\tau^N(M, f_1, f_2), w_g) = \tau^N(M, f_1, f_2)$. In particular, this holds if (M, f_1, f_2) is a $\mathbb{Z}$-cobordism with bottom homeomorphic to S^2, and hence also for any $(M, \emptyset, f)$ such that $\widehat{M}$ is a $\mathbb{Z}$-homology sphere. The statement now follows from Part 1) of Theorem 3.2 and Lemma 3.7. □

Proof of Theorem 2.9. 1) By construction, the inverse limits $\lim_{\infty \leftarrow N} \mathcal{A}^{\leq N}(\emptyset) = \mathcal{A}(\emptyset)$ and $\lim_{\infty \leftarrow N} \mathcal{A}^{\leq N}(\Gamma^{g_1}, \Gamma^{g_2}) = \mathcal{A}(\Gamma^{g_1}, \Gamma^{g_2})$. Let us show that the following diagram is commutative for every $N \in \mathbb{N}$:

$$\begin{array}{ccc} \mathcal{A}^{\leq N+1}(\Gamma^{g_1}, \Gamma^{g_2}) \otimes \mathcal{A}^{\leq N+1}(\Gamma^{g_2}, \Gamma^{g_3}) & \overset{vdeg_{\leq N}}{\rightarrow} & \mathcal{A}^{\leq N}(\Gamma^{g_1}, \Gamma^{g_2}) \otimes \mathcal{A}^{\leq N}(\Gamma^{g_2}, \Gamma^{g_3}), \\ \downarrow \ell^{\leq N+1} & & \downarrow \ell_N \\ \mathcal{A}^{\leq N+1}(\Gamma^{g_1}, \Gamma^{g_3}) & \overset{vdeg_{\leq N}}{\rightarrow} & \mathcal{A}^{\leq N}(\Gamma^{g_1}, \Gamma^{g_3}), \end{array} \tag{3.17}$$

where the horizontal arrows are the maps that forget the degrees $N + 1$ parts. Let $\alpha = \tau^{N+1}(M_1)$, $\beta = \tau^{N+1}(M_2)$ for some $\mathbb{Q}$-cobordisms M_1 and M_2 from category $\mathfrak{Q}$. Then, as previously observed $vdeg_{\leq N}\tau^{N+1}(M_i) = \tau^N(M_i)$, $i = 1, 2$, i.e. $vdeg_{\leq N}\alpha =$

$\tau^N(M_1)$, $vdeg_{\leq N}\beta = \tau^N(M_2)$. By the gluing formula (3.4) we then have $\tau^{N+1}(M_2 \cup M_1) = \ell^{\leq N+1}(\alpha, \beta)$ and $\tau^N(M_2 \cup M_1) = \ell_N(vdeg_{\leq N}\alpha, vdeg_{\leq N}\beta)$. Now again, using $vdeg_{\leq N}\tau^{\leq N+1}(M_2 \cup M_1) = \tau^N(M_2 \cup M_1)$, we get $vdeg_{\leq N}\left(\ell^{\leq N+1}(\alpha, \beta)\right) = \ell_N(vdeg_{\leq N}\alpha, vdeg_{\leq N}\beta)$. Hence the diagram (3.17) is commutative for α, β as above. By Part 2) of Theorem 3.2 and Lemma 3.7, the diagram is then commutative for arbitrary α, β.

Therefore there exists a well-defined $\mathcal{A}(\emptyset)$-bilinear map $\ell : \mathcal{A}(\Gamma^{g_1}, \Gamma^{g_2}) \otimes \mathcal{A}(\Gamma^{g_2}, \Gamma^{g_3}) \to \mathcal{A}(\Gamma^{g_1}, \Gamma^{g_2})\tilde{\otimes}\mathcal{A}(\Gamma^{g_2}, \Gamma^{g_3}) \to \mathcal{A}(\Gamma^{g_1}, \Gamma^{g_3})$, such that when restricting to the vacuum degree $\leq N$ parts, one obtains the map ℓ_N.

