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LOCAL RULES FOR QUASIPERIODIC TILINGS

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Abstract We discuss local rules theory for quasiperiodic tilings. Two versions of local rules, with or without decorations, are distinguished. Weak local rules are also considered. For the classes of tilings obtained by the canonical projection method, we present necessary conditions and sufficient conditions for the existence of local rules. Every set of quasiperiodic tilings obtained from the canonical strip projection method and based on quadratic irrationalities always admits local rules *after decoration*. In many cases there exist local rules *without decoration*. Examples of pentagonal tilings and 2-dimensional quasiperiodic tilings, obtained by the projection method from 4-dimensional space, are considered in detail. We prove that the existence of (even weak) local rules without decoration implies that the projection plane is based on algebraic irrationalities. The topology of sets of tilings obtained by projection methods is described.

1. Introduction

The aim of the paper is to give a survey of recent results on the theory of local rules for quasiperiodic tilings obtained by the projection method.

One of the most interesting problems in tiling theory is to find sets of building blocks, say polyhedra, and rules which state which building block can be put next to another one, such that every tiling obeying these rules is *aperiodic* and/or quasi-periodic in some sense.

For example, consider two "arrowed" rhombi in Figure 1 (in Section 2) as building blocks. The acute angles of these rhombi are $\pi/5$ and $2\pi/5$, and their edges have the same length. The rule is that only edges with the same kind of arrows can be matched. There are uncountably many tilings of the plane obeying this rule; every such tiling is ***aperiodic*** in the sense that it is different from every nontrivial translate of itself. Moreover these tilings are quasi-periodic in a sense that will be explained later. These tilings are known as Penrose tilings (Penrose, 1978; de Bruijn, 1981).

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A general form of this rule will be called a local rule; some authors call "matching rule" or "local matching rule". A "local rule", in some sense, contains information in a local finite radius. It is far from trivial to decide when a local rule forces a tiling to be aperiodic, or quasiperiodic, since the latter are global properties.

The first local rule enforcing aperiodicity was found in (Berger, 1966), but it contains too many building blocks, and the tilings are just the infinite checker board with some decoration. The best known example is the above mentioned Penrose local rule. Until recently there were known only a few local rules which force quasiperiodicity. We will present here infinitely many such local rules.

We can begin with a set of *aperiodic tilings* and ask whether this set of tilings admits a local rule, i.e., if there is a local rule such that this set of tilings is exactly the set of tilings obeying the local rule. This question also has importance for physics. It seems that only sets of tilings with local rules can serve as model for the *real* quasicrystals discovered in 1984.

There are two methods for generating aperiodic (quasiperiodic) tilings: the substitution method (see, for example, (Grümbaum and Shephard, 1987; Senechal, 1995; Danzer, 1991)) and the projection method (and its modifications, see (de Bruijn, 1981; Kramer and Neri, 1984; Kramer and Schlottmann, 1989; Oguey et al., 1988; Gähler and Rhyner, 1986)). Tilings obtained by the projection method seem closer to periodic tilings. Tilings obtained by the substitution method may have more exotic structures. The Penrose tilings can be obtained by either method.

The main question of this paper is when a set of tilings obtained by the *canonical* projection method admits a local rule (see Section 3). This question has been investigated in many special cases, see, for example, (Baake et al., 1990; Burkov, 1988; Danzer, 1989; Ingersent, 1991; Katz, 1988; Klitzing et al., 1993; Socolar, 1989; Socolar, 1990). We will give a survey of known necessary and sufficient conditions for the existence of local rules. Among other things, we prove that a necessary condition is that the projection plane must be based on *algebraic irrationality*. We also describe in detail the topology of sets of tilings obtained by the projection method.

For local rules for tilings obtained by substitution methods, see (Danzer, 1991; Radin, 1994; Senechal, 1995). There are several sets of tilings obtained from noncanonical projection methods and admitting local rules, see (Baake et al., 1990, 1991; Danzer et al., 1993; Klitzing et al., 1993; Klitzing and Baake, 1994).

We will distinguish between two types of local rules, one without decoration, and one with decoration. The first type is stronger than the second, and local rules of the first type are much rarer that the second. There are many sets of tilings which admit local rules of the second type but do *not* admit any local rules of the first type.

We will introduce basic definitions about local rules and mutual local derivability in Section 2, together with examples of Penrose and Ammann local rules which illustrate the difference between the two types of local rule.

In Section 3 we will first recall the projection method and its equivalent form, the canonical cut method, following (Oguey et al., 1988). Then we describe in detail the space of these tilings and its topology, and formulate a simple criterion when two sets of tilings are mutually locally derivable.

Important results concerning local rules are surveyed in Sections 4–7.

In Section 4, we review known necessary and sufficient conditions for the existence of local rules (both types) for the sets of pentagonal tilings, or generalized Penrose tilings; including a candidate for local rules for the set of *all* pentagonal tilings. The ideas of the proofs are discussed.

In Section 5 two general sufficient conditions for the existence of local rules *after decoration* are given. This provides infinitely many local rules, enforcing quasi-periodicity, in any dimension greater than 1, and includes all previously known local rules (for canonical projection tilings). These local rules always select tilings in a single *local isomorphism class* (see Section 3.3).

In Section 6 we present Levitov's SI condition as a necessary condition for the existence of a local rule without decoration, and formulate a much stronger necessary condition about algebraicity, which is proved in Section 8.

In Section 7 we give necessary and sufficient conditions for the existence of local rules *without decoration* for the case when the superspace has dimension 4. The results are much fuller than in higher dimensional cases. An infinite series of sets of tilings, previously unknown, admitting local rules *without decoration* is given.

In Section 8 we give a proof of the fact that if the set of tilings obtained by the projection method admits a local rule without decoration, or even a *weak* local rule, then the projection plane must have algebraic slope. The proof is based on Tarski's theory of real algebras. Other related notions, weak local rules and r-volumes, are also discussed.

2. Definitions and preliminary facts

For the purpose of this paper, many definitions have stricter meaning than generally accepted.

A ***decorated polyhedron*** is a pair (P, j) where P is a polyhedron in a Euclidean space, and j is an arbitrary element, called the ***decoration*** of this polyhedron. Two decorated polyhedra are ***congruent*** if their decorations are the same and the second is an image of the first under an isometry of the Euclidean space. Two decorated polyhedra are ***t-congruent*** if their decorations are the same and the second is a translate of the first. We always distinguish between two congruent polyhedra.

A ***tiling*** of $\mathbb{R}^k$ is a family of k-dimensional polyhedra which covers $\mathbb{R}^k$ without overlaps such that up congruence there are only a finite number of polyhedra in this family. A polyhedron of this family is called a ***tile***. A tiling is ***face-to-face*** if the intersection of every two polyhedra is a common facet of lower dimension, if not empty. In this paper tilings are always assumed to be face-to-face unless otherwise stated.

A ***decorated tiling*** is a tiling whose tiles are decorated polyhedra such that up to congruence there are only a finite number of tiles. A nondecorated (or plain) tiling can be regarded as the decorated tiling with exactly one decoration.

Definition 2.1 *An **r-map** is an arbitrary collection of **decorated** polyhedra intersecting a ball of radius r, where $r \geq 0$ is a real number.*

Two r-maps are t-congruent if the second is a translate of the first and the corresponding decorations of polyhedra are the same. We are interested only in r-maps whose polyhedra fit together and cover the r-ball.

Let T be a decorated tiling and v a vertex of T. The *r-map* of T at v is the collection of decorated tiles of T intersecting the ball centered at v and of radius r. An r-map of T means any r-map of T at some vertex.

For example, a 0-map of a tiling T at a vertex v is the collection of tiles incident to this vertex. A 0-map is also called a ***vertex configuration***.

Now we can introduce the main definition:

Definition 2.2 (Levitov, 1988) *An **r-rule** is any finite set $\mathcal{A}$ of decorated r-maps. A* decorated tiling *T satisfies the r-rule $\mathcal{A}$ if every r-map of T is t-congruent to an r-map in $\mathcal{A}$.*

Remark 2.1 For the tilings in this paper, it is more convenient to consider t-congruence instead of the usual congruence.

