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Local rules for multi-dimensional quasicrystals

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Abstract: We prove that quasiperiodic tilings of $\mathbb{R}^k$, obtained by the strip projection method when the linear embedding of $\mathbb{R}^k$ in $\mathbb{R}^{2k}$ has quadratic coefficients, always admit local rules with decorations.

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Introduction

Quasicrystals are quasiperiodic tilings of the Euclidean space $\mathbb{R}^k$ by a finite (up to translations) number of polyhedra. For the history and reviews we refer to [2, 6, 7]. One of the most important problems in the theory of quasicrystals is the question what kind of quasiperiodic order might appear in the Nature as a property of real materials. It seems sound to require the "physical" quasicrystals to admit restoration by means of only information of its local structure (i.e. on the finite number of admissible configurations of (possibly decorated) tiles). A well-known example of such Local (matching) Rules are the Penrose–de Bruijn arrowed rhombi. According to [2], quasiperiodic is any tiling of the plane by these rhombi, providing the obvious matching condition is met, that the common edges of neighboring rhombi are to have definite arrows on them.

The purely mathematical problem to find sets of prototiles such that every tiling of the k-dimensional Euclidean space by these prototiles must be aperiodic, or more difficult, quasiperiodic but not periodic seems interesting and investigated by many authors (cf. [1–5, 7–10, 12–19]). The term "Local Rules" was suggested by A. Katz [9] and L.S. Levitov [12] for the matching prescriptions, enforcing quasiperiodicity.

We fix standard Euclidean coordinates in $\mathbb{R}^n$. Let $\mathbf{E}$ be a k-dimensional plane in $\mathbb{R}^n$ going through the origin $\mathbf{0}$, α be a vector of $\mathbb{R}^n$. For generic α there is a quasiperiodic

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tiling T_α on $\mathbf{E}$ which appears from the so called strip projection procedure (cf. [6,18]). In our previous paper [13] it was proved that if $k = 2, n = 4$ and $\mathbf{E}$ is quadratic (that is, it is spanned by vectors with coordinates in $\mathbb{Z}\sqrt{D}$) and is totally irrational (that is, $\mathbf{E}$ dose not contain any integer points except $\mathbf{0}$) then the tiling T_α has a local rule. Here we generalize this result for multi-dimensional cases. The algorithm of construction of these LR is also presented.

The paper is organized as follows: In Section 1 the definitions and notations are introduced. In Section 2 we recall the cut method and the strip projection method and explain how to define matching rules. In Section 3 some technical results, important in our construction of LR, are proved. In Section 4 the existence of quasiperiodic LR is established.

1. Basic definitions and notations

1.1. A *colored polyhedron* is a pair (P, j) where P is a polyhedron, j is an integer, called the color of this polyhedron. Two colored polyhedra are *congruent* if their colors are the same and the second is a translate of the first. In this paper we always make distinction between two congruent polyhedra.

A *tiling of* $\mathbb{R}^k$ is a family of k-dimensional polyhedra which covers $\mathbb{R}^k$ without holes and overlaps such that up to translations there is only a finite number of polyhedra in this family. A polyhedron of a tiling is called its *tile*. A tiling is called *special* if the intersection of every two its polyhedra is a common facet of lower dimension, if not empty. In this paper tilings are always assumed to be special unless the case of family $\mathcal{O}$ and its refinements that appear later. A *colored tiling* is a tiling whose tiles are colored polyhedra such that up to congruence there is only a finite number of tiles.

An *r-map* is an arbitrary collection of colored polyhedra lying inside a ball with radius r, where r is a real number. Two r-maps are *congruent* if the second is a translate of the first and the corresponding colors of polyhedra are the same. If T is a colored tiling and v is a vertex of T then the *r-map of T at v* is the collection of colored tiles of T lying inside the ball with center at v and radius r.

A *local rule $\mathcal{A}$ of radius r* is any finite set of r-maps. A colored tiling T *satisfies* this local rule $\mathcal{A}$ if the r-map of T at every vertex is congruent to an r-map from $\mathcal{A}$.

A special type of local rule is the vertex type defined as follows. A *star* is a collection of colored polyhedra having a common vertex. Two stars are *congruent* if the second is a translate of the first and the corresponding colors are the same. A finite set $\mathcal{A}$ of stars is called a *local rule of vertex type*. In this paper we shall encounter only local rules of vertex type. A *star-configuration* of a tiling T at a vertex is the collection of all the tiles incident to this vertex. A colored tiling T *satisfies local rule $\mathcal{A}$ of vertex type* if the star-configuration of T at any vertex is congruent to one of $\mathcal{A}$.