2) By the proof of 1), $\tilde{\ell}(w_g) = \lim_{\infty \leftarrow N} \tilde{\ell}_N(w_g^N)$. By Proposition 2.8 the operators $\tilde{\ell}_N(w_g^N)$ are identities, hence so is the limit. □

Proof of Theorem 3.2.4). Theorem 2.9.1) shows that ℓ_N are the $vdeg_{\leq N}$-truncations of ℓ. The result then follows from (3.4). □

Proof of Theorem 3.2.5). Proposition 2.8 proves the first formula, the normalization of the truncated TQFTs $\mathfrak{Q} \to \mathcal{A}^{\leq N}$ and $\mathfrak{Z} \to \mathcal{A}^{\leq N}$. Theorem 2.9.2) in particular implies the second formula, the normalization of the non-truncated TQFTs $\mathfrak{Q} \to \mathcal{A}$ and $\mathfrak{Z} \to \mathcal{A}$. □

Lemma 3.8 (The continuity of ℓ). *Let $\beta \in \mathcal{A}(\Gamma^{g_2}, \Gamma^{g_3})$, and let α_n, $n \in \mathbb{N}$ be a sequence of elements of $\mathcal{A}(\Gamma^{g_1}, \Gamma^{g_2})$, such that for every n, $deg_{\leq n}(\alpha_n) = deg_{\leq n}(\alpha_m)$, $\forall m > n$. Then, both sides of the following equality are well-defined, and the equality holds:*

$$\ell(\lim_n \alpha_n, \beta) = \lim_n \ell(\alpha_n, \beta). \tag{3.18}$$

A similar property holds for the rôle of two arguments of ℓ reversed.

Proof. ℓ_N are the $vdeg_{\leq N}$-truncations of ℓ. Apply (3.16) and pass to the limit (keeping, for example $n = (2g + 1)N$). □

Proof of Theorem 3.2.6). The conjugation in $\mathcal{A}(\emptyset)$ can be extended to $\mathcal{A}(\Gamma^g)$ and $\mathcal{A}(\Gamma^{g'}, \Gamma^g)$ as follows. For an arbitrary chord diagram D, define $\overline{D} = D$, if $vdeg(D) = even$, and $\overline{D} = -D$, if $vdeg(D) = odd$. They induce maps also for $vdeg_{\leq N}$-truncations.

Let (L, G) be the embedding of a link and (disjoint union of) chain graph(s) in S^3, and let $\overline{(L, G)}$ denote its mirror image. Then $\tilde{\iota}_N\check{Z}\overline{(L, G)} = (-1)^{|L|N}\overline{\tilde{\iota}_N\check{Z}(L, G)}$ (compare with [17, Prop. 5.2]). This is true for the Murakami-Ohtsuki extension of Z because a and b from [21], and hence Z(vicinity of a trivalent vertex) are "mirrors" of themselves, which is easy to check. For the extension of Z using an even associator (Sect. 2.4), this property is obvious. In the proof of [17, Prop. 5.2] it is shown that $c_- = \overline{c_+}$. Hence for any $N \in \mathbb{N}$:

$$\tau^N(-M) = \frac{(-1)^{\sigma'_+ N}}{d^N} vdeg_{\leq N}\left(\frac{\tilde{\iota}_N(\check{Z}\overline{(L, G)})}{(c_+^{[\leq N]})^{\sigma'_+} \cdot (c_-^{[\leq N]})^{\sigma'_-}}\right)$$
$$= \frac{(-1)^{\sigma_+ N}}{d^N} vdeg_{\leq N}\left(\frac{\overline{\tilde{\iota}_N(\check{Z}(L, G))}}{\overline{(c_+^{[\leq N]})}^{\sigma_+} \cdot \overline{(c_-^{[\leq N]})}^{\sigma_-}}\right) = \overline{\tau^N(M)}.$$

Therefore also $\tau(-M) = \overline{\tau(M)}$, for any $\mathbb{Q}$-cobordism M. Using this formula and (3.5), it follows that for $\alpha = \tau(M_1)$, $\beta = \tau(M_2)$, where M_i are 3-cobordisms in the category $\mathfrak{Q}$, we have $\overline{\ell(\alpha,\beta)} = \overline{\ell(\tau(M_1),\tau(M_2))} = \overline{\tau(M_2 \circ M_1)} = \tau(-(M_2 \circ M_1)) = \tau((-M_1) \circ (-M_2)) = \ell(\tau(-M_2),\tau(-M_1)) = \ell(\overline{\tau(M_2)},\overline{\tau(M_1)}) = \ell(\overline{\beta},\overline{\alpha})$. By Theorem 3.2.2) and the continuity (3.18) of ℓ, $\overline{\ell(\alpha,\beta)} = \ell(\overline{\beta},\overline{\alpha})$ holds for arbitrary α, β. In particular, it remains true if we add $vdeg_{\leq N}$. The property is, therefore, verified for truncated and non-truncated TQFTs. □