A ***facet-configuration*** is a collection of two decorated polyhedra of the same dimension sharing a common facet of codimension 1.

Definition 2.3 *A **facet-rule** is a finite set of facet-configurations. A tiling T satisfies a facet-rule $\mathcal{A}$ if every facet-configuration of T is t-congruent to a facet-configuration in $\mathcal{A}$.*

We are interested in r-rules such that every tiling satisfying this r-rule is quasi-periodic in some sense.

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Definition 2.4 *A set $\mathcal{T}$ of decorated tilings admits a* ***local rule*** *if there is an r-rule such that $\mathcal{T}$ is exactly the set of all decorated tilings satisfying this r-rule.*

A set $\mathcal{T}$ of decorated tilings admits a ***strict local rule*** *if there is a facet-rule such that $\mathcal{T}$ is the set of all tilings satisfying this facet-rule.*

Recall that the case with "plain" tilings corresponds to the case when there is only one kind of decoration.

Certainly if a set admits a strict local rule, then it admits a local rule.

An interesting question in tiling theory is: when does a given set of (decorated) tilings admit a local rule? Usually this set of tilings is constructed by some method, say a substitution method (cf. (Grümbaum and Shephard, 1987)) or a projection method.

We can reformulate the question as follows. Let $\mathcal{T}(r)$ be the set of all tilings every r-map of which is t-congruent to an r-map of a tiling in $\mathcal{T}$. Then $\mathcal{T}$ admits a local rule if and only if there is some r such that $\mathcal{T} = \mathcal{T}(r)$.

Obviously $\mathcal{T}(r') \subset \mathcal{T}(r)$ if $r' > r$, and every $\mathcal{T}(r)$ contains $\mathcal{T}$. Hence

$$\overline{\mathcal{T}} := \bigcap_{r \in \mathbb{R}, r>0} \mathcal{T}(r)$$

contains $\mathcal{T}$. We call $\overline{\mathcal{T}}$ the ***closure*** of $\mathcal{T}$.

If $\mathcal{T}$ admits a local rule, then it is closed, i.e., $\mathcal{T} = \overline{\mathcal{T}}$.

2.1. TWO EXAMPLES

1. THE PENROSE TILINGS: The best known examples of quasiperiodic tilings are the Penrose tilings (cf. Penrose, 1978; de Bruijn, 1981).

(a) *Decorated Penrose tilings.* Let us consider the two decorated rhombi in Figure 1. The acute angles of the rhombi are $\pi/5$ and $2\pi/5$. The sides

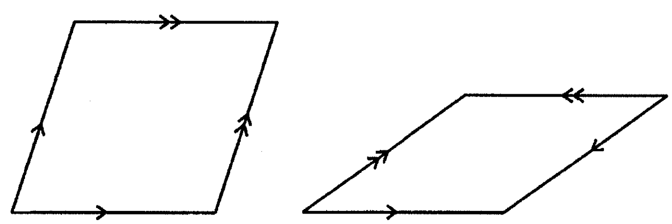

Figure 1. Building blocks of Penrose tilings

have length 1, and are equipped with single or double arrows; this information can be easily converted into the decoration of the rhombi.

Let $\mathcal{A}$ be the facet-rule consisting of all facet-configurations; each is formed by 2 copies such that arrows on the common edge are the same. Here a copy is an image of one of the two decorated rhombi by a translation and a rotation by $m\pi/5, m \in \mathbb{Z}$.

A tiling satisfying this facet-rule is called a ***Penrose tiling***. It is a nontrivial fact that the set of Penrose tilings has uncountably many elements which are pairwise noncongruent, see (Penrose, 1978). Every Penrose tiling is *aperiodic* and *quasiperiodic* in a sense which is explained later. Penrose tilings can be obtained by a substitution method (de Bruijn, 1981).

(b) *Plain Penrose tilings.* A ***plain Penrose tiling*** is a nondecorated tiling obtained from a Penrose tiling by ignoring the decoration (i.e., erasing the arrows). It is known that the set of plain Penrose tilings admits a local rule of radius 2, see (Senechal, 1995). Actually, every tiling, whose 2-maps are t-congruent to those of plain Penrose tilings, can be decorated by arrows so that the resulting decorated tilings satisfying the facet-rule described above.

Plain Penrose tilings can be obtained by the canonical projection method (de Bruijn, see below). Every plain Penrose tiling can be decorated in exactly one way to become a Penrose tiling.

2. Ammann Octagonal Tilings: These are analogs of Penrose tilings.

(a) *Decorated Ammann tilings.* Consider the decorated rhombus and the decorated square in Figure 2; the acute angle of the rhombus is $\pi/4$, and sides of the rhombus and the square have length 1.

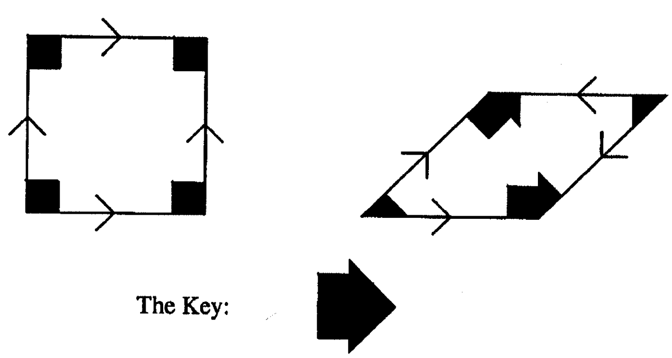

Figure 2. Building blocks of the Ammann tilings and the key

Consider the local rule consisting of 0-maps (vertex configurations) which can be formed by copies of the decorated rhombi and squares with the following constraints: matching at edges and matching at vertices. Here a copy is an image of the decorated rhombus or square under a combination of translations, mirror reflection with respect to a horizontal line, and rotations by $\pi/4$; matching at an edge means the arrows are the same; matching at a vertex means the marking at the vertex must form the key.

An ***Amman tiling*** is any tiling satisfying this 0-rule.

There are uncountably many pairwise noncongruent Ammann tilings, see (Ammann et al., 1992). All of them are aperiodic and quasi-periodic.

(b) *Plain Ammann tilings.* A ***plain Ammann tiling*** is a nondecorated tiling obtained from an Ammann tiling by ignoring the decoration. These tilings were studied in (Beenker, 1982; Burkov, 1988; Le, 1993; Socolar, 1989).

It is known that the set of all plain Ammann tilings does *not* admit any local rule, see (Burkov, 1988). Plain Ammann tilings can be obtained by the projection method (see below).

An extremely interesting fact is that there are two different Ammann tilings which are the same if the decorations are ignored; in fact, there are infinitely many such pairs. Hence some plain Ammann tilings can be decorated in many different ways (in fact 1, 2, 4 or 8 different ways, see (Le, 1993)); and the set of plain Ammann tilings which can be decorated in more than 1 way is of measure 0.

This is very different from the Penrose case: every plain Penrose tiling can be decorated in exactly one way.

2.2. LOCAL RULES AFTER DECORATING

Many tilings constructed by geometrical methods are not decorated, and since a "decorated local" rule in some sense is more powerful than a non-decorated one, we introduce the following:

Definition 2.5 *Suppose $\mathcal{T}$ is a set of* nondecorated *tilings. We say that $\mathcal{T}$ admits a **local rule after decoration** if there is a set $\mathcal{T}^c$ of decorated tilings which admits a local rule such that when ignoring the decorations, the two sets $\mathcal{T}$ and $\mathcal{T}^c$ are coincident.*

Note that there may be two different tilings in $\mathcal{T}^c$ which are the same if we ignore the decorations.

For example, as noted above, the set of plain Ammann tilings admits a local rule after decoration; but it does not admit any local rule!

Some authors do not distinguish between local rules and local rules after decoration, and the existence of a local rule *after decoration* is sometimes considered as the existence of local rules.

The existence of a local rule for a set of plain tilings is a much rarer phenomenon than the existence of a local rule after decoration.