

# AN INVARIANT OF INTEGRAL HOMOLOGY 3-SPHERES WHICH IS UNIVERSAL FOR ALL FINITE TYPE INVARIANTS

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ABSTRACT. In [LMO] a 3-manifold invariant  $\Omega(M)$  is constructed using a modification of the Kontsevich integral and the Kirby calculus. The invariant  $\Omega$  takes values in a graded Hopf algebra of Feynman 3-valent graphs. Here we show that for homology 3-spheres the invariant  $\Omega$  is *universal* for all finite type invariants, i.e.  $\Omega_n$  is an invariant of order  $3n$  which dominates all other invariants of the same order. This shows that the set of finite type invariants of homology 3-spheres is equivalent to the Hopf algebra of Feynman 3-valent graphs. Some corollaries are discussed. A theory of groups of homology 3-spheres, similar to Gusarov's theory for knots, is presented.

## 1. INTRODUCTION

In a previous work [LMO] (joint with J. Murakami and T. Ohtsuki) we defined a 3-manifold invariant  $\Omega$  with values in, as expected in perturbative Chern-Simon theory (see [AS1, AS2, Ko2, Roz, Wit]), the graded Hopf algebra  $\mathcal{D}$  of vertex-oriented 3-valent graphs. The construction uses a modification of the Kontsevich integral and the Kirby calculus. Actually,  $\Omega$  corresponds only to the trivial connection in the perturbation theory. One can expect that  $\Omega$  plays an important role in the set of homology 3-sphere invariants.

The first non-trivial degree part  $\Omega_1$  is essentially the Casson invariant; and  $\Omega$  can be regarded as a far generalization of the Casson invariant. The degree  $n$  part of  $\Omega$  is constructed using finite type invariants of links of order  $\leq (l+1)n$ , where  $l$  is the number of components of the link.

Finite type invariants for *knots and links* were introduced by Vassiliev, see [Vas, BL], and proved to be very useful in knot theory. Kontsevich [Ko1] introduced a knot invariant, called the *Kontsevich integral*, with values in a graded algebra of chord diagrams. The grading  $n$  part of the Kontsevich integral is a finite type invariant of order  $n$ ; and it is universal, since it dominates all other invariants of the same order. All quantum invariants, including the Jones polynomial, are special values of the Kontsevich integral, obtained using weight systems coming from semi-simple Lie algebras (see Theorem 10 of [LM2] and Theorem XX.8.3 of [Kas]).

Using  $sl_2$  quantum invariants at roots of unity, Reshetikhin and Turaev [RT] constructed quantum invariants of 3-manifolds. There are generalizations for other quantum groups. The algebraic part of the construction is the complicated theory of *modular* Hopf algebra and quantum groups *at roots of unity*, see [Tur].

Since quantum invariants of links are special values of the Kontsevich integral, there arose a question of constructing 3-manifold invariants from the Kontsevich integral. In [LMO], we succeeded in constructing such an invariant  $\Omega$ . Actually we used a modification of the Kontsevich integral, called the universal Kontsevich-Vassiliev invariant for *framed* links and tangles (see [LM1, LM2]).

Invariants of finite type for *homology 3-spheres* were introduced by Ohtsuki [Oh1], in analogy with the knot case. The theory was developed further in [GL1, GL2, GO1, GO2, Hab, Lin] and others. In particular, it was proved that orders of finite type invariants are multiples of 3 (see [GL1, GO1]), and that the space of invariants of a fixed order is finite-dimensional (see [Oh1]).

Both finite type invariants for links and for homology 3-spheres can be considered as a way to organize the set of invariants in a systematical way. While a lot has been known about the knot case, much less is for the manifold case.

The aim of this paper is to show that for homology 3-spheres,  $\Omega$  is as powerful as the set of all finite type invariants; i.e.  $\Omega$  is a counterpart of the Kontsevich integral in knot theory. We show that the degree  $n$  part of  $\Omega$  is an invariant of degree  $3n$  which dominates all other invariants of degree  $3n$ . This implies that the set of finite type invariants of homology 3-spheres is equivalent to the (purely combinatorial) Hopf algebra  $\mathcal{D}$  of Feynman 3-valent diagrams. For example, the number of linearly independent invariants of degree  $\leq n$  is equal the number of linearly independent 3-valent diagrams of degree  $\leq n$ . The study of finite type invariants of homology 3-spheres is reduced to the study of  $\Omega$ .

The proof of the main theorem is much more complicated than that of the similar theorem in the knot case. We have to use many results of the theory of the (modified) Kontsevich integral. We show that every weight system, defined in [GO1], can be integrated to a finite type invariant, like in the knot case (and hence give an affirmative answer to Question 1 there).

We also show that for every  $n$ , there is a homology 3-sphere  $M$  which cannot be distinguished from  $S^3$  by any invariant of order  $< 3n$ , but can be distinguished from  $S^3$  by an invariant of order  $3n$ . An operation on homology 3-spheres which does not alter values of invariants of order less than a fixed number is presented. A theory of *groups of homology 3-spheres*, similar to Gusarov's groups of knots, is sketched. Some other corollaries of the main theorem are presented.

The paper is organized as follows. In Sections 2 we recall the universal Kontsevich-Vassiliev invariant for *framed oriented links and tangles*, considered in [LM1, LM2]. We also show some properties of this invariant, needed later. In Section 3 we give a definition of  $\Omega$ , and recall some fundamental concepts in the theory of finite type invariants of homology 3-spheres. In Section 4 we present the main results and discuss some corollaries and related problems. In Section 5 we prove the lemmas needed in the proof of the main theorem.

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## 2. CHORD DIAGRAMS AND FINITE TYPE INVARIANTS FOR FRAMED TANGLES

**2.1. Chord diagrams.** Note that our definition of chord diagrams is more general than that of [BN1, LM1, LM2]. All vector spaces are over the field  $\mathbb{Q}$  of rational numbers.

A *uni-trivalent graph* is a graph every vertex of which is either univalent or trivalent. A uni-trivalent graph is *vertex-oriented* if at each trivalent vertex a cyclic order of edges is fixed. A 3-valent (resp. 1-valent) vertex is called an *internal* (*external*) vertex.

Let  $X$  be a compact oriented 1-dimensional manifold whose components are *numbered*. A *chord diagram* with support  $X$  is the manifold  $X$  together with a vertex-oriented

uni-trivalent graph whose external vertices are on  $X$ ; and the graph does not have any connected component homeomorphic to a circle.

In figures components of  $X$  are depicted by solid lines, while the graph is depicted by dashed lines, with the convention that the orientation at every vertex is counterclockwise. Each chord diagram has a natural topology. Two chord diagrams  $\xi, \xi'$  on  $X$  are regarded as equal if there is a homeomorphism  $f : \xi \rightarrow \xi'$  such that  $f|_X$  is a homeomorphism of  $X$  which preserves components and orientation and the restriction of  $f$  to the dashed graph is a homeomorphism preserving orientation at every vertex.

There may be connected components of the dashed graph which do not have univalent vertices, and hence do not connect to any solid lines.

Let  $\mathcal{A}(X)$  be the vector space spanned by chord diagrams with support  $X$ , subject to the AS, IHX and STU relations. The AS (*anti-symmetry*) condition says that  $\xi_1 + \xi_2 = 0$  for any two chord diagrams which are identical except for the orientation at a vertex. The IHX (or Jacobi) relation says that  $\xi_1 = \xi_2 + \xi_3$ , for every three chord diagrams identical outside a ball in which they differ as in Figure 1.

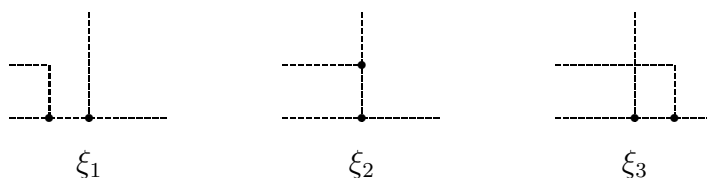


FIGURE 1.

The STU relation is depicted in Figures 2.

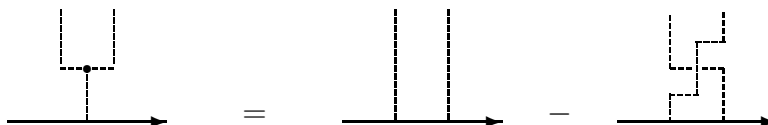


FIGURE 2. The STU relation

The *degree* of a chord diagram is half the number of (external and internal) vertices of the *dashed graph*. Since the relations AS, IHX and STU respect the degree, there is a grading on  $\mathcal{A}(X)$  induced by this degree. We also use  $\mathcal{A}(X)$  to denote the completion of  $\mathcal{A}(X)$  with respect to the grading.

Let  $\mathcal{D}_n$  be the vector space spanned by all vertex-oriented 3-valent (without 1-valent vertex) graphs of degree  $n$  subject to the AS and IHX relations. Let  $\mathcal{D}_0$  be the ground field  $\mathbb{Q}$ , and let  $\mathcal{D}$  be the direct product of all  $\mathcal{D}_n, n = 0, 1, 2, \dots$ . This vector space  $\mathcal{D}$  is a commutative algebra in which the product of 2 graphs is just the union of them. All  $\mathcal{A}(X)$  can be regarded as graded  $\mathcal{D}$ -modules, where the product of a 3-valent graph  $\xi \in \mathcal{D}$  and a chord diagram  $\xi' \in \mathcal{A}(X)$  is their disjoint union. In [LMO],  $\mathcal{D}$  is denoted by  $\mathcal{A}(\emptyset)$ .

Suppose  $C$  is a component of  $X$ . Reversing the orientation of  $C$ , from  $X$  we get  $X'$ . Let  $S_{(C)} : \mathcal{A}(X) \rightarrow \mathcal{A}(X')$  be the linear mapping which transfers every chord diagram  $\xi$  in  $\mathcal{A}(X)$  to  $S_{(C)}(\xi)$  obtained from  $\xi$  by reversing the orientation of  $C$  and multiplying by  $(-1)^m$ , where  $m$  is the number of vertices of the dashed graph on the component  $C$ .

We also define the linear operator  $\Delta_{(C)}$  which is described in Figure 3. The definition is explained as follows.

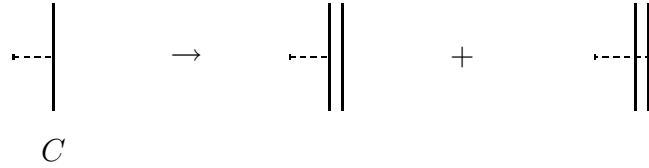


FIGURE 3.