We have shown that $\tau : \mathfrak{Q} \to \mathcal{A}$, $\tau^N : \mathfrak{Q} \to \mathcal{A}^{\leq N}$ are functors, and the TQFTs are non-degenerate. The full TQFT induces a linear representation $\mathcal{L}_g \to GL_{\mathcal{A}(\emptyset)}(\mathcal{A}(\Gamma^g))$. The truncated TQFTs induce linear representations $\mathcal{L}_g \to GL_{\mathcal{A}^{\leq N}(\emptyset)}(\mathcal{A}^{\leq N}(\Gamma^g))$. It is known [8] that any $\mathbb{Z}$HS can be obtained as filling of a parametrized 3-cobordism $(\Sigma_g \times I, w, id)$ for some $g \geq 0$ and some $w \in \mathcal{T}_g$, the Torelli group of genus g. Furthermore [20], it even suffices to consider only $w \in \mathcal{K}_g$, the kernel of the Johnson homomorphism, or topologically, the subgroup of $\mathcal{T}_g$ generated by Dehn twists on bounding simple closed curves. Our TQFTs, of course, induce linear representations of both these subgroups of $\mathcal{L}_g$. The group $\mathcal{L}_g$ has not been studied before, no explicit set of generators, less so one of relations, is known.

Note, that Theorem 3.2.1) and 2) address the so-called realization problem for links, string links, three-dimensional manifolds, and chain graphs, by showing (see also 3.3, 3.4) that Z(links), τ(closed 3-manifolds), Z(string links), and τ(3-manifolds with boundary), in the closure, generate the corresponding combinatorial spaces of chord diagrams: $\mathcal{A}(\circlearrowleft \dots \circlearrowleft)$, $\mathcal{A}(\emptyset)$, $\mathcal{A}(\uparrow_g)$, and $\mathcal{A}(\Gamma^{g_1},\Gamma^{g_2})$. Without proving Theorem 3.2.1) and 2), even partial results of this sort were hard to obtain, as can be seen from the following

Proposition 3.9. *For every $N \geq 0$ and every $\mathbb{Q}$-cobordism (M, f_1, f_2) from g to g, $\tilde{\ell}_N(\tau^N(M, f_1, f_2))$ sends the $\mathcal{A}^{\leq N}(\emptyset)$-submodule of $\mathcal{A}^{\leq N}(\uparrow_g)$ generated by* $\exp(\alpha)$*, $\forall\alpha \in vdeg_{\leq N}\mathfrak{a}(\uparrow_g)$, to itself.*

Proof. By Proposition 3.1.3) $\tau^N(M, f_1, f_2)$ is group-like. Observe that Δ commutes with $*$, and repeating the argument from the proof of 3.1.3) for $\tilde{\iota}_N$ in the definition of ℓ_N, we can see that $\tilde{\ell}_N(\tau^N(M, f_1, f_2))$ takes a group-like element of $\mathcal{A}^{\leq N}(\Gamma^{g_1})$ of the form $1 + h.o.t.$ to a group-like element of $\mathcal{A}^{\leq N}(\Gamma^{g_2})$ of the form $1 + h.o.t.$ Now apply Proposition 2.3.5) and 3) for the truncated case. Hence it sends the $\mathcal{A}^{\leq N}(\emptyset)$-submodule of $\mathcal{A}^{\leq N}(\uparrow_g)$ generated by $\exp(\alpha)$, $\alpha \in vdeg_{\leq N}\mathfrak{a}(\uparrow_g)$ to itself. □

Remarks. 1) This construction of TQFT can be done also in the language of the Aarhus integral.
2) A combinatorial formula for the pairing $\ell : \mathcal{A}(\Gamma^{g_1},\Gamma^{g_2}) \otimes \mathcal{A}(\Gamma^{g_2},\Gamma^{g_3}) \to \mathcal{A}(\Gamma^{g_1},\Gamma^{g_3})$ is given in [7].

In [17], Le, Murakami and Ohtsuki have introduced the chord-KII move to mirror the second Kirby move for links, which then allowed them to define Z^{LMO}. However, it is well-known that handle canceling can not be obtained solely by Kirby-2, and would require in addition Kirby-1. But no corresponding chord-KI move exists, the invariance of Z^{LMO} under Kirby-1 is achieved via normalization. Therefore there is no a priori reason to suspect that a chord-canceling-handle relation is true for arbitrary chord diagrams.