A local rule is called quasiperiodic if it is not trivial, i.e. at least one colored tiling satisfies it, and every colored tiling satisfying it has to be quasiperiodic. The exact definition of quasiperiodicity will be given later in Section 2.1.

A family $\mathcal{U}$ of colored tilings is called to *admit a local rule* $\mathcal{A}$ if $\mathcal{U}$ is the set of all colored tilings satisfying this local rule.

1.2. In the Euclidean space $\mathbb{R}^n$ with origin $\mathbf{0}$ we fix a standard basis $\varepsilon_1, \ldots, \varepsilon_n$. Let $\mathbb{Z}^n$ be the integer lattice. For a set of vectors $v_1, \ldots, v_m$ from $\mathbb{R}^n$ let

$$\mathrm{Pol}(v_1, \ldots, v_m) = \left\{ \sum_{i=1}^{m} \lambda_i v_i \mid \lambda_i \in [0,1] \right\}.$$

The set $\gamma = \mathrm{Pol}(\varepsilon_1, \ldots, \varepsilon_n)$ is called the *unit cube.* Let M_j be the set of multi-indices $(i_1, \ldots, i_j)$ such that $1 \leqslant i_1 < i_2 < \ldots < i_j \leqslant n$. If $I \in M_j$ let I^c be the multi-index of M_{n-j} such that $I \cup I^c$ is $\{1, 2, \ldots, n\}$. For $I = (i_1, \ldots, i_j) \in M_j$ the set $\gamma_I = \mathrm{Pol}(\varepsilon_{i_1}, \ldots, \varepsilon_{i_j})$ and its translates by integer vectors (i.e. vectors from $\mathbb{Z}^n$) are called j-facets of the lattice $\mathbb{Z}^n$.

We shall always have to do with two planes: a k-dimensional plane (or briefly k-plane) $\mathbf{E}$ and an $(n-k)$-plane $\mathbf{E}'$ in $\mathbb{R}^n$ such that $\mathbf{E} \cap \mathbf{E}' = \{\mathbf{0}\}$. Denote by $\mathbf{p}$ the projector along $\mathbf{E}'$ on $\mathbf{E}$ and $\mathbf{p}'$ the projector along $\mathbf{E}$ on $\mathbf{E}'$. Put $e_i = \mathbf{p}(\varepsilon_i)$, $e'_i = \mathbf{p}'(\varepsilon_i)$, $i = 1, \ldots, n$. A set X is called an $\mathbf{E}'$*-prism* (or briefly prism) if $X = \mathbf{p}(X) + \mathbf{p}'(X)$. Let $\mathbf{E}^{\perp}$ be the $(n-k)$-plane perpendicular to $\mathbf{E}$.

A k-plane going through $\mathbf{0}$ is called *totally irrational* if there are no integer points lying on it except $\mathbf{0}$. A k-plane $\mathbf{E}$ is called *rational* (resp. *quadratic*) if it goes through $\mathbf{0}$ and is spanned by k vectors $v_1, \ldots, v_k$ with coordinates belonging to $\mathbf{Q}$ (resp. $\mathbf{Q}\sqrt{D}$, where D is a natural number). In the case $\mathbf{E}$ is quadratic, v_i must be quadratic, that is, $v_i = a_i + b_i\sqrt{D}$, where a_i and b_i are vectors with rational coordinates. Let $\bar{\mathbf{E}}$ be the plane spanned by $\bar{v}_i = a_i - b_i\sqrt{D}$, for $i = 1, \ldots, k$.

Proposition 1.1. *Suppose that* $\mathbf{E}$ *is quadratic and totally irrational. Then:*
a) $2k$ *vectors* $a_1, \ldots, a_k, b_1, \ldots, b_k$ *are linearly independent.*
b) $\dim \mathbf{E} = \dim \bar{\mathbf{E}} = k$.
c) $\mathbf{E} \cap \bar{\mathbf{E}} = \{\mathbf{0}\}$.
d) $\bar{\mathbf{E}}$ *is totally irrational.*