Replacing  $C$  by 2 copies of  $C$ , from  $X$  we get  $X^{(2,C)}$ , with a projection  $p : X^{(2,C)} \rightarrow X$ . If  $x$  is a point on  $C$  then  $p^{-1}(x)$  consists of 2 points, while if  $x$  is a point of other components, then  $p^{-1}(x)$  consists of one point. Let  $\xi$  be a chord diagram on  $X$ , with the dashed graph  $G$ . Suppose that there are  $m$  univalent vertices of  $G$  on  $C$ . Consider all possible new chord diagrams on  $X^{(2,C)}$  with the same dashed graph  $G$  such that if a univalent vertex of  $G$  is attached to a point  $x$  on  $X$  in  $\xi$ , then this vertex is attached to a point in  $p^{-1}(x)$  in the new chord diagram. There are  $2^m$  such chord diagrams, and their sum is denoted by  $\Delta_{(C)}(\xi)$ . It is easy to check that these linear mappings  $S_{(C)}$  and  $\Delta_{(C)}$  are well-defined.

Suppose that  $X$  has a distinguished component  $C$ . Let  $\xi \in \mathcal{A}(X)$  and  $\xi' \in \mathcal{A}(S^1)$  be two chord diagrams. From each of  $C$  and  $S^1$  we remove a small arc which does not contain any vertices. The remaining part of  $S^1$  is an arc which we glue to  $C$  in the place of the removed arc so that the orientations are compatible. The new chord diagram is called the *connected sum of  $\xi$  and  $\xi'$  along  $C$* ; it does not depend on the locations of the removed arcs. The proof is the same as in the case  $X = S^1$ , considered in [BN1].

When  $X = S^1$ , the connected sum defines an algebra structure on  $\mathcal{A}(S^1)$  which is known to be commutative (see [BN1]).

We now define a co-multiplication  $\hat{\Delta}$  in  $\mathcal{A}(X)$ . A *chord sub-diagram* of a chord diagram  $\xi$  with dashed graph  $G$  is any chord diagram obtained from  $\xi$  by removing some connected components of  $G$ . The *complement chord sub-diagram* of a chord sub-diagram  $\xi'$  is the chord sub-diagram obtained by removing components of  $G$  which are in  $\xi'$ . Let

$$\hat{\Delta}(\xi) = \sum \xi' \otimes \xi''.$$

Here the sum is over all chord sub-diagrams  $\xi'$  of  $\xi$ , and  $\xi''$  is the complement of  $\xi'$ . This co-multiplication is co-commutative.

A similar co-product is defined for  $\mathcal{D}$ , by the same formula, and  $\mathcal{D}$  becomes a commutative co-commutative Hopf algebra. Hence  $\mathcal{D}$  is the polynomial algebra on *primitive* elements, i.e. elements  $x$  such that  $\hat{\Delta}(x) = 1 \otimes x + x \otimes 1$ . It is easy to see that an element is primitive if and only if it is a linear combination of *connected* 3-valent graphs.

**2.2. A universal invariant for framed tangles: the modified Kontsevich integral.** We fix an oriented 3-dimensional Euclidean space  $\mathbb{R}^3$  with coordinates  $(x, y, t)$ . A *tangle* is a smooth one-dimensional compact oriented manifold  $L \subset \mathbb{R}^3$  lying between two horizontal planes  $\{t = a\}, \{t = b\}$ ,  $a < b$  such that all the boundary points are lying on two lines  $\{t = a, y = 0\}, \{t = b, y = 0\}$ , and at every boundary point  $L$  is orthogonal to these two planes. These lines are called the top and the bottom lines of the tangle.

A *normal vector field* on a tangle  $L$  is a smooth vector field on  $L$  which is nowhere tangent to  $L$  (and, in particular, is nowhere zero) and which is given by the vector  $(0, -1, 0)$  at every boundary point. A *framed tangle* is a tangle enhanced with a normal vector field. Two framed tangles are isotopic if they can be deformed by a 1-parameter family of diffeomorphisms into one another within the class of framed tangles.

Framed oriented links are special framed tangles when there is no boundary point. The empty link, or empty tangle, by definition, is the empty set.

One assigns a symbol  $+$  or  $-$  to all the boundary points of a tangle according to whether the tangent vector at this point directs downwards or upwards. Then on the top boundary line of a tangle diagram we have a word  $w_t$  of symbols consisting of  $+$  and  $-$ . Similarly on the bottom boundary line there is also a word  $w_b$  of symbols  $+$  and  $-$ .

A non-associative word on  $+, -$  is an element of the *free magma* generated by  $+, -$  (see [Ser]). A *q-tangle* (or *non-associative tangle*) is a tangle whose top and bottom words  $w_t, w_b$  are equipped with some non-associative structure.

Every tangle in this paper will also be regarded as a *q-tangle*, with *standard* non-associative structure on  $w_t, w_b$ : product is taking successively from left to right.

If  $T_1, T_2$  are tangles such that  $w_b(T_1) = w_t(T_2)$  we can define the product  $T = T_1 T_2$  by placing  $T_1$  on top of  $T_2$ . In this case, if  $\xi_1 \in \mathcal{A}(T_1), \xi_2 \in \mathcal{A}(T_2)$  are chord diagrams, then the *product*  $\xi_1 \xi_2$  is a chord diagram in  $\mathcal{A}(T)$  obtained by placing  $\xi_1$  on top of  $\xi_2$ .

For any two tangles  $T_1, T_2$  with the same top and bottom lines, we can define their *tensor product*  $T_1 \otimes T_2$  by putting  $T_2$  to the right of  $T_1$ . Similarly, if  $\xi_1 \in \mathcal{A}(T_1), \xi_2 \in \mathcal{A}(T_2)$  are chord diagrams, then one defines  $\xi_1 \otimes \xi_2 \in \mathcal{A}(T_1 \otimes T_2)$  by the same way.

For example, if  $L_1, L_2$  are two links, then  $L_1 L_2 = L_1 \otimes L_2 = L_1 \sqcup L_2$ , the disjoint union of  $L_1$  and  $L_2$ .

Let  $T$  be a framed tangle, then there exists an element  $\hat{Z}(T) \in \mathcal{A}(T)$  called the *universal Kontsevich-Vassiliev invariant* of  $T$ . This invariant satisfies

$$\text{if } T = T_1 T_2 \text{ then } \hat{Z}(T) = \hat{Z}(T_1) \hat{Z}(T_2). \quad (2.1)$$

The definition of this invariant  $\hat{Z}$  is given in, say, [LM1, LM2] (see also [BN2]); in [LM1, LM2]  $\hat{Z}$  is denoted by  $\hat{Z}_f$ . It is a regularization of the original Kontsevich integral, regularized so that it can be defined for *framed links*. For framed links,  $\hat{Z}$  is equivalent to the set of all finite type invariants.

Note that the definition of  $\hat{Z}$  depends on an element called *associator*. For the theory of associators in our sense, see, for example, [LM2]. For links the values of  $\hat{Z}$  do not depend on what associator to choose (see Theorems 7,8 in [LM2]), but the values of  $\hat{Z}$  of a framed tangle with non-empty boundary do depend on the associator. For technical convenience, we will use the normalization of  $\hat{Z}$  as in [LM3] and the associator in [LM3], since this associator has rational coefficients and admits a lot a symmetry (more symmetry than the associator coming from the Knizhnik-Zamolodchikov equations). Note that the normalization of  $\hat{Z}$  is chosen so that  $\hat{Z}$  of an empty link is 1.

Let  $\mathcal{P}_n$  be the space of chord diagrams with support being  $n$  numbered straight vertical lines, pointing downwards. There is a structure of a (non-commutative, if  $n > 1$ ,) algebra on  $\mathcal{P}_n$ , where the product of two chord diagrams is obtained by placing the first on top of the second. Together with the co-multiplication  $\hat{\Delta}$ , we get a co-commutative Hopf algebra  $\mathcal{P}_n$ . It is known that the subspace of primitive elements in  $\mathcal{P}_n$  is spanned by chord diagrams with *connected dashed graph*.

In what follows, element  $1 \in \mathcal{A}(X)$  is always the chord diagram without dashed graph.

**Definition .** We say that an element  $\xi \in \mathcal{A}(X)$  has *i-filter*  $\geq n$  if  $\xi$  is a linear combination of chord diagrams, each has at least  $n$  internal vertices.

If  $\xi$  is a chord diagram with *connected* dashed graph and of degree  $n$ , then it is easy to see that  $\xi$  has at least  $(n - 1)$  internal vertices. Hence we have the following

**Proposition 2.1.** *If  $\xi \in \mathcal{P}_n$  is primitive and is a linear combination of chord diagrams of degree  $\geq n$ , then  $\xi$  has i-filter  $\geq (n - 1)$ .*

It is known that the associator is an element in  $\mathcal{P}_3$  of the form  $\exp(\xi)$ , where  $\xi$  is *primitive* and is a linear combination of chord diagrams of degree  $\geq 2$ . It follows that

**Lemma 2.2.** *The associator has the form  $1 + (\text{elements of i-filter} \geq 1)$ .*

Unlike invariants of tangles coming from co-associative quantum groups, in general, if  $T = T_1 \otimes T_2$  then  $\hat{Z}(T) \neq \hat{Z}(T_1) \otimes \hat{Z}(T_2)$ . But one has

**Lemma 2.3.** *For some elements  $a, b \in \mathcal{A}(T_1 \otimes T_2)$  of the form  $1 + (\text{elements of i-filter} \geq 1)$ , one has*

$$\hat{Z}(T_1 \otimes T_2) = a[\hat{Z}(T_1) \otimes \hat{Z}(T_2)]b.$$

Moreover, if  $T_2$  is a trivial tangle, then  $a = b = 1$ .

*Proof.* The general method of computing  $\hat{Z}$  in [LM2] says that

$$\hat{Z}(T_1 \otimes T_2) = a[\hat{Z}(T_1) \otimes \hat{Z}(T_2)]b,$$

where  $a, b$  are derived from the associator by some operators; and these operators do not decrease i-filters.  $\square$

For a framed tangle  $T$  and its component  $C$ , let  $T^{(2,C)}$  be obtained from  $T$  by adding a string  $C'$  parallel to  $C$  with respect to the framing.

**Theorem 2.4.** [LM3] *Let  $C$  be a component of a framed tangle  $T$ .*

a) *One has*

$$\hat{Z}(T^{(2,C)}) = \Delta_{(C)}(\hat{Z}(T)) + (\text{elements of i-filter} \geq 1).$$

b) *Let  $T'$  be obtained from  $T$  by reversing the orientation of  $C$ , then*

$$\hat{Z}(T') = S_{(C)}(\hat{Z}(T)).$$

Actually, Theorem 4.2 of [LM3] (the proof of which requires the special associator mentioned above) says that we have the identity in a), without any extra terms of elements of i-filter  $\geq 1$ , but for another non-associative structure of  $T^{(2,C)}$ . Here we have the standard order; and the values of  $\hat{Z}$  of two tangles differed by non-associative structure differ by an element of i-filter  $\geq 1$ , by the same reason as in the proof of Lemma 2.3. Part b) follows easily from the definition of  $\hat{Z}$ , see Theorem 4 of [LM2].

### 3. THE UNIVERSAL INVARIANT $\Omega(M)$ ; FINITE TYPE INVARIANTS

For a graded space  $A$ , we will denote by  $\text{Grad}_n(A)$  the subspace of grading  $n$ ; by  $\text{Grad}_{\leq n}(A)$  the subspace of grading  $\leq n$ . For an element  $x \in A$ , we will denote by  $\text{Grad}_n(x)$  and  $\text{Grad}_{\leq n}(x)$ , respectively, the projection of  $x$  on  $\text{Grad}_n(A)$  and  $\text{Grad}_{\leq n}(A)$ .

3.1. **The mapping  $\iota_n$ .** The mappings  $\iota_n$  plays an important role in the theory; it helps to convert chord diagrams with supports to chord diagrams without support.

We denote by  $\mathring{\mathcal{A}}(X)$  the space of chord diagrams with support  $X$ , subject to the AS, IHX and STU relations, as in  $\mathcal{A}(X)$ ; but *the dashed graph may contain some finite number of dashed components which are loops*. Certainly  $\mathcal{A}(X)$  is a subspace of  $\mathring{\mathcal{A}}(X)$ , and both are graded  $\mathcal{D}$ -modules.

Let  $(L < 2n)$  be the equivalence relation in  $\mathring{\mathcal{A}}(X)$  generated by: a chord diagram with less than  $2n$  external vertices on some solid component is equivalent to 0.

Suppose on the boundary of a ball  $B$  there are  $2n$  distinct points. Connect every point of these  $2n$  points with exactly one other point by a dashed line; there will be  $(2n - 1)!!$  ways to do that, each produces a 1-valent dashed graph in  $B$ . The formal sum of these  $(2n - 1)!!$  1-valent graphs is denoted by  $T_{2n}^n$ . Let  $R_n$  be the equivalence relation  $T_{2n}^n = 0$ ; i.e. if we have  $(2n - 1)!!$  chord diagrams identical everywhere, except for a ball  $B$  in which they are exactly terms of  $T_{2n}^n$ , then the sum of these chord diagrams is 0 (in [LMO],  $R_n$  is denoted by  $P_n$ ).

Let  $O_n$  be the equivalence relation in  $\mathring{\mathcal{A}}(X)$  generated by: if  $\xi$  is a chord diagram which is a union of  $\xi'$  and a dashed component which is just a loop, then  $\xi = -2n \xi'$ . With this relation one can remove all loop dashed components.

**Proposition 3.1.** *The quotient space of  $\text{Grad}_{\leq n(l+1)}[\mathring{\mathcal{A}}(\sqcup^l S^1)]$ , where  $\sqcup^l S^1$  is the union of  $l$  solid loops, by relations  $(L < 2n)$ ,  $R_{n+1}$  and  $O_n$  is the free  $\text{Grad}_{\leq n}(\mathcal{D})$ -module generated by  $(x_n)^{\otimes l}$ , where  $(x_n)^{\otimes l}$  is the chord diagram which is the disjoint union of  $l$  solid loops, each has  $n$  isolated chords.*

Here an isolated chord is a dashed line (without any 3-valent vertex on it) connecting two neighboring external vertices on a solid line or loop. This proposition was proved in [LMO] (see Lemma 3.1 there).

Suppose  $\xi$  is a chord diagram in  $\mathring{\mathcal{A}}(\sqcup^l S^1)$ . Projecting  $\xi$  to  $\text{Grad}_{\leq n(l+1)}[\mathring{\mathcal{A}}(\sqcup^l S^1)] / (L < 2n)(R_{n+1})(O_n)$ , we get an element  $\tilde{\iota}_n(\xi) \times (x_n)^{\otimes l}$ . Here  $\tilde{\iota}_n(\xi) \in \text{Grad}_{\leq n}(\mathcal{D})$ . Let

$$\iota_n(\xi) = [(-2)^n n!]^l \tilde{\iota}_n(\xi) \in \text{Grad}_{\leq n}(\mathcal{D}).$$

Note that  $\iota_n$  decreases the degree of chord diagrams by  $ln$ , where  $l$  is the number of solid loops. Another, more geometric, definition of  $\iota_n$  is given in §3.3.

3.2. **Definition of the universal invariant  $\Omega$ ; some properties.** Suppose  $M$  is an oriented closed 3-manifold obtained from  $S^3$  by surgery on a framed  $l$ -component unoriented link  $L$ . Providing  $L$  with an arbitrary orientation, we can define  $\hat{Z}(L)$ . Let

$$\check{Z}(L) = (\nu^{\otimes l}) \# \hat{Z}(L).$$

This means,  $\check{Z}(L)$  is obtained from  $\hat{Z}(L)$  by successively taking connected sum of  $\hat{Z}(L)$  with  $\nu$  along each component of  $L$ . Here  $\nu \in \mathcal{A}(S^1)$  is  $\hat{Z}$  of the unknot with framing 0.

We will construct an invariant  $\Omega(M) \in \mathcal{D}$ , using  $\check{Z}(L)$ . The degree  $n$  part  $\text{Grad}_n(\Omega)$  is constructed using  $\text{Grad}_{\leq (l+1)n}[\hat{Z}(L)]$ .

Let  $U_+$  (resp.  $U_-$ ) be the unknot with framing  $+1$  (resp.  $-1$ ). Suppose the linking matrix of  $L$  has  $\sigma_+$  positive eigenvalues and  $\sigma_-$  negative eigenvalues. Define ([LMO])

$$\Omega_n(L) = \frac{\iota_n(\check{Z}(L))}{(\iota_n(\check{Z}(U_+)))^{-\sigma_+}(\iota_n(\check{Z}(U_-)))^{-\sigma_-}} \in \text{Grad}_{\leq n}(\mathcal{D}). \quad (3.1)$$

In [LMO] we proved that  $\iota_n(\check{Z}(U_{\pm}))$  are of the form  $(\mp 1)^n + (\text{elements of degree } \geq 1)$ , hence their inverses exist.

**Theorem 3.2.** ([LMO])  $\Omega_n(L)$  does not depend on the orientation of  $L$  and does not change under the Kirby moves. Hence  $\Omega_n(L)$  is an invariant of the 3-manifold  $M$ .

Recall that  $\mathcal{D}$  is a commutative co-commutative Hopf algebra. Let

$$\Omega(M) = 1 + \text{Grad}_1(\Omega_1(M)) + \cdots + \text{Grad}_n(\Omega_n(M)) + \cdots \in \mathcal{D}.$$

**Proposition 3.3.** ([LMO])  $\Omega(M)$  is a group-like element, i.e.

$$\hat{\Delta}(\Omega(M)) = \Omega(M) \otimes \Omega(M).$$

Hence  $\ln(\Omega(M))$  is a linear combination of connected 3-valent vertex-oriented graphs.

In general,  $\Omega_n(M)$  is not equal to  $\text{Grad}_{\leq n}(\Omega(M))$ . Let  $d(M)$  be the cardinality of  $H_1(M, \mathbb{Z})$  if the first Betti number of  $M$  is 0, otherwise let  $d(M) = 0$ .

**Proposition 3.4.** ([LMO]) We have that  $\text{Grad}_{\leq n}\Omega_{n+1}(M) = d(M)\Omega_n(M)$ . Hence if  $M$  is an integral homology 3-sphere, then

$$\Omega_n(M) = \text{Grad}_{\leq n}(\Omega(M)).$$

It was proved in [LMMO] that  $\Omega_1$  is, in essential, the Casson invariant.

**Proposition 3.5.** ([LMO]) If  $M_1, M_2$  are integral homology 3-spheres, then

$$\Omega(M_1 \# M_2) = \Omega(M_1) \times \Omega(M_2).$$

In general, if  $M_1, M_2$  are rational homology 3-spheres, then Proposition 3.5 does not hold true. However, if we modify  $\Omega$ :

$$\Omega'(M) = 1 + \frac{\text{Grad}_1(\Omega_1(M))}{d(M)} + \cdots + \frac{\text{Grad}_n(\Omega_n(M))}{d(M)^n} + \cdots$$

Then  $\Omega'(M_1 \# M_2) = \Omega'(M_1) \times \Omega'(M_2)$ , and  $\hat{\Delta}(\Omega'(M)) = \Omega'(M) \otimes \Omega'(M)$ .

For every Lie algebra  $\mathfrak{g}$  with a non-degenerate invariant bilinear form (say, a semi-simple Lie algebra), one can define a weight system (see, for example, [BN1, Ko1]) which transforms  $\Omega(M)$  into a formal power series on a variable  $h$ . The relation of this formal power series with quantum invariants coming from quantum groups at roots of unity is yet to discover.



**3.3. Another definition of  $\iota_n$ .** In [LMO], we also use another definition of  $\iota_n$  which is more geometric. Actually, we first came to this form of the definition. The aim of  $\iota_n$  is to remove solid loops and replace them by appropriate dashed graphs.

Let  $\mathcal{A}(m)$ , for any positive integer  $m$ , be the vector space spanned by uni-trivalent vertex-oriented graphs with exactly  $m$  1-valent vertices located at  $m$  numbered points, subject to the AS and IHX relations. The vector space  $\mathcal{A}(m)$  is a  $\mathcal{D}$ -module.

We would like to replace a solid loop with  $m$  external vertices on it by an element  $\eta_m \in \mathcal{A}(m)$ , i.e. we remove the solid loop, then glue the external vertices of  $\eta_m$  to the 1-valent vertices of the removed solid loop. Then the sequence  $\eta_m$  must satisfy

The diagram shows three circular components. The first component is a circle labeled  $\eta_{m-1}$  with a single vertical dashed line extending upwards from its top vertex. The second component is a circle labeled  $\eta_m$  with two vertical dashed lines extending upwards from its top two vertices. The third component is a circle labeled  $\eta_m$  with a more complex dashed graph structure extending upwards from its top two vertices, consisting of a vertical line that branches into two horizontal lines, which then meet at a central point. The equation is  $\eta_{m-1} = \eta_m - \text{(third component)}$ .

which follows from the STU relation (we do not draw the other  $m - 2$  external vertices of  $\eta_m, \eta_{m-1}$ ). In addition,

$$\eta_m \text{ is invariant under cyclic permutation of the external vertices.} \tag{p2}$$

But it seems impossible to have a sequence  $\eta_m \in \mathcal{A}(m), m = 1, 2, \dots$ , with the help of which one can directly transform  $\hat{Z}$  into something which is invariant under the Kirby moves. What we did in [LMO] is, motivated by low degree cases and other thoughts, to construct the transformation from  $\mathcal{A}(\sqcup^l S^1)$  to  $\mathcal{D}$  step by step, each is for chord diagrams up to some degree.

So we consider, for each fixed positive integer  $n$ , a sequence  $T_m^n \in \mathcal{A}(m), m = 1, 2, \dots$  with properties: (p1), (p2) (in which  $\eta_m$  is replaced by  $T_m^n$ ), and

$$T_m^n = 0 \quad \text{if } m < 2n. \tag{p3}$$

Certainly if the sequence  $T_m^n, m = 1, 2, \dots$  satisfies these conditions, then the sequence  $aT_m^n$  also satisfies these conditions, for any  $a \in \mathcal{D}$ .

It can be proved that up to this kind of ambiguity, there exists a unique sequence  $T_m^n \in \mathcal{A}(m)/(R_{n+1})$ , satisfying the above conditions. For the construction of  $T_m^n$ , see [LMO]. Actually, we will need only the element  $T_{2n}^n$ , which had been described in §3.1. The element  $T_m^n \in \mathcal{A}(m)$  is a linear combination of uni-tri-valent graphs with  $m$  external vertices and  $m - 2n$  internal vertices. Note that  $T_{2n}^n$  is invariant under any permutation of the  $2n$  external vertices.

Now the mapping  $\iota_n$  is defined as follows. For  $\xi \in \mathcal{A}(\sqcup^l S^1)$ , a *chord diagram* with  $m_i$  external vertices on the  $i$ -th solid component, let us replace the  $i$ -th solid loop by  $T_{m_i}^n$ , then we get a linear combination 3-valent graphs which may contains some loop components. Using relation  $O_n$  (which says that a loop component is equal to  $-2n$ ), and deleting the part of degree  $> n$ , we obtain  $\iota_n(\xi)$  which is in  $\text{Grad}_{\leq n} \mathcal{D}$ . From this definition we get

**Proposition 3.6.** *Suppose  $\xi$  is a chord diagram with  $k$  internal vertices, then  $\iota_n(\xi)$  has at least  $k$  vertices, hence has degree  $\geq k/2$ .*

**3.4. Finite type invariants for homology 3-spheres.** We briefly recall here some fundamental concepts, referring the reader to [Oh1, GO1, GL1].

Let  $\mathcal{M}$  be the vector space over  $\mathbb{Q}$  spanned by the set of all integral homology oriented 3-spheres. Every invariant  $I$  of homology 3-spheres with values in a *vector space* can be uniquely extended to a linear mapping on  $\mathcal{M}$ .

For a framed link  $L$  in  $M$ , let  $M_L$  denote the 3-manifold obtained by performing Dehn surgery on  $L$ . If  $K$  is a formal linear combination  $\sum_i c_i L_i$ , where  $L_i$  are framed links and  $c_i$  are numbers, then  $M_K$ , by definition, is  $\sum_i c_i M_{L_i} \in \mathcal{M}$ .

When  $M$  is an oriented homology 3-sphere, there is a natural way to identify the set of framings of a knot in  $M$  with the set  $\mathbb{Z}$  of integers; and we will use this identification.

A framed link  $L$  in  $M$  is *unit-framed* if the framing of each component is  $\pm 1$ ;  $L$  is *algebraically split* if the linking number of every two components is 0.

Let  $|L|$  be the number of components of  $L$ . Define

$$\delta(L) = \sum_{L' \subset L} (-1)^{|L'|} L', \quad (3.2)$$

where the sum is over all sublinks  $L' \subset L$ , including the empty link.

Consider the following decreasing filtration in  $\mathcal{M}$ . Let  $\mathcal{F}_n(\mathcal{M})$  be the vector space generated by  $M_{\delta(L)}$ , where  $M$  is an arbitrary homology 3-sphere and  $L$  is a unit-framed and algebraically split  $n$ -component link in  $M$ .

**Definition .** ([Oh1]) *An invariant  $I$  of integral homology 3-spheres with values in a vector space is of order  $\leq n$  if  $I(\mathcal{F}_{n+1}(\mathcal{M})) = 0$ .*

Ohtsuki showed that  $\mathcal{M}/\mathcal{F}_n(\mathcal{M})$  has finite dimension, see [Oh1]; this means the space of invariants of order less than  $n$  is finite-dimensional.

An important result (see [GL1, GO1]) is that  $\mathcal{F}_{3n}(\mathcal{M}) = \mathcal{F}_{3n+1}(\mathcal{M}) = \mathcal{F}_{3n+2}(\mathcal{M})$ . Hence the set of invariants of order  $3n$  is the same as the set of invariants of order  $3n+2$ .

Every invariant  $I$  of order  $3n$  with values in  $\mathbb{Q}$  defines a linear form on  $\mathcal{M}/\mathcal{F}_{3n+1}(\mathcal{M})$ , hence restricts to a linear form on  $\mathcal{F}_{3n}(\mathcal{M})/\mathcal{F}_{3n+1}(\mathcal{M})$ . As in knot theory, it's important to understand the structure of  $\mathcal{F}_{3n}(\mathcal{M})/\mathcal{F}_{3n+1}(\mathcal{M})$ .

In [GO1], a *surjective* linear mapping  $O_{3n}^* : \text{Grad}_n(\mathcal{D}) \rightarrow \mathcal{F}_{3n}/\mathcal{F}_{3n+1}(\mathcal{M})$  was constructed. For a definition of  $O_{3n}^*$ , see §5.1 below. It follows that if  $I$  is an invariant of order  $3n$ , then by combining with  $O_{3n}^*$ ,  $I$  defines a linear mapping from  $\text{Grad}_n(\mathcal{D})$  to  $\mathbb{Q}$ , called the *corresponding linear form* of  $I$ . Note that if the corresponding linear form is 0, then the invariant is of order  $< 3n$ .

Is every linear mapping from  $\text{Grad}_n(\mathcal{D})$  to  $\mathbb{Q}$  the linear form of some invariant of order  $\leq 3n$ ? (Question 1 in [GO1]). We will see that the answer is positive. This is equivalent to the fact that the above mapping  $O_{3n}^*$  is an isomorphism of vector spaces.

## 4. MAIN RESULTS

**4.1. Universality of  $\Omega$ .** We will show the following

**Theorem 4.1.** (Main Theorem)

- a) For homology 3-spheres,  $\Omega_n$  is an invariant of order  $\leq 3n$ .
- b) The mapping  $O_{3n}^*$  is an isomorphism, i.e. every linear form from  $\text{Grad}_n(\mathcal{D})$  to  $\mathbb{Q}$  is a linear form of some finite invariant of order  $3n$ .
- c)  $\Omega_n$  is a universal invariant of order  $3n$ , i.e. if  $\Omega_n(M) = \Omega_n(M')$  then  $I(M) = I(M')$  for every invariant  $I$  of order  $\leq 3n$ .

We need the following two lemmas, whose proofs will be presented in Section 5.

**Lemma 4.2.** *Suppose that  $\Gamma$  is a 3-valent vertex-oriented graph in  $\mathcal{D}$  of degree  $n$ . Then for any representative  $\alpha(\Gamma)$  of  $O_{3n}^*(\Gamma)$  in  $\mathcal{F}_{3n}(\mathcal{M})$ , one has*

$$\Omega_n(\alpha(\Gamma)) = (-1)^n \Gamma + \Omega_n(M_1), \quad (4.1)$$

where  $M_1$  is in  $\mathcal{F}_{3n+3}(\mathcal{M})$ .

**Lemma 4.3.** *If  $N \in \mathcal{F}_{6n+1}(\mathcal{M})$ , then  $\Omega_n(N) = 0$ .*

This lemma is certainly weaker than part *a*) of Main Theorem.

*Proof.* [of Main Theorem]

a) Suppose  $N \in \mathcal{F}_{3n+1}(\mathcal{M}) = \mathcal{F}_{3n+3}(\mathcal{M})$  we have to show that  $\Omega_n(N) = 0$ .

Since  $O_{3n+3}^* : \text{Grad}_{n+1}(\mathcal{D}) \rightarrow \mathcal{F}_{3n+3}(\mathcal{M})/\mathcal{F}_{3n+6}(\mathcal{M})$  is surjective, there is  $\Gamma_1 \in \text{Grad}_{n+1}(\mathcal{D})$  such that

$$N - \alpha(\Gamma_1) = N_1 \in \mathcal{F}_{3n+6}(\mathcal{M}),$$

where  $\alpha(\Gamma_1)$  is any representative of  $O_{3n+3}^*(\Gamma_1)$  in  $\mathcal{F}_{3n+3}(\mathcal{M})$ . Apply  $\Omega_{n+1}$  to both sides, using Lemma 4.2 with  $n$  replaced by  $n+1$ ; we see that, for some  $M_1 \in \mathcal{F}_{3n+6}(\mathcal{M})$ ,

$$\Omega_{n+1}(N) = \pm \Gamma_1 + \Omega_{n+1}(N_1) + \Omega_{n+1}(M_1) = \pm \Gamma_1 + \Omega(N_1 + M_1).$$

Since  $\Omega_n = \text{Grad}_{\leq n} \Omega_{n+1}$  (see Proposition 3.4) and  $\Gamma_1$  is of degree  $n+1$ , one has

$$\Omega_n(N) = \Omega_n(M_1 + N_1).$$

Here  $(M_1 + N_1)$  is in  $\mathcal{F}_{3n+6}$ , while  $N$  is in  $\mathcal{F}_{3n+3}(\mathcal{M})$ . Repeat this argument  $n$  times, one finds  $N'$  in  $\mathcal{F}_{6m+3}(\mathcal{M})$  such that

$$\Omega_n(N) = \Omega_n N'.$$

Now Lemma 4.3 says that  $\Omega_n(N') = 0$ . This completes the proof of *a*).

b) We will show an inverse of  $O_{3n}^* : \text{Grad}_n(\mathcal{D}) \rightarrow \mathcal{F}_{3n}(\mathcal{M})/\mathcal{F}_{3n+3}(\mathcal{M})$ . Let us consider equation (4.1) again. Since  $M_1$  in the equation is in  $\mathcal{F}_{3n+3}(\mathcal{M})$ , by part *a*), we have  $\Omega_n(M_1) = 0$ . Hence

$$\Omega_n(\alpha(\Gamma)) = (-1)^n \Gamma.$$

This is true for every representative  $\alpha(\Gamma)$  of  $O_{3n}^*(\Gamma)$  in  $\mathcal{F}_{3n}(\mathcal{M})$ , hence  $(-1)^n \Omega_n$  restricts to a well-defined linear mapping from  $\mathcal{F}_{3n}(\mathcal{M})/\mathcal{F}_{3n+3}(\mathcal{M})$  to  $\text{Grad}_n \mathcal{D}$ , which, by the above identity, is inverse to  $O_{3n}^*$ .

c) We use induction on  $n$ . Suppose  $\Omega_n(M) = \Omega_n(M') \in \text{Grad}_{\leq n}(\mathcal{D})$  and  $I$  is an invariant of order  $n$  with values in  $\mathbb{Q}$ . Let  $W : \text{Grad}_n(\mathcal{D}) \rightarrow \mathbb{Q}$  be the corresponding linear form of  $I$ . Let  $I_1 = W(\text{Grad}_n(\Omega_n))$ . Then  $I_1$  is an invariant of order  $\leq 3n$  and  $I_1(M) = I_1(M')$ .

Note that  $I - I_1$  is an invariant of order  $\leq 3n$ . It is not hard to verify that the corresponding linear form of  $I - I_1$  from  $\text{Grad}_n(\mathcal{D})$  to  $\mathbb{Q}$  is 0, hence  $I - I_1$  is of order  $\leq (3n - 1)$ . By induction,  $(I - I_1)(M) = (I - I_1)(M')$ . Hence  $I(M) = I(M')$ .  $\square$

Let the *product* of two homology 3-spheres be their connected sum. The unit is  $S^3$ . Define a co-product:  $\Delta(M) = M \otimes M$ , and co-unit:  $\varepsilon(M) = 1$ , for every homology 3-sphere  $M$ . Then  $\mathcal{M}$  becomes a commutative co-commutative Hopf algebra.

It is easy to see that the product is compatible with the filtration

$$\mathcal{M} = \mathcal{F}_0 \mathcal{M} \supset \mathcal{F}_3 \mathcal{M} \supset \cdots \supset \mathcal{F}_{3n}(\mathcal{M}) \dots$$

This means, if  $N_1 \in \mathcal{F}_{n_1}(\mathcal{M})$  and  $N_2 \in \mathcal{F}_{n_2}(\mathcal{M})$ , then their product is in  $\mathcal{F}_{n_1+n_2}(\mathcal{M})$ .

Hence the algebra structure of  $\mathcal{M}$  induces an algebra structure on the associated (complete) graded vector space

$$\mathcal{GM} = \prod_{i=0}^{\infty} \mathcal{F}_{3i}(\mathcal{M}) / \mathcal{F}_{3i+3}(\mathcal{M}).$$

A nice interpretation of the main theorem is the following.

**Theorem 4.4.** *a) The mapping  $\Omega : \mathcal{M} \rightarrow \mathcal{D}$  is a Hopf algebra homomorphism which maps  $\mathcal{F}_{3n}(\mathcal{M})$  to  $\text{Grad}_{>n}\mathcal{D}$ .*

*b)  $\Omega$  induces an algebra isomorphism between  $\mathcal{GM}$  and  $\mathcal{D}$ .*

This theorem follows immediately from the previous one and Propositions 3.3, 3.5. Part b) shows that  $\mathcal{GM}$  has a structure of a commutative co-commutative Hopf algebra.

#### 4.2. Some further properties of $\Omega$ .

**Proposition 4.5.** *a)  $\Omega_n$  is of order exactly  $3n$ .*

*b) For every  $n$  there is an integral homology 3-sphere  $M_n$  such that  $\Omega(M_n) = 1 + \text{degree} \geq n$ , and the  $n$ -th degree part is not 0.*

*Proof.* a) This is equivalent to the fact that  $\text{Grad}_n(\mathcal{D})$  has positive dimension. It's enough to show that there is a non-zero linear form on  $\text{Grad}_n(\mathcal{D})$ . One can use the weight system coming from simple Lie algebra, as defined in [Ko1, BN1, LM2], to define non-zero linear form. The details are left for the reader. a) also follows from b).

b) Consider a solid loop and a dashed loop, each has  $2n$  marked points  $x_1, x_2, \dots, x_{2n}$  and  $y_1, y_2, \dots, y_{2n}$ , counting counterclockwise. Connect  $x_i$  with  $y_i$  by a dashed line, for every  $i = 1, 2, \dots, 2n$ , we get a chord diagram  $\xi_n$  on a solid loop. In [Ng], a knot  $K$  was constructed with property that for every knot invariant  $I$  of degree  $< 2n$ , we have  $I(K) = I(\text{unknot})$  and for every invariant  $I$  of degree  $2n$  we have  $I(K) = I(\xi_n)$ . It follows that if  $K$  has framing 0, then

$$\hat{Z}(K) = \nu(1 + \xi_n) + (\text{elements of degree } > 2n).$$

Hence if  $K'$  is the knot  $K$  with framing 1, we have

$$\check{Z}(K') = \nu^2 e^{\theta/2} (1 + \xi_n) + (\text{elements of degree } > 2n).$$

Here  $\theta \in \mathcal{A}(S^1)$  is the chord diagram of degree 1 with one isolated chord;  $e^{\theta/2}$  is the contribution of framing 1 (see [LM2], Theorem 3). Let  $M_n$  be obtained by Dehn surgery on  $K'$ , then, by definition

$$\Omega_n(M_n) = \frac{\iota_n(\check{Z}(K'))}{\iota_n(U_+)} = \frac{\iota_n[\nu^2 e^{\theta/2} (1 + \xi_n)]}{\iota_n(\nu^2 e^{\theta/2})}.$$

Since  $\xi_n$  is of degree  $2n$ ,  $\iota_n$  annihilates any chord diagram in  $\mathcal{A}(S^1)$  of degree  $> 2n$ , and  $\iota_n(U_+) = \iota_n(\nu^2 e^{\theta/2}) = (-1)^n + \dots$ , we see that  $\Omega_n(M_n) = 1 + (-1)^n \iota_n(\xi_n)$ .

Using the definition of  $\iota_n$  (the one with  $T_{2n}^n$  described in §3.3), combining with  $sl_2$ -weight system, one can prove that  $\iota_n(\xi_n) \neq 0$ .  $\square$

In [Lin], X. S. Lin asked whether there exists an operation, similar to the Stanford one in the knot case, on homology 3-spheres which does not alter the values of any invariant of order  $\leq n$ . We now present such an operation.

Suppose  $M$  is obtained by Dehn surgery on a unit-framed algebraically split link  $L$ . Suppose that  $T$  is a part of  $L$  which is a trivial tangle with  $m$  strands. Let  $G_j(P_m)$ ,  $j =$

$0, 1, 2, \dots$  be the lower central series of the pure braid group  $P_m$  on  $m$  strands, defined by  $G_0(P_m) = P_m, G_{j+1}(P_m) = [G_j(P_m), P_m]$ . Replacing  $T$  by an element  $\gamma$  in the  $(2n + 2)$ -th group  $G_{2n+2}(P_m)$ , we obtain a unit-framed algebraically split link  $L'$ . The homology 3-sphere  $S^3_{L'}$  is said to be obtained from  $M$  by an  $s_n$  operation.

**Proposition 4.6.** *If  $M'$  is obtained from  $M$  by an  $s_n$  operation, then  $\Omega_n(M) = \Omega_n(M')$ . Hence the values of any invariant of order  $\leq 3n$  on  $M$  and on  $M'$  are the same.*

*Proof.* We first consider  $\hat{Z}(\gamma) \in \mathcal{P}_m$ . By a result of Stanford [Sta] (see also [Koh]), the values of any pure braid invariant of order  $\leq (2n + 1)$  on  $T$  (the trivial tangle) and  $\gamma$  are the same. Hence

$$\text{Grad}_{\leq(2n+1)}\hat{Z}(\gamma) = 1.$$

From Theorem 4.2 of [LM3], it follows that  $\hat{Z}(\gamma)$  is a group-like element, i.e.  $\hat{Z}(\gamma) = \exp(\xi)$ , where  $\xi$  is primitive. The above identity shows that  $\xi$  is a linear combination of chord diagrams of degree  $\geq (2n + 2)$ . Hence from Proposition 2.1 one sees that

$$\hat{Z}(\gamma) = 1 + (\text{elements of i-filter } \geq (2n + 1)).$$

Since  $\iota_n$  annihilates any element of i-filter  $\geq (2n + 1)$ , we have that  $\Omega_n(M) = \Omega_n(M')$ .  $\square$

Note that we proved this fact only for invariants with *rational* values. It would be interesting to give a direct geometric proof of this fact, and to establish the same fact for invariants with values in any *abelian* group.

From this proposition, it is easy to construct, for any given homology 3-sphere  $M$  and any positive integer  $n$ , infinitely many homology 3-spheres which are indistinguishable from  $M$  by all invariants of order  $\leq n$ .

### 4.3. Some other corollaries; remarks. X. S. Lin proved (see [Lin])

**Proposition 4.7.** *The space of finite type invariants of homology 3-spheres with rational values is a commutative co-commutative Hopf algebra.*

Here this fact is a consequence of Theorem 4.4, b), as observed in [GO1], Remark 1.10.

Suppose  $I$  is an invariant of integral homology 3-spheres,  $K$  is a knot in  $S^3$ . Let us define  $\lambda_I(K) = I(S^3_K)$ , where we provide  $K$  with framing 1.

**Proposition 4.8.** *If  $I$  is a homology 3-sphere invariant of order  $3n$ , then  $\lambda_I$  is a knot invariant of order  $2n$ .*

This had been conjectured by Garoufalidis, and proved by Habegger, see [Hab]. Here this fact follows easily from the main theorem: for a knot  $K$ , the universal invariant  $\Omega_n(S^3_K)$  is computed using  $\text{Grad}_{\leq 2n}(\hat{Z}(K))$ , hence  $I_{\Omega_n}(K)$  is derived from  $\text{Grad}_{\leq 2n}\hat{Z}(K)$  which is a knot invariant of order  $\leq 2n$ .

**Remark . 1.** The main theorem can be reformulated and proved (in a similar way) for *rational* homology 3-spheres, see [GO2] for theory of finite type invariants of rational homology 3-spheres. The details will appear elsewhere.

2. The construction of  $\Omega$  suggests that the relation between  $\Omega$ , combined with  $sl_2$  weight, and  $sl_2$  quantum invariant at roots of unity maybe expressed through Ohtsuki polynomial in [Oh2], or the invariant defined in [Oh3].

**4.4. Groups of homology 3-spheres.** In analogy with the knot case (see [Gus, Ng, NgS]), we say that two integral homology spheres  $M_1, M_2$  are  $V_n$ -equivalent if they are indistinguishable by any *rational* invariant of order  $\leq n$ ; or the same as:  $(M_1 - M_2) \in \mathcal{F}_{3n+1}(\mathcal{M})$ , or  $\Omega_n(M_1) = \Omega_n(M_2)$ .

Let  $\mathcal{VM}_n$  be the set of all homology 3-spheres, regarded up to the  $V_n$ -equivalence relation. This set  $\mathcal{VM}_n$  is a semigroup, where the product is the connected sum.

**Theorem 4.9.** *For every connected 3-valent vertex-oriented graph  $\Gamma$  of degree  $n$ , there are homology 3-spheres  $M^\pm(\Gamma)$  such that*

$$\Omega(M^\pm(\Gamma)) = 1 + \pm\Gamma + (\text{elements of degree } > n).$$

Moreover,  $M^\pm(\Gamma)$  are obtained by Dehn surgery on links of  $n$  components.

This is stronger than proposition 4.5. From Theorem 4.9, one gets

**Theorem 4.10.** *The semi-group  $\mathcal{VM}_n$  is a group. This means, for every homology 3-sphere  $M$ , there is another homology 3-spheres  $M'$  such that  $M\#M'$  is  $V_n$ -equivalent to 0.*

*The group  $\mathcal{VM}_n$  is free abelian of rank equal to the dimension of the subspace of  $\text{Grad}_n(\mathcal{D})$ , spanned by connected 3-valent vertex-oriented graphs.*

This answers the first part of question 2 in [Lin].

Proofs of Theorems 4.9 and 4.10 will appear elsewhere.

## 5. PROOFS

**5.1. The mapping  $O_{3n}^*$ .** Let's consider a 3-valent vertex-oriented graph  $\Gamma$  of degree  $n$  in  $\mathbb{R}^3$ . Like with links, we can represent  $\Gamma$  using a generic projection on the plane with decoration at every double point to indicate over/under-crossing. We suppose that  $\Gamma$  is equal to this projection everywhere except for a neighborhood of the double points, that the orientation at every vertex is given by counterclockwise direction of the plane, and that the three edges incident to a 3-valent vertex is going downwards in a neighborhood of the vertex as in Figure 4. Here edges of  $\Gamma$  are depicted by solid lines.

$\Gamma$  has  $2n$  trivalent vertices and  $3n$  edges. Using  $2n$  pairs of horizontal lines on the plane, each pair consists of a line above and a line below a vertex, we can decompose  $\Gamma$  as:

$$\Gamma = T_1V_1T_2V_2 \dots T_{2n}V_{2n}T_{2n+1}. \quad (5.1)$$

Here  $T_i$  are some non-oriented tangles while  $V_i$  are trivial tangles with a 3-valent vertex part as in Figure 4; and product in the right hand side means "placing on top".

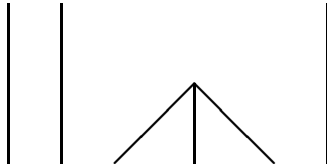


FIGURE 4.  $V_i$

Now we define  $\beta(T_i)$  and  $\beta(V_i)$  as follows. Let  $\beta(T_i)$  be the *tangle* obtained by replacing each component of  $T_i$  by a pair of parallel push-offs (on the plane) of this component. The orientation on  $\beta(T_i)$  is chosen so that at boundary points of a pair of push-offs, the left boundary point is pointing upwards, and the right boundary point downwards.

$\beta(V_i)$  is not a *single tangle*, but rather the *formal difference of two tangles*,  $u - v$ . The first tangle  $u$  is obtained from  $V_i$  by replacing each vertical component of  $V_i$  by a pair of its parallel push-offs and replacing the 3-valent vertex by 3 arcs linked with each other as in the Borromean ring; in the second tangle  $v$ , the 3-valent vertex is replaced by 3 arcs which do not link with each other. For example, the  $V_i$  of Figure 4 has  $\beta(V_i)$  as in Figure 5. The orientation is chosen so that at boundary points of a pairs of push-offs, or at boundary points of a non-vertical arc, the left point is pointing upwards, and the right point is pointing downwards.



FIGURE 5.  $\beta(V_i)$

$\beta(T_i)$  will be regarded as a *framed oriented tangle*, while  $\beta(V_i)$  will be regarded as the difference of two *framed oriented tangles*, where the framing vector at every point is given by the vector perpendicular to the page and pointing at us.

We define

$$\beta(\Gamma) = \beta(T_1)\beta(V_1) \dots \beta(V_{2n})\beta(T_{2n+1}).$$

This  $\beta(\Gamma)$  is not a single link, but a *formal linear combination* of  $2^{2n}$  algebraically split framed oriented links, each has  $3n$  components with framing 0.

Changing the framing of each link component to 1, from  $\beta(\Gamma)$  we get  $\tilde{\beta}(\Gamma)$  which is a linear combination of  $2^{2n}$  unit-framed algebraically split oriented links.

Then  $\alpha(\Gamma) = S_{\delta(\tilde{\beta}(\Gamma))}^3$  is an element in  $\mathcal{F}_{3n}(\mathcal{M})$ . Recall that the operator  $\delta$  is given by (3.2). Define  $O_{3n}^*(\Gamma)$  as the image of  $\alpha(\Gamma)$  in  $\mathcal{F}_{3n}(\mathcal{M})/\mathcal{F}_{3n+1}(\mathcal{M})$ . In [GO1] it was proved that  $O_{3n}^*$  is a well-defined *surjective* linear mapping from  $\text{Grad}_n(\mathcal{D})$  to  $\mathcal{F}_{3n}(\mathcal{M})/\mathcal{F}_{3n+1}(\mathcal{M})$ .

**Lemma 5.1.** *One has  $\delta(\tilde{\beta}(\Gamma)) = \tilde{\beta}(\Gamma)$ . Hence  $\alpha(\Gamma) = S_{\tilde{\beta}(\Gamma)}^3$ .*

This combinatorial lemma follows easily from the definition and a simple induction.

**5.2. Proof of Lemma 4.2.** We will prove

**Lemma 5.2.** *For every 3-valent vertex-oriented graph  $\Gamma$  in  $\text{Grad}_n(\mathcal{D})$  one has:*

$$\Omega_n(S_{\tilde{\beta}(\Gamma)}^3) = (-1)^n \Gamma. \tag{5.2}$$

Note that Lemma 4.2 follows from this lemma, since  $\alpha(\Gamma) = S_{\tilde{\beta}(\Gamma)}^3$  (by Lemma 5.1) and any two representatives of  $O_{3n}^*(\Gamma)$  differ by an element in  $\mathcal{F}_{3n+3}(\mathcal{M})$ .

For a formal linear combination  $\sum c_i L_i$  of framed oriented links, we define  $\check{Z}(\sum c_i L_i)$  as  $\sum c_i \check{Z}(L_i)$ . Then  $\check{Z}(\tilde{\beta}(\Gamma)) \in \mathcal{A}(\sqcup^{3n} S^1)$ . The left hand side of (5.2), by definition, is

$$\Omega_n(S_{\tilde{\beta}(\Gamma)}^3) = \frac{\iota_n[\check{Z}(\tilde{\beta}(\Gamma))]}{\iota_n(U_+)^{-3n}}.$$

Since  $\iota_n(U_+)$  has the form  $(-1)^n +$  elements of degree  $\geq 1$  (see [LMO]), in order to prove (5.2) it suffices to show that

$$\iota_n[\check{Z}(\tilde{\beta}(\Gamma))] = \Gamma. \quad (5.3)$$

For  $V_i$  in the decomposition (5.1) of  $\Gamma$ ,  $\beta(V_i) = u - v$ , where  $u - v$  are tangles in Figure 5. Note that as abstract 1-manifolds,  $u$  and  $v$  are homeomorphic, and  $\hat{Z}(\beta(V_i)) = \hat{Z}(u) - \hat{Z}(v)$  is in  $\mathcal{A}(v)$ . Let  $\xi(V_i)$  be the *chord diagram* in  $\mathcal{A}(v)$  with exactly one dashed component with 1 internal vertex which is connected to the 3 non-vertical solid lines of  $v$ , as Figure 6. Then  $\xi(V_i)$  has i-filter 1.

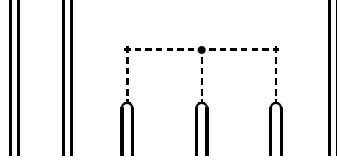


FIGURE 6.  $\xi(V_i)$

**Lemma 5.3.** *One has  $\hat{Z}(\beta(V_i)) = \hat{Z}(u) - \hat{Z}(v) = \xi(V_i) + (\text{elements of i-filter } \geq 2)$ .*

A proof of the lemma is given in §5.6, after a proof of a similar fact.

Let  $\xi(\Gamma)$  be the *chord diagram* with support  $3n$  solid loops obtained as follows:

$$\xi(\Gamma) = T_1 \xi(V_1) T_2 \xi(V_2) \dots T_{2n} \xi(V_{2n}) T_{2n+1}.$$

On every solid loop of  $\xi(\Gamma)$  there are exactly two external vertices. It is important to notice that if we remove every solid loop of  $\xi(\Gamma)$  and connect the two external vertices of each solid loop by a dashed line, then we get exactly  $\Gamma$ .

Now we consider

$$\hat{Z}(\beta(\Gamma)) = \hat{Z}(\beta(T_1)) \hat{Z}(\beta(V_1)) \dots \hat{Z}(\beta(V_{2n})) \hat{Z}(\beta(T_{2n+1})).$$

Recall that  $\Omega_n$  annihilates any chord diagram of i-filter  $> 2n$ .

Using Lemma 5.3 we see that  $\hat{Z}(\beta(\Gamma))$  is, modulo elements of i-filter  $> 2n$ , a linear combination of chord diagrams, each contains  $\xi(\Gamma)$  as a subdiagram, i.e. each is  $\xi(\Gamma)$  plus some extra chords. The extra chords are contributions from  $\hat{Z}(\beta(T_i))$ . Note that every  $\beta(T_i)$  is obtained by first taking parallel push-offs of every component of the tangle  $T_i$ , then putting orientations on components such that the two push-offs are of opposite orientation. Hence Theorem 2.4 says that  $\hat{Z}(\beta(T_i))$  is  $1+$  a linear combination of terms of the form  $\xi - \xi'$ , where  $\xi$  and  $\xi'$  are chord diagrams (in  $\mathcal{A}[\beta(T_i)]$ ) identical everywhere, except for a ball in which they are as in Figure 7.



FIGURE 7.

The two solid lines here are of the same component of the support of chord diagrams in  $\mathcal{A}(\sqcup^{3n} S^1)$ . Using the STU relation we can always move any external vertex around the



solid loop; when exchanging two external vertices, we have to add a chord diagram with an extra *internal* vertex. Hence we can cancel  $\xi$  and  $\xi'$ , and get

$$\hat{Z}(\beta(\Gamma)) = \xi(\Gamma) + (\text{elements of i-filter } > 2n). \quad (5.4)$$

Now we prove (5.3). The left hand side of (5.3) is, by the definitions of  $\check{Z}$  and  $\tilde{\beta}$ ,

$$\iota_n[\check{Z}(\tilde{\beta}(\Gamma))] = \iota_n[(\nu e^{\theta/2})^{\otimes 3n} \# \hat{Z}(\beta(\Gamma))],$$

where  $\theta$  is the chord diagram on a solid loop with exactly one isolated chord;  $e^{\theta/2}$  is the contribution of the framing 1 (see Theorem 3 of [LM2]).

Note that  $\xi(\Gamma)$  already has  $2n$  internal vertices and  $\nu$  has the form  $1 + (\text{i-filter } \geq 1)$ , hence we can delete all the  $\nu$ 's in the formula.

Every solid component of  $\xi(\Gamma)$  has 2 external vertices. Since  $\iota_n$  annihilates any chord diagram with less than  $2n$  external vertices on a solid component, we see that

$$\iota_n[(\nu e^{\theta/2})^{\otimes 3n} \# \hat{Z}(\beta(\Gamma))] = \iota_n\left\{\left[\frac{1}{2^{n-1}(n-1)!}\right]^{3n} \tilde{\xi}(\Gamma)\right\}, \quad (5.5)$$

where  $\tilde{\xi}(\Gamma)$  is obtained from  $\xi(\Gamma)$  by adding  $(n-1)$  isolated chords to each solid component. The isolated chords and the coefficients  $\frac{1}{2^{n-1}(n-1)!}$  are contributions of  $e^{\theta/2}$ .

$\tilde{\xi}(\Gamma)$  is a chord diagram with  $2n$  external vertices on each solid loops,  $(2n-2)$  of them are external vertices of the  $(n-1)$  isolated chords.

**Lemma 5.4.** *One has*

$$\iota_n(\text{---} \langle \text{---} \vdots \text{---} \rangle \text{---}) = (-2)^{n-1}(n-1)! (\text{-----}).$$

Here the chord diagram in the left hand side has  $(n-1)$  vertical chords. The result holds true if these vertical dashed chords are replaced by isolated chords.

*Proof.* By definition, we have to replace the solid loop by element  $T_{2n}^n$  (see §3.3), with the convention that a dashed loop is equal  $-2n$ , by relation  $O_n$ . Because of the symmetry of  $T_{2n}^n$ , the left hand side does not depend on whether the  $(n-1)$  dashed lines are vertical chords or isolated chords. The proof of the Lemma now follows by an induction.  $\square$

Note that if we remove every solid loop of  $\xi(\Gamma)$  and connect the two external vertices of each solid loop by a dashed line, then we get exactly  $\Gamma$ . Hence from Lemma 5.4 we get

$$\iota_n(\tilde{\xi}(\Gamma)) = [(-2)^{n-1}(n-1)!]^{3n} \Gamma.$$

This, together with (5.5) proves (5.3), and hence Lemma 4.2.

**5.3. Beginning of Proof of Lemma 4.3.** Let  $K$  be a unit-framed algebraically split link in a homology 3-sphere  $M$  with  $(6n+1)$  components. We have to show that  $\Omega_n(M_{\delta(K)}) = 0$ . The following is well-known and can be proved using the Kirby calculus for links in 3-manifolds and Lemma 1.1 of [Hos].

**Lemma 5.5.** *There is a unit-framed and algebraically split link  $L$  in  $S^3$  which is a union of two links  $L_1, L_2$  such that after Dehn surgery on  $L_1$ , from  $S^3$  we get  $M$  and from  $L_2$  we get  $K$ . ( $L_1$  and  $L_2$  may link with each other in  $S^3$ ).*

Let

$$\tilde{\delta}_{L_2}(L) = \sum_{L'_2 \subset L_2} (-1)^{|L'_2|} (L_1, L''_2),$$

where the sum is over the set of all sublinks  $L'_2$  of  $L_2$ ; and  $(L_1, L''_2)$  is the link obtained from  $L = (L_1, L_2)$  by replacing every component of  $L_2 \setminus L'_2$  by a trivial knot with the same framing; the trivial knots are not linked with each other nor with  $L$ .

Let  $(L_1, L'_2)$  be obtained from  $(L_1, L_2)$  by replacing  $L_2$  with  $L'_2$ .

By definition,

$$\Omega_n(M_{\delta(L)}) = \sum_{L'_2 \subset L_2} (-1)^{|L'_2|} \Omega_n[S^3_{(L_1, L'_2)}] = \sum_{L'_2 \subset L_2} (-1)^{|L'_2|} \frac{\iota_n(\check{Z}(L_1, L'_2))}{\iota_n(U_+)^{-\sigma_+(L_1, L'_2)} \iota_n(U_-)^{-\sigma_-(L_1, L'_2)}}.$$

Note that  $U_{\pm}$  are the trivial knots with framing  $\pm 1$ , and that  $\iota_n(a \sqcup b) = \iota_n(a)\iota_n(b)$  for two links  $a, b$ . Using the same denominator  $\iota_n(U_+)^{-\sigma_+} \iota_n(U_-)^{-\sigma_-}$ , where  $\sigma_{\pm} = \sigma_{\pm}(L_1, L_2)$ , for all the terms of the last sum, we arrive at the following important formula:

$$\Omega_n(M_{\delta(K)}) = \frac{\iota_n[\check{Z}(\check{\delta}_{L_2}(L_1, L_2))]}{\iota_n(U_+)^{-\sigma_+} \iota_n(U_-)^{-\sigma_-}}.$$

Hence in order to prove Lemma 4.3, it's sufficient to prove that

$$\iota_n[\check{Z}(\check{\delta}_{L_2}(L_1, L_2))] = 0 \tag{5.6}$$

Here  $(L_1, L_2)$  is a unit-framed algebraically split link in  $S^3$  with  $|L_2| = 6n + 1$ .

The proof of (5.6) may look complicated, but the idea is simple. The point is that  $\check{\delta}_{L_2}(L)$  is a repeated difference formula. The simplest case is when  $|L_2| = 1$  (only one step). In this case  $\check{\delta}_{L_2}(L) = L - L'$ , where  $L'$  is obtained from  $L$  by replacing the only component of  $L_2$  by a trivial knot with the same framing. Then it can be proved that  $\check{Z}(L - L')$  is of i-filter  $\geq 1$ . (The reason for this phenomenon is that the linking number of  $L_2$  and other components is 0).

When  $|L_2| > 1$  we have a repeated difference sum for  $\check{Z}(\check{\delta}_{L_2}(L_1, L_2))$ ; we expect that  $\check{Z}(\check{\delta}_{L_2}(L_1, L_2))$  has i-filter  $\geq (2n + 1)$ , if  $|L_2|$  is sufficiently large ( $|L_2| = 6n + 1$  is enough). It follows then  $\iota_n[\check{Z}(\check{\delta}_{L_2}(L_1, L_2))] = 0$ . A rigorous proof requires a lot of technical stuff.

**5.4. Some preparations.** Let  $Y$  be a submanifold of  $X$ ;  $Y$  consists of several components of  $X$ . We say that an element  $\xi \in \mathcal{A}(X)$  is *i-near*  $Y$  if  $\xi$  is a linear combination of chord diagrams, each has internal vertices *near* every component of  $Y$ . Here an internal vertex is near a solid component if the vertex is connected to the solid component by a dashed line of the dashed graph.

Recall that  $\mathcal{P}_m$ , the space of chord diagrams on  $m$  numbered solid lines, is a co-commutative Hopf algebra. The space of primitive elements is spanned by chord diagrams with *connected* dashed graph.

A tangle consisting of  $m$  strings (no loops) such that the two end points of each string project vertically to the same point is called a *string link*. Suppose that  $T$  is an  $m$ -component framed string link whose orientation at every boundary points is downward. A corollary of Theorem 4.2 of [LM3] is that  $\hat{Z}(T) \in \mathcal{P}_m$  is group-like. Hence  $\hat{Z}(T) = \exp(\xi)$ , where  $\xi$  is primitive.

Suppose  $T'$  is obtained from  $T$  by removing a component  $C$ . Let  $\varepsilon_C : \mathcal{P}_m \rightarrow \mathcal{P}_{m-1}$  be the linear mapping which send a chord diagram  $\xi \in \mathcal{P}_m$  to  $\varepsilon_C(\xi)$  obtained from  $\xi$  by removing the solid line  $C$  if there is no external vertex on  $C$ ; otherwise  $\varepsilon_C(\xi) = 0$ .

The relation between  $\hat{Z}(T)$  and  $\hat{Z}(T')$  is expressed by (see [LM1, LM2]):

$$\hat{Z}(T') = \varepsilon_C[\hat{Z}(T)]. \tag{5.7}$$

For a string link  $T$ , one defines the *closure link* of  $T$  as in the braid case, see Figure 8.

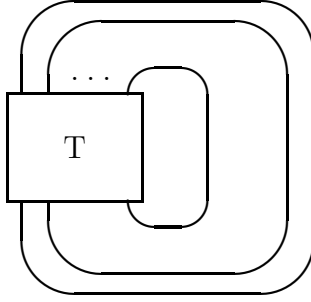


FIGURE 8. The closure link of a string link  $T$

Suppose that a *framed* string link  $T$  is represented by a tangle diagram on the plane, where the framing vector at every boundary point is given by the vector perpendicular to the page and pointing to us. In the closure link of  $T$ , we suppose that the framing at every point outside  $T$  is given by the vector perpendicular to the page and pointing to us. By this way we can identify the set of framings of a component of  $T$  with  $\mathbb{Z}$ ; say,  $C$  has framing 0 if its closure has framing 0. We say that  $T$  is algebraically split if the closure link is algebraically split.

Let  $d_C(T)$  be obtained from  $T$  by replacing component  $C$  by a vertical line connecting the two boundary point of  $C$  and over-passing any other component of the tangle diagram. The new component is assumed to have the same framing (in  $\mathbb{Z}$ ) as  $C$  has.

**Lemma 5.6.** *Suppose that a framed string link  $T$  is algebraically split. Then  $\hat{Z}(T) - \hat{Z}(d_C(T)) \in \mathcal{A}(T)$  is i-near  $C$ .*

*Proof.* We suppose that  $C$  is the last component of  $T$ , counting from left to right. The other cases are similar (and also follow from this case and Lemma 2.3). Without loss of generality, we can assume that  $C$  has framing 0.

Let  $\hat{Z}(T) = \exp(a + b)$ , where  $\varepsilon_C(a) = 0$  and  $\varepsilon_C(b) \neq 0$ . Both  $a$  and  $b$  are primitive.

Let  $T'$  be obtained from  $T$  by removing  $C$ . Then (with  $1 \in \mathcal{P}_1$ )

$$\hat{Z}(d_C(T)) = \hat{Z}(T') \otimes 1 = \exp[\varepsilon_C(a + b)] \otimes 1 = \exp(\varepsilon_C(b)) \otimes 1 = \exp(b);$$

the first equality follows from Lemma 2.3.

Hence  $\hat{Z}(T) - \hat{Z}(d_C(T)) = \exp(a + b) - \exp(b)$  is a sum of terms, each is a product with  $a$  being one of the factor.

Since  $\varepsilon_C(a) = 0$ , and  $a$  is primitive, it follows that  $a$  is a linear combination of chord diagrams whose dashed graph is connected and has an external vertex on  $C$ .

Since the coefficients of first degree chord diagrams in  $\hat{Z}(T)$  are the linking numbers or self-linking numbers (i.e. framing numbers), we conclude that  $a$  does not have any chord diagrams of degree 1.

It is clear that if  $\xi$  is a chord diagram whose graph is connected, has an external vertex on  $C$  and  $\xi$  has degree  $\geq 2$ , then  $\xi$  has an internal vertex near  $C$ . Hence  $a$  is i-near  $C$ .

It follows that  $\hat{Z}(T) - \hat{Z}(d_C(T))$  is i-near  $C$ . □

**5.5. End of proof of Lemma 4.3.** An  $m$ -marked Chinese character is a vertex-oriented 3-valent graph whose external vertices are partitioned into  $m$  sets  $\Theta_1, \dots, \Theta_m$ . Consider

the space  $\mathcal{E}_m$  spanned by  $m$ -marked Chinese characters, subject to the AS and IHX relations.

Let  $s$  be a subset of  $\{1, 2, \dots, m\}$ . An element  $\xi \in \mathcal{E}_m$  is *i-near*  $s$  if  $\xi$  is a linear combination of  $m$ -marked Chinese characters, each has internal vertices near  $\Theta_j$  for every  $j \in s$ . Here an internal vertex is near  $\Theta_j$  if it is connected to an external vertex in  $\Theta_j$  by a dashed line.

Consider the linear mapping  $\chi : \mathcal{E}_m \rightarrow \mathcal{P}_m$ , defined as follows. Suppose an  $m$ -marked Chinese character  $\xi$  has  $k_i$  external vertices in  $\Theta_i, i = 1, 2, \dots, m$ . There are  $k_i!$  ways to put the vertices from  $\Theta_i$  on the  $i$ -th solid string; and each of the  $k_1! k_2! \dots k_m!$  possibilities give us a chord diagram in  $\mathcal{P}_m$ . Summing up all such chord diagrams, we get  $\chi(\xi)$ .

It is known that  $\chi$  is an isomorphism of vector spaces (see Theorem 9 in [LM2]). The case when  $m = 1$  was first proved by Kontsevich; for a detailed proof of the case  $m = 1$  see [BN1] (Theorem 6). The inverse of  $\chi$  is obtained by a symmetrizing process.

It follows from the definitions of  $\chi$  and its inverse that

**Proposition 5.7.** *For the  $j$ -th component  $C$  of the support of  $\mathcal{P}_m$ ,  $\chi : \mathcal{E}_m \rightarrow \mathcal{P}_m$  maps isomorphically the subspace of *i-near*  $j$  elements to the subspace of *i-near*  $C$  elements.*

An important fact is that the AS and IHX relations preserve the property to have an internal vertex near  $\Theta_j$ . Hence the space spanned by *m-marked Chinese characters not i-near*  $j$  has 0 intersection with the space of elements *i-near*  $j$ , for every  $j \in \{1, 2, \dots, m\}$ . The similar fact does not hold true for  $\mathcal{P}_m$ , due to the STU relation. It follows that

**Proposition 5.8.** *Suppose that  $\xi \in \mathcal{E}_m$  is *i-near*  $j$  for each  $j \in s$ , where  $s$  is a subset of  $\{1, 2, \dots, m\}$ , then  $\xi$  is *i-near*  $s$ .*

From this proposition and Proposition 5.7 we have

**Proposition 5.9.** *If  $\xi \in \mathcal{P}_m$  is *i-near*  $C$  for each component  $C$  of  $Y$ , then  $\xi$  is *i-near*  $Y$ .*

Now we prove Lemma 4.3. We need to prove (5.6). We will first prove that  $\check{Z}(\check{\delta}_{L_2}(L))$  is *i-near*  $L_2$ , i.e.,  $\check{Z}(\check{\delta}_{L_2}(L))$  is a linear combination of chord diagrams, each has internal vertices near every component of  $L_2$ . Since each internal vertex can be near at most 3 different solid components, and since  $|L_2| = 6n + 1$ , it then follows that  $\check{Z}(\check{\delta}_{L_2}(L))$  has *i-filter*  $\geq 2n + 1$ . Hence  $\Omega_n[\check{Z}(\check{\delta}_{L_2}(L))] = 0$ .

We can represent  $L = (L_1, L_2)$  as the closure of an  $m$ -component string link  $T$ . We suppose  $T$  is the union of  $T_i, i = 1, 2$ , where the components of  $T_i$  correspond to the components of  $L_i$ . For a sub-tangle  $T'_2 \subset T_2$ , let  $(T_1, T''_2)$  be the tangle obtained from  $T = (T_1, T_2)$  by successively applying  $d_C$  to  $T$ , where  $C$  runs the set of all components of  $T_2 \setminus T'_2$  and  $d_C$  is as in Lemma 5.6. Let

$$\check{\delta}_{T_2}(T) = \sum_{T'_2 \subset T_2} (-1)^{|T'_2|} (T_1, T''_2).$$

It is easy to see that for every component  $C$  of  $T_2$ , the sum  $\hat{Z}(\check{\delta}_{T_2}(T))$  can be represented as a finite sum of terms, each is of the form  $\hat{Z}(V) - \hat{Z}(d_C(V))$ . Hence by Lemma 5.6,  $\hat{Z}(\check{\delta}_{T_2}(T)) \in \mathcal{P}_m$  is *i-near*  $C$ . Since this is true for every component  $C$  in  $T_2$ , Proposition 5.9 says that  $\hat{Z}(\check{\delta}_{T_2}(T))$  is *i-near*  $T_2$ .

Observe that  $\check{\delta}_{T_2}(T)$  is a linear combination of framed string links; and the closure of  $\check{\delta}_{T_2}(T)$  is exactly  $\check{\delta}_{L_2}(L)$ . By Lemma 2.3 and property (2.1) we have

$$\check{Z}(\check{\delta}_{L_2}(L)) = a[\hat{Z}(\check{\delta}_{T_2}(T)) \otimes 1]b,$$

where  $1 \in \mathcal{P}_m$  is the chord diagram without dashed graph, and  $a, b$  are some linear combinations of chord diagrams. It follows that  $\check{Z}(\check{\delta}_{L_2}(L))$  is  $i$ -near  $L_2$ . As argued above, since  $|L_2| \geq 6n + 1$ , we get  $\Omega_n[\check{Z}(\check{\delta}_{L_2}(L))] = 0$ . This completes the proof of (5.6), and hence that of Lemma 4.3.

**5.6. Proof of Lemma 5.3.** By Lemma 2.3, it suffices to consider the case when  $V_i$  does not contain any vertical line. Then  $\beta(V_i) = u - v$ , where  $v$  is 3 arcs not linked with each other, while  $u$  is 3 arcs linked with each other like in the Borromean link.

These two tangles  $u$  and  $v$  can also be described as follows. The tangle  $u$  is obtained from the tangle in Figure 9 by replacing the box by  $\gamma_{123}$ , where  $\gamma_{123}$  is the tangle described in Figure 10; and  $v$  is obtained by replacing the box by the trivial tangle on 3 strands. The orientations of  $u, v$  are chosen so that all the lines in the box are upward.

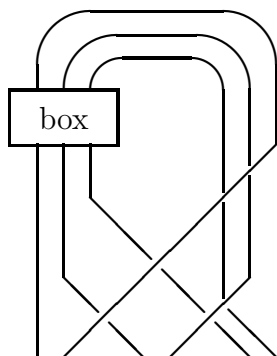


FIGURE 9.

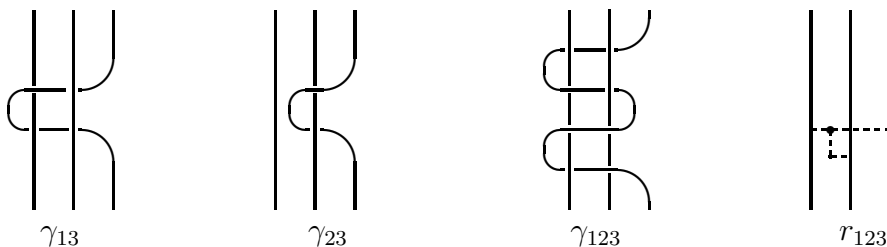


FIGURE 10.  $\gamma_{13}, \gamma_{23}, \gamma_{123} = [\gamma_{13}^{-1}, \gamma_{23}]$ , and  $r_{123}$

Let  $\gamma_{13}, \gamma_{23}$  be elements of the pure braid group  $P_3$  on 3 strands depicted in Figure 10. We suppose the orientation of each strand is upwards. An important observation is that  $\gamma_{123} = \gamma_{13}^{-1} \gamma_{23} \gamma_{13} \gamma_{23}^{-1}$ , a commutator.

An easy exercise in the theory of the Kontsevich integral is that, for  $ij = 12$  or  $13$ ,

$$\hat{Z}(\gamma_{ij}) = \exp(r_{ij} + \xi_{ij}), \tag{5.8}$$

where  $r_{ij}$  is the chord diagram of degree 1 with one dashed line connecting the  $i$ -th and  $j$ -th strands, and  $\xi_{ij}$  is of degree  $\geq 2$ . Hence for  $\gamma_{123} = \gamma_{13}^{-1} \gamma_{23} \gamma_{13} \gamma_{23}^{-1}$ , one has

$$\hat{Z}(\gamma_{123}) = 1 - r_{13}r_{23} + r_{23}r_{13} + (\text{elements of degree } \geq 3),$$

By the STU relation  $-r_{13}r_{23} + r_{23}r_{13} = r_{123}$ , where  $r_{123}$  is the chord diagram in Figure 10. Hence we have

$$\hat{Z}(\gamma_{123}) = \exp(r_{123} + \xi),$$

where  $\xi$  is primitive and of degree  $\geq 3$ . It follows from Proposition 2.1 that

$$\hat{Z}(\gamma_{123}) = 1 + r_{123} + (\text{elements of i-filter} \geq 2). \quad (5.9)$$

Note that if we replace the box in Figure 9 by  $r_{123}$ , then we get  $\xi(V_i)$ . Hence

$$\hat{Z}(\beta(V_i)) = \hat{Z}(u) - \hat{Z}(v) = \xi(V_i) + (\text{elements of i-filter} \geq 2).$$

This completes the proof.

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