Proposition 3.10. (Chord-handle canceling) *The chord-handle-canceling relation, schematically depicted in Fig. 10 holds for arbitrary $\beta \in \mathcal{A}(\uparrow_g)$. (The upper part of each F_i should be read as Z(drawn tangle).)*

Proof. For arbitrary β, F_1 differs from F_2 by a chord-KII move. (An argument similar to the one in [17, Prop. 3.2] works.) But now $F_2 = \ell(\beta, w_g) = \beta = F_3$. □

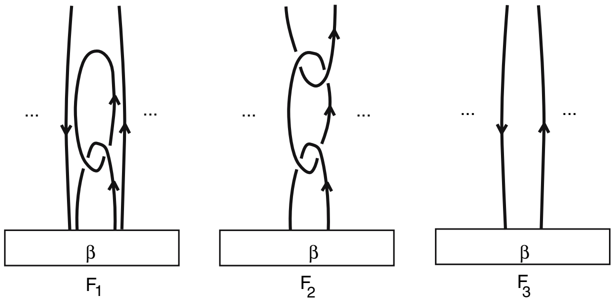


Fig. 10. Chord-handle-canceling relation $F_1 = F_3$

4. A TQFT for the Casson-Walker-Lescop Invariant

The term of degree one of Z^{LMO} of a 3-manifold is $(-1)^{b_1(M)}\frac{\lambda(M)}{2}\theta$, where $b_1(M)$ is the first Betti number, $\lambda(M)$ is the Casson invariant (in Walker-Lescop extension), and θ is the (only) open chord diagram of degree 1, which looks like the symbol θ [17]. Recall the definition and basic properties of the Casson invariant. Let K be a knot in an oriented $\mathbb{Z}$-homology 3-sphere M, and $\Delta_K(t) = a_0+a_1(t+t^{-1})+a_2(t^2+t^{-2})+\ldots$ be its Alexander polynomial normalized such that $\Delta_K(1) = 1$. Denote $\lambda'(K) = \frac{1}{2}\Delta''(1) = \sum_n n^2 a_n$.

Theorem 4.1 (Casson). *There is an integer-valued invariant λ for oriented integer homology 3-spheres such that:*

(1) *λ mod 2 is the Rohlin invariant,*
(2) *$\lambda(M) = 0$ for any homotopy 3-sphere,*
(3) *$\lambda(-M) = -\lambda(M)$,*
(4) *$\lambda(M_1\#M_2) = \lambda(M_1) + \lambda(M_2)$,*
(5) *If K is a knot in an oriented integer homology 3-sphere M, and $M(K, \frac{1}{n})$ denotes the integer homology 3-sphere obtained from M by a $\frac{1}{n}$-surgery on K, then $\lambda(M(K, \frac{1}{n})) = \lambda(M) + n\lambda'(K)$.*

Property (5) from this theorem, for $n = \pm 1$, and normalization $\lambda(S^3) = 0$ determine λ uniquely, since any integer homology 3-sphere can be obtained from S^3 by a succession of ± 1-surgeries on knots. λ was extended to rational homology 3-spheres by Walker, and corresponding properties (4) and (5) were given by Lescop [18]:

(4′) $\lambda(M_1\#M_2) = |H_1(M_2,\mathbb{Z})|\lambda(M_1) + |H_1(M_1,\mathbb{Z})|\lambda(M_2)$,

(5′) $\lambda(M(L, \frac{p_1}{q_1}, \ldots, \frac{p_{|L|}}{q_{|L|}})) = \frac{|H_1(M(L,\frac{p_1}{q_1},\ldots,\frac{p_{|L|}}{q_{|L|}}),\mathbb{Z})|}{|H_1(M,\mathbb{Z})|}\lambda(M) + \mathcal{F}_M(L, \frac{p_1}{q_1}, \ldots, \frac{p_{|L|}}{q_{|L|}})$,

where $M(L, \frac{p_1}{q_1}, \ldots, \frac{p_{|L|}}{q_{|L|}})$ is the manifold obtained from M by performing rational surgery with indicated coefficients on the components of the link L, and $\mathcal{F}_M(L, \frac{p_1}{q_1}, \ldots, \frac{p_{|L|}}{q_{|L|}})$ is a certain function on the set of surgery presentations in M, in fact a function of the linking matrix, homology, and Alexander polynomial [18].

4.1. The vacuum-degree $N \leq 1$ truncation. of the TQFT defined in Sect. 3 is a TQFT for the Casson-Walker-Lescop invariant. Indeed, let $R = \mathcal{A}^{\leq 1}(\emptyset) = (\{r + s\theta | r, s \in$

$\mathbb{Q}\}$, $[\leq 1]$-multiplication) $\cong \mathbb{Q}[\theta]/(\theta^2) \cong \left\{ \begin{pmatrix} r & s \\ 0 & r \end{pmatrix} | r, s \in \mathbb{Q} \right\}$. Observe [17], that $c_+ = 1 - \frac{\theta}{16} + h.o.t.$, $c_- = 1 + \frac{\theta}{16} + h.o.t.$, hence $c_+ c_- =$ 1+terms of degree ≥ 2, and therefore $z_g^1 = (-1)^{g \cdot 1} vdeg_{\leq 1} \left(\frac{Z(T_g) \otimes (\nu^{1/2})^{\otimes 2g}}{c_+ c_-} \right) = (-1)^g Z(T_g) \otimes (\nu^{1/2})^{\otimes 2g}$. If $\alpha \in \mathcal{A}^{\leq 1}(\Gamma^{g_1}, \Gamma^{g_2})$ and $\beta \in \mathcal{A}^{\leq 1}(\Gamma^{g_2}, \Gamma^{g_2 3})$, then:

$$\ell_1(\alpha, \beta) = vdeg_{\leq 1} \left(\widetilde{\iota_1}(\alpha * z_{g_1}^1 * \beta) \right). \tag{4.1}$$

If $g = 0$, ℓ_1 is the disjoint union. A formula for $Z(T_1)$ and $Z(W_1)$ is given for example in [9]. Using an even associator it is easy to write down z_1^1 and w_1 explicitly in low degrees.

Let $\kappa(L, G_1, G_2) = (M, f_1, f_2)$. Then, keeping in mind that $c_+ c_- =$ 1+terms of degree ≥ 2, denote

$$c(M, f_1, f_2) := \tau^1(M, f_1, f_2) = \frac{(-1)^{\sigma_+}}{d(M)} vdeg_{\leq 1} \left(\widetilde{\iota_1}(\check{Z}(L, G_1, G_2)) \right), \tag{4.2}$$

where $d(M) = |H_1(\widehat{M}, \mathbb{Z})|$ and $\sigma_+ = sign_+(L)$, is an invariant of 3-cobordisms of category $\mathfrak{Q}$. In particular, for cobordisms between S^2 and S^2, $c(M, id_{S^2}, id_{S^2}) = \frac{(-1)^{\sigma_+}}{d(M)} vdeg^{\leq 1}(\widetilde{\iota_1}(\check{Z}(L))) = vdeg_{\leq 1} Z^{LMO}(M) = 1 + \frac{\lambda(M)}{2}\theta$, where we have identified $\mathcal{A}^{\leq 1}(\Gamma^0, \Gamma^0) \equiv R = \mathcal{A}^{\leq 1}(\emptyset)$. The filling of the composition of two 3-cobordisms between S^2 and S^2 is the connected sum of the fillings. Hence $c(M_2 \cup M_1, S^2, S^2) = c(M_1, S^2, S^2) c(M_2, S^2, S^2)$ implies property (4′) of the Casson invariant (the generalized version). By results of Sect. 3, the following axioms of TQFT hold:

$$c(M_2 \cup_{f_2 \circ (f_1')^{-1}} M_1, f_1, f_2') = \ell_1(c(M_1, f_1, f_1'), c(M_2, f_2, f_2')), \tag{4.3}$$

$$c(\Sigma_g \times [0, 1], p_1, p_2) = id_{\mathcal{A}^{\leq 1}(\Gamma^g)}, \tag{4.4}$$

$$c(-M, -f_2, -f_1) = \overline{c(M, f_1, f_2)}, \tag{4.5}$$

where the notations are obvious. R, $\mathcal{A}^{\leq 1}(\Gamma^0, \Gamma^g)$, and $\mathcal{A}^{\leq 1}(\Gamma^{g_1}, \Gamma^{g_2})$ are $\mathbb{Z}_2$-graded by the vacuum degree. In particular (4.5) implies property (3) of the CWL invariant.

Unfortunately, explicit calculations for $c(M, f_1, f_2)$, as expected, are hard to do. We show below that the induced representation $\mathcal{L}_g \to GL_R(\mathcal{A}^{\leq 1}(\Gamma^g))$ descends to Morita's homomorphism $\lambda^* : \mathcal{K}_g \to \mathbb{Z}$. ($\lambda^*$ extends to $\mathcal{L}_g$, but fails to be a homomorphism there.)

Proposition 4.2. *1) Let $\mathfrak{B}$ be the completion of the $\mathbb{Q}$-vector subspace of $\mathcal{A}(\Gamma^{g_1} \sqcup (\sqcup_m S^1) \sqcup \Gamma^{g_2})$ generated by finite sums of chord diagrams which intersect $\Gamma^{g_1} \sqcup \Gamma^{g_2}$. Then $p : \mathcal{A}(\Gamma^{g_1} \sqcup (\sqcup_m S^1) \sqcup \Gamma^{g_2}) \to \mathcal{A}(\sqcup_m S^1)$, the natural map "erase Γ^{g_1} and Γ^{g_2} from a chord diagram", if it does not intersect $\Gamma^{g_1} \sqcup \Gamma^{g_2}$, and set $= 0$, otherwise, is well-defined, and the following sequence is short exact:*

$$0 \to \mathfrak{B} \to \mathcal{A}(\Gamma^{g_1} \sqcup (\sqcup_m S^1) \sqcup \Gamma^{g_2}) \xrightarrow{p} \mathcal{A}(\sqcup_m S^1) \to 0.$$

We will denote also by p the induced maps on the vacuum degree $\leq N$ parts. They have the same property.

2) *Denote by r the maps similar to p from 1), corresponding to the case* $m = 0$. *Then the following diagram is commutative:*

$$\begin{array}{ccccc} \mathcal{A}(\Gamma^{g_1} \sqcup (\sqcup_m S^1) \sqcup \Gamma^{g_2}) & \xrightarrow{\tilde{\iota}_N} & \mathcal{A}(\Gamma^{g_1} \sqcup \Gamma^{g_2}) & \longrightarrow & \mathcal{A}^{\leq N}(\Gamma^{g_1} \sqcup \Gamma^{g_2}) \\ \downarrow p & & \downarrow r & & \downarrow r \\ \mathcal{A}(\sqcup_m S^1) & \xrightarrow{\iota_N} & \mathcal{A}(\emptyset) & \longrightarrow & \mathcal{A}^{\leq N}(\emptyset) \end{array}$$

3) *For every embedding* $L \cup G \hookrightarrow S^3$, *such that G as an abstract graph is* $\Gamma^{g_1} \sqcup \Gamma^{g_2}$, $p\check{Z}(L \cup G) = \check{Z}(L)$.
4) *For every embedding* $L \cup G \hookrightarrow S^3$, *such that G as an abstract graph is* $\Gamma^{g_1} \sqcup \Gamma^{g_2}$, *and every* $N \geq 1$, $p(\tau^N(\kappa(L \cup G))) = deg_{\leq N} Z^{LMO}(\kappa(\widehat{L \cup G}))$. *In particular (if* $N = 1$*),* $p(c(M, f_1, f_2)) = 1 + \frac{\lambda(\widehat{M})}{2}\theta$.
5) *If* $\varphi_1, \varphi_2 \in \mathcal{K}_g$, *then* $p(c(\Sigma_g \times I, \varphi_2 \circ \varphi_1, id)) = p(c(\Sigma_g \times I, \varphi_1, id))p(c(\Sigma_g \times I, \varphi_2, id))$.

Proof. 1) The following argument works for every fixed degree, and since all relations are homogeneous, we can use the universality property of the direct product as mentioned after Proposition 2.1 to obtain the result. Consider the corresponding diagram before introducing relations:

$$0 \to \mathfrak{B}' \to \mathcal{D}(\Gamma^{g_1} \sqcup (\sqcup_m S^1) \sqcup \Gamma^{g_2}) \xrightarrow{p'} \mathcal{D}(\sqcup_m S^1) \to 0.$$

The terms of any relation for diagrams on $\Gamma^{g_1} \sqcup (\sqcup_m S^1) \sqcup \Gamma^{g_2}$, either all intersect $\Gamma^{g_1} \sqcup \Gamma^{g_2}$, or none does. Hence, if we denote by R_1 the $\mathbb{Q}$-vector space generated by relations of the first type, by R_2 - the space generated by relation of the second type, and by R - the one generated by all relations, then $R/R_1 \cong R_2$. All in all we get a diagram:

$$\begin{array}{ccccccccc} 0 \to & R_1 & \longrightarrow & & R & \longrightarrow & R_2 & \longrightarrow 0 \\ & \downarrow & & & \downarrow & & \downarrow & \\ 0 \to & \mathfrak{B}' & \to & \mathcal{D}(\Gamma^{g_1} \sqcup (\sqcup_m S^1) \sqcup \Gamma^{g_2}) & & \xrightarrow{p'} & \mathcal{D}(\sqcup_m S^1) & \to 0 \\ & \downarrow & & & \downarrow & & \downarrow & \\ 0 \to & \mathfrak{B} & \xrightarrow{i} & \mathcal{A}(\Gamma^{g_1} \sqcup (\sqcup_m S^1) \sqcup \Gamma^{g_2}) & & \xrightarrow{p} & \mathcal{A}(\sqcup_m S^1) & \to 0 \end{array}$$

where all columns and the first two rows are short exact. The arrows i and p in the third row are then induced and make the diagram commutative. They clearly are the maps described in the statement. The exactness in the third row follows from the exactness in the second.

2) Let $\alpha \in \mathcal{A}(\Gamma^{g_1} \sqcup (\sqcup_m S^1) \sqcup \Gamma^{g_2})$ and β be such that $\tilde{\varphi}(\beta) = \alpha$. Recall that $\tilde{\iota_N} = \tilde{q_N} \circ \tilde{\kappa_N} \circ \tilde{\varphi}^{-1}$. A chord diagram x from the expression of β connects to $\Gamma^{g_1} \sqcup \Gamma^{g_2}$, if and only if its image via φ is a sum y of chord diagrams expressing α, all connected to $\Gamma^{g_1} \sqcup \Gamma^{g_2}$. Again, using the fact that the terms in any relation either all connect, or

all do not, $p(y) = 0$ implies (in fact $\Leftrightarrow$) that $\tilde{q_N} \circ \tilde{\kappa_N}(x)$ connects to $\Gamma^{g_1} \sqcup \Gamma^{g_2}$, i.e. $r(\tilde{q_N} \circ \tilde{\kappa_N} \circ \tilde{\varphi}^{-1}(y)) = r(\tilde{q_N} \circ \tilde{\kappa_N}(x)) = 0$.

Now, if we decompose $\beta = \beta_1 + \beta_2$ such that all terms in β_1 connect to $\Gamma^{g_1} \sqcup \Gamma^{g_2}$, and all terms in β_2 do not, the result follows: $(r \circ \tilde{\iota_N})(\alpha) = r(\varphi(\beta_1) + \varphi(\beta_2)) = r(\tilde{q_N} \circ \tilde{\kappa_N}(x_1) + \tilde{q_N} \circ \tilde{\kappa_N}(x_2)) = \tilde{q_N} \circ \tilde{\kappa_N}(x_2) = \tilde{\iota_N}(\varphi(\beta_2)) = \tilde{\iota_N}(\varphi(p(\beta))) = \tilde{\iota_N}(p(\varphi(\beta))) = (\tilde{\iota_N} \circ p)(\alpha)$.

3) Decompose $L \cup G$ into elementary pseudo-quasi-tangles. Observe that for every one, except  's and  's, possibly with multiple strands, Z either returns diagrams, which are either all in $\mathfrak{B}$, or all have no intersection between the dashed graph and $\Gamma^{g_1} \sqcup \Gamma^{g_2}$. Thus, suppressing G for these elementary tangles corresponds precisely to applying p.

The remaining cases. Observe, first, that one can "lift L above G", leaving only some "fingers" from L attached to G. To see this, from a generic plane projection of $L \cup G$ on $\mathbb{R}^2 \subset \mathbb{R}^3$ obtain an isotopic embedding of $G \cup L$ in $\mathbb{R}^3$, such that G is in an ε-neighborhood of the plane $\{z = 0\} \in \mathbb{R}^3$, and L, except for some fingers that correspond to intersections between G and L in the original plane projection, lies in an ε-neighborhood of the plane $\{z = 1\} \in \mathbb{R}^3$. Hence, by "opening the two-page book", we can find such a tangle decomposition, that all occurring associator-tangles are of one of the following three types:

(A) refer only to G or only to L;

(B) a single middle strand, which comes from L, the left-most strand (with "big" multiplicity) comes from G, the right-most strand (also with "big" multiplicity) comes from L. Moreover, if such an associator-tangle occurs, its inverse (on the same strands) will occur "soon";

(C) one of the left-most two strands is a single strand coming from L, all other strands come from G.

We will assume that he associator Φ is horizontal, i.e. it is a formal series in two non-commuting variables r_{12}, r_{23}, which correspond to a dashed line joining these indicated strands [17].

(A) If all strands are from G or none are from G, then all terms of $\Phi^{\pm 1} = Z(\text{tangle})$, connect, respectively do not connect, to $\Gamma^{g_1} \sqcup \Gamma^{g_2}$.

(C) If two of the three stands come from G, $\Phi^{\pm 1} = Z(\text{tangle})$ will have all terms connected to $\Gamma^{g_1} \sqcup \Gamma^{g_2}$. Then, eliminating G corresponds precisely to replacing this tangle-associator by the single strand from L, i.e. corresponds to applying p in this case.

(B) If exactly one (multiple) strand comes from G, this corresponds to setting one of the two non-commutative variables r_{12}, r_{23} zero. But, as mentioned, such tangles occur in pairs with their opposite. Hence, both Φ and $\Phi^{321} = \Phi^{-1}$ occur. Setting one of r_{12}, r_{23} zero in this case leaves a series, and its inverse (elementary exercise). Thus, eliminating G corresponds again to applying p.

4) Recall the definitions of τ^N (3.3) and Z^{LMO} (2.2). Apply p and use the result of part 3). Then, use the commutativity of the diagram from part 2) to obtain the desired relation.

5) Applying p to (4.3), $p(c(\Sigma_g \times I, \varphi_2 \circ \varphi_1, id)) = p(\ell_1(c(\Sigma_g \times I, \varphi_1, id), c(\Sigma_g \times I, \varphi_2, id)))$. Using part 4), $p(c(\Sigma_g \times I, \varphi_2 \circ \varphi_1, id)) = 1 + \frac{\lambda(W_{\varphi_2 \circ \varphi_1})}{2}\theta$, $p(c(\Sigma_g \times I, \varphi_1, id)) = 1 + \frac{\lambda(W_{\varphi_1})}{2}\theta$, and $p(c(\Sigma_g \times I, \varphi_2, id)) = 1 + \frac{\lambda(W_{\varphi_2})}{2}\theta$, where $W_{\varphi_i} = (\widehat{\Sigma_g \times I, \varphi_i}, id)$. But $1 + \frac{\lambda(W_{\varphi_2 \circ \varphi_1})}{2}\theta = \left(1 + \frac{\lambda(W_{\varphi_1})}{2}\theta\right)\left(1 + \frac{\lambda(W_{\varphi_2})}{2}\theta\right)$ in R, because $\lambda^* : \mathcal{K}_g \to \mathbb{Z}$, $\lambda^*(\varphi) := \lambda(W_\varphi)$ satisfies $\lambda^*(\varphi_2 \circ \varphi_1) = \lambda(\varphi_1) + \lambda(\varphi_2)$ by [20]. $\square$

Remarks. Using expression (4.1) for ℓ, and observing that p commutes with $\tilde{\iota_1}$, and with $vdeg_{\leq 1}$ by Proposition 4.2.2), we can re-write $p(\ell_1(c(\Sigma_g \times I, \varphi_1, id), c(\Sigma_g \times I, \varphi_2, id)))$ as $vdeg_{\leq 1}\tilde{\iota_1}(p(c(\Sigma_g \times I, \varphi_1, id) * z_g^1 * c(\Sigma_g \times I, \varphi_2, id)))$. Expressing $c(\Sigma_g \times I, \varphi_i, id)$ by (4.2), and keeping in mind the definition of p and properties of $\tilde{\iota_1}$, we get to having to apply p on $\left(\check{Z}(L_1, G_1, G_1') * z_g^1 * \check{Z}(L_2, G_2, G_2')\right)$, respectively to apply p on $\left(\check{Z}(L_i, G_i, G_i')\right)$, $i = 1, 2$. On the other hand, it is possible to show directly that $\tilde{\iota_1} p \left(\check{Z}(L_1, G_1, G_1') * z_g^1 * \check{Z}(L_2, G_2, G_2')\right) = p\left(\check{Z}(L_1, G_1, G_1')\right) \cdot p\left(\check{Z}(L_2, G_2, G_2')\right)$, for suitably chosen L_i in the triplets. This gives another proof of Proposition 4.2.5). Thus the fact that $\lambda^* : \mathcal{K}_g \to \mathbb{Z}$ is a homomorphism follows from the Kontsevich integral.

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