Proof. a) Since a_i, b_i are rational vectors, they are dependent over $\mathbb{R}$ if and only if they are dependent over $\mathbb{Q}$. Suppose there are rational numbers m_i, n_i, $i = 1, \ldots, n$ such that $\sum_{i=1}^{n}(m_i a_i + n_i b_i) = \mathbf{0}$. Consider vector $v = \sum_{i=1}^{n}(m_i\sqrt{D} + n_i)v_i$. Then $v \in \mathbf{E}$ and

$$\begin{aligned} v &= \sum_{i=1}^{n}(m_i\sqrt{D} + n_i)(a_i + b_i\sqrt{D}) \\ &= \sum_{i=1}^{n}\left(\sqrt{D}(m_i a_i + n_i b_i) + (m_i D + n_i\sqrt{D})\right) \\ &= \sum_{i=1}^{n}(m_i D b_i + n_i a_i). \end{aligned}$$

This means that v is rational, hence $m_i = n_i = 0$ for every $i = 1, \dots, n$.

b) This is a corollary of the previous.

c) If $x \in \mathbf{E} \cap \bar{\mathbf{E}}$ then x is quadratic, $x = a + b\sqrt{D}$ with a, b rational. But in this case $\bar{x} = a - b\sqrt{D}$ also belongs to $\mathbf{E} \cap \bar{\mathbf{E}}$ and so $a = (x + \bar{x})/2$ belongs to $\mathbf{E} \cap \bar{\mathbf{E}}$. It follows that $a = b = \mathbf{0}$.

d) If $v \in \bar{\mathbf{E}}$ and v is rational then $\bar{v} = v$ is rational and belongs to $\mathbf{E}$, hence $v = \mathbf{0}$. $\square$

As a corollary we see that in the condition of Proposition 1.1 we always have $n \geqslant 2k$.

Proposition 1.2. *Suppose that* $\mathbf{E}$ *is a totally irrational k-plane and* $\mathbf{E}'$ *is an* $(n-k)$*-plane with* $\mathbf{E} \cap \mathbf{E}' = \{\mathbf{0}\}$. *Then:*

a) *The projection of* $\mathbb{Z}^n$ *on* $\mathbf{E}'$ *(by* $\mathbf{p}'$*) is one-to-one.*

b) *If* F *is a rational plane containing* $\mathbf{E}$ *then* F *contains* $\mathbf{E} \oplus \bar{\mathbf{E}}$.

c) *If in addition* $n = 2k$ *then the set* $\mathbf{p}'(\mathbb{Z}^n)$ *is dense in* $\bar{\mathbf{E}}$.

The proof is trivial, and we omit it.

In the whole paper we always assume that $\mathbf{E}$ is totally irrational and quadratic.

2. The strip method, the cut method and local rules

2.1. *The strip method and the cut method*

Let us briefly recall these methods used to construct quasiperiodic tilings. The reader is referred to [6, 11, 18] for full expositions on these subjects.

Let $\mathbf{E}$ be a totally irrational k-plane in $\mathbb{R}^n$. We obtain a strip in $\mathbb{R}^n$ by shifting the cell γ along an affine k-plane parallel to $\mathbf{E}$:

$$S_\alpha = \mathbf{E} + \gamma + \alpha, \quad \alpha \in \mathbf{E}'.$$

It is proved in [18] that for translation α such that the boundary of the strip does not contain any point of $\mathbb{Z}^n$ (in this case α is called *regular*) the strip contains exactly a unique k-dimensional surface built up of k-facets of the lattice $\mathbb{Z}^n$ lying in $\mathbf{S}_\alpha$. This surface goes through all the vertices of the lattice $\mathbb{Z}^n$ falling inside $\mathbf{S}_\alpha$ and has an obvious polyhedral structure. By projecting along $\mathbf{E}^\perp$ on $\mathbf{E}$ this polyhedral structure we get a tiling T_α of $\mathbf{E}$. Note that there are no overlaps: the restriction of the projector along $\mathbf{E}^\perp$ on this surface is one-to-one. The prototiles are the projections of k-dimensional facets of the lattice $\mathbb{Z}^n$. If instead of projection along $\mathbf{E}^\perp$ we take projection along $\mathbf{E}'$, overlaps may happen. But if there are no overlaps (for example when $\mathbf{E}'$ is near to $\mathbf{E}^\perp$) we get a new quasiperiodic tiling of $\mathbf{E}$, which is topologically equivalent to the old one: only shapes of prototiles are changed, while the order of tiles is the same.

Let $\mathcal{T}_E$ be the set of all tilings of the form T_α with regular $\alpha \in \mathbf{E}'$.

If α is not regular then the family of all k-facets of $\mathbb{Z}^n$ falling in the strip $\mathbf{S}_\alpha$ does not form a proper surface. We can delete, in many way, some facets from this family such that the remaining ones form a surface which projects one-to-one by the projector along $\mathbf{E}^{\perp}$ onto $\mathbf{E}$, and a tiling of $\mathbf{E}$ is received. This tiling is called a tiling defined by α. An irregular α defines many tilings, sometime even an infinite number of tilings. But only some of them will be considered quasiperiodic.

Definition. A sequence of tilings $T_i, i = 1, \dots, m, \dots$ of $\mathbf{E}$ *converges* to a tiling T if for every $r > 0$ there exists a number N such that T_i coincides with T for $i > N$ inside the ball $\mathbf{U}_r$ with center at $\mathbf{0}$ and radius r.

Let $\overline{\mathcal{T}}_E$ be the set of all tilings which are the limits of sequences of tilings belonging to $\mathcal{T}_E$.

A tiling congruent to a tiling from $\overline{\mathcal{T}}_E$ is called a *quasiperiodic tiling associated with* $\mathbf{E}$.

In general if α is not regular then not all the tilings defined by α are quasiperiodic, but only some of them are (cf. [14]).

Let's now consider another construction of these tilings, known as the cut method [18]. Put $P_I = \mathbf{p}(\gamma_I), P_I^c = -\mathbf{p}'(\gamma_{I^c})$ and $C_I = P_I + P_I^c$, $C_{I,\xi} = C_I + \xi$, $I \in M_k$.

Each $C_{I,\xi}$ is a prism. If a k-plane $\mathbf{E} + \alpha$ intersects with a prism $C_{I,\xi}$ then the intersection is congruent to P_I. For a prism X we define $\partial^{||}(\mathrm{X}) = \mathbf{p}(X) + \partial(\mathbf{p}'(X))$ and $\partial'(\mathrm{X}) = \partial(\mathbf{p}(X)) + \mathbf{p}'(X)$ where ∂Y is the boundary of the set Y in $\mathbf{E}$ or in $\mathbf{E}'$. The sets $\partial^{||}(\mathrm{X})$ and $\partial'(\mathrm{X})$ are called the parallel and the complement boundaries of the prism X respectively. The parallel (resp. complement) boundary of a family of prisms, by definition, is the union of the parallel (resp. complement) boundary of all the prisms of this family.

Consider the family $\mathcal{O} = \{C_{I,\xi};\, I \in M_k,\, \xi \in \mathbb{Z}^n\}$. Its parallel and complement boundaries are denoted respectively by $\mathbf{B}$ and $\mathbf{B}'$. All the set $\mathcal{O}$, $\mathbf{B}$, $\mathbf{B}'$ depend on $\mathbf{E}'$ when $\mathbf{E}$ is fixed.

At first we assume that $\mathbf{E}' = \mathbf{E}^{\perp}$. Then the family $\mathcal{O}$ covers the whole $\mathbb{R}^n$ without overlaps and holes, i.e. it is a partition of $\mathbb{R}^n$ (cf. [18]). This partition in called "oblique periodic tiling" of $\mathbb{R}^n$ in [18] because it is invariant under translations from $\mathbb{Z}^n$. The union of $\binom{n}{k}$ prisms C_I, $I \in M_k$ is a fundamental domain of the group $\mathbb{Z}^n$. One can regard this union as a rearrangement of the unit cell γ. Every k-plane $\mathbf{E} + \alpha$, where $\mathbf{E} + \alpha$ does not meet $\mathbf{B}$, inherits a unique tiling from the family $\mathcal{O}$. This tiling, when projected on $\mathbf{E}$ by the projector along $\mathbf{E}^{\perp}$, is exactly the tiling T_α obtained by the strip method.

Now suppose that $\mathbf{E}^{\perp} \neq \mathbf{E}'$, then in the family $\mathcal{O}$ there may have overlaps. For every α we consider the collection of intersections of $\mathbf{E} + \alpha$ with family $\mathcal{O}$.

Theorem 2.1. *If $C \in \mathcal{O}$ is a prism such that $P = C \cap (\mathbf{E} + \alpha)$ is not empty then there is a k-facet Q of $\mathbb{Z}^n$ lying in the strip $\mathbf{S}_\alpha$ such that $\mathbf{p}(P) = \mathbf{p}(Q)$. A point $\alpha \in \mathbf{E}'$ is regular if and only if $\mathbf{E} + \alpha$ does not meet the parallel boundary $\mathbf{B}$.*

This theorem points out the equivalence between the strip method and the cut method. The proof is in fact contained in [18] although only the case $\mathbf{E}^{\perp} = \mathbf{E}'$ is considered there.

From now on we always assume that we have to do with a pair of planes $\mathbf{E}$ and $\mathbf{E}'$ such that the family $\mathcal{O}$ has no overlaps.

The set $\mathbf{Ir}$ of irregular points is $\mathbf{p}'(\mathbf{B})$ and can be described as follows. Let f_I for $I = (i_1, \ldots, i_{n-k-1}) \in M_{n-k-1}$ be the $(n-k-1)$-plane spanned by $e'_{i_1}, \ldots, e'_{i_{n-k-1}}$. Then the set of irregular points in $\mathbf{E}'$ is the union of $\binom{n}{n-k-1}$ families of parallel $(n-k-1)$-planes, each of the form $f_I + \mathbf{p}'(\mathbb{Z}^n)$ (cf. [18]). Each family is dense in $\mathbf{E}'$ but the union of its members has measure 0.

2.2. *Refinements of the family $\mathcal{O}$*

Suppose that the polyhedra P_I^c are somehow divided into smaller convex polyhedra, $P_I^c = \bigcup_{j=1}^{j(I)} P_I^{c,j}$. Then the prisms C_I are also divided into smaller prisms, $C_I = \bigcup_{j=1}^{j(I)} C_I^j$, where $C_I^j = P_I + P_I^{c,j}$. We spread out this division to other prisms of $\mathcal{O}$ by translations. Instead of $\mathcal{O}$ we get a new family $\tilde{\mathcal{O}} = \{C_I^j + \xi \mid I \in M_k,\ j = 1, \ldots, j(I),\ \xi \in \mathbb{Z}^n\}$. Let $\tilde{\mathbf{B}}$ be the parallel boundary of the new family $\tilde{\mathcal{O}}$. A point α is called regular with respect to $\tilde{\mathcal{O}}$ (or to $\tilde{\mathbf{B}}$) if $\mathbf{E}+\alpha$ does not meet $\tilde{\mathbf{B}}$. We color the prism C_I^j by color j and spread out the coloring to other prisms of $\tilde{\mathcal{O}}$ by translation. Then each regular (with respect to $\tilde{\mathcal{O}}$) α defines a colored tiling, called the quasiperiodic colored tiling defined by α.

The set of irregular points with respect to $\tilde{\mathcal{O}}$ is $\widetilde{\mathbf{Ir}} = \mathbf{p}'(\tilde{\mathbf{B}})$.

Proposition 2.2. *Suppose $\mathbf{E}'$ is totally irrational. If (P, j) is a colored polyhedron congruent to (P_I, j) and its vertices lie in $\mathbf{p}(\mathbb{Z}^n)$ then there is a unique colored prism C from the refinement $\tilde{\mathcal{O}}$ such that $\mathbf{p}(C) = P$ and the colors of C and P are the same.*

Proof. This proposition follows immediately from the construction. The uniqueness follows from the total irrationality of $\mathbf{E}'$. □

The colored prism C in this proposition is called *the lift* of the colored polyhedron P. If T is a colored tiling of $\mathbf{E}$ such that it has the same set of prototiles as T_α then after a shift we may assume that all the vertices of T are in $\mathbf{p}(\mathbb{Z}^n)$. Then all the tiles of T have lifts. The family of lifts of all the tiles of T is called the *lift* of T and will be denoted by $\mathcal{L}_T$. Sometime we also use the notion $\mathcal{L}_T$ for the union of all the prisms of family $\mathcal{L}_T$.

A *section* Ω will be referred to as a k-dimensional surface in $\mathbb{R}^n$ such that the restriction of $\mathbf{p}$ on Ω is a homeomorphism between Ω and $\mathbf{E}$. If this section does not meet the parallel boundary $\tilde{\mathbf{B}}$ of the refinement then by projecting the intersections of Ω with the family $\tilde{\mathcal{O}}$ and preserving colors we get a colored tiling of $\mathbf{E}$. The lift of this tiling is the set of all prisms from $\tilde{\mathcal{O}}$ meeting Ω. This tiling may be not quasiperiodic. But it is easy to see that every star of this tiling is the same as the star of a colored tiling T_α with an α regular with respect to $\tilde{\mathcal{O}}$.

Proposition 2.3. *Let $\mathcal{L}_T$ be the lift of a colored tiling T. Suppose that for every finite number of prisms $D_1, \ldots, D_m$ of $\mathcal{L}_T$ the polyhedra $\mathbf{p}'(D_1), \ldots, \mathbf{p}'(D_m)$ in $\mathbf{E}'$ have a* *common interior point. Then T is a quasiperiodic colored tiling, that is, when ignoring colors $T \in \bar{\mathcal{T}}_E$.*

Proof. We number the prisms of the refinement $\tilde{\mathcal{O}} = \{C_1, C_2, \ldots\}$ in such a way that for every r the disk $\mathbf{U}_r$ with center at $\mathbf{0}$ and radius r in $\mathbf{E}$ is covered by the first N prisms, here N depends on r. Because the polyhedra $\mathbf{p}'(C_1), \mathbf{p}'(C_2), \ldots, \mathbf{p}'(C_N)$ have non-empty interior intersection, the intersection of these polyhedra is an $(n-k)$-dimensional polyhedron. There is a regular (with respect to $\tilde{\mathcal{O}}$) point α_r belonging to the interior of this polyhedron. By the construction the colored tiling defined by α_r is the same as T inside $\mathbf{U}_r$. We can choose $r_1, r_2, \ldots$ which tends to infinity. Then the quasiperiodic tilings defined by α_{r_i} converge to T. □

We shall construct a local rule such that any colored tiling satisfying this local rule is a tiling defined by some section not meeting $\tilde{\mathbf{B}}$. This local rule depends on the refinement of $\mathcal{O}$. Then we shall choose a refinement of $\mathcal{O}$ such that every section not meeting the parallel boundary defines a quasiperiodic tiling (the previous proposition is a criterion). This is the second part, more difficult.

The first problem is solvable for any refinement. Suppose a refinement $\tilde{\mathcal{O}}$ is fixed. Denote the set of all star-configurations of tilings from $\mathcal{T}_E$ by $\mathcal{A}$. This set is finite up to a congruence, and it defines a local rule of vertex type. It is easy to see that if β is regular then the colored tiling T_β satisfies this local rule. The local rule $\mathcal{A}$ is called the *local rule defined by the refinement $\tilde{\mathcal{O}}$.*

Theorem 2.4. *Suppose that $\mathbf{E}'$ is totally irrational. Then every colored tiling satisfying the local rule defined by a refinement $\tilde{\mathcal{O}}$ is a tiling defined by some section not meeting the parallel boundary of this refinement.*

Proof. Suppose a colored tiling T satisfies the local rule $\mathcal{A}$, and $\mathcal{L}_T$ is its lift. Let v be a vertex of T and $P_1, \ldots, P_m$ be the tiles incident to v. Then the lifts of these tiles must have a common point $\tilde{v}$ because the collection of these lifts is a translation of the corresponding collection of lifts of a star-configuration of T_α. More precisely, the intersection is an $(n-k)$-dimensional polyhedron. We have also $\mathbf{p}(\tilde{v}) = v$. We now refine the polyhedral structure T of $\mathbf{E}$ into a simplicial structure by simply putting in some diagonal facets of every tile of T. By means of $v \to \tilde{v}$ and linearity we take $\mathbf{E}$ up into $\mathbb{R}^n$ and get a simplicial complex $\tilde{T}$ which is a k-dimensional surface lying in $\mathcal{L}_T$ and is the surface to find. □

Remark. This theorem holds true without the assumption that $\mathbf{E}'$ is totally irrational.

3. Some technical constructions

From now on we assume that $n = 2k, \mathbf{E}' = \bar{\mathbf{E}}$ and that the following conditions hold:
(*) The family $\mathcal{O}$, generated by $\mathbf{E}$ and $\bar{\mathbf{E}}$ is non-overlapping.
(**) Every k vectors from $e'_1, \ldots, e'_n$ are linearly independent.

In fact these assumptions are not essential, but without them the proof would be technically more complicated.

3.1. *Completeness*

Proposition 3.1. *If v is a rational vector then the 2-plane F spanned by $\mathbf{p}(v)$ and $\mathbf{p}'(v)$ is rational.*

Proof. Since $\mathbf{E}$ and $\bar{\mathbf{E}}$ are quadratic, both $\mathbf{p}(v)$ and $\mathbf{p}'(v)$ are also quadratic. If $\mathbf{p}(v) = a + b\sqrt{D}$ with rational vectors a, b then it is easy to check that $\mathbf{p}'(v) = a - b\sqrt{D}$. Then F is spanned by a and b and hence is rational. □

Quadraticity of $\mathbf{E}$ means the following:

Proposition 3.2. *There exist* rational *2-planes $H_1, \ldots, H_{k+2}$ in $\mathbb{R}^n$ such that:*
a) $\dim(H_i \cap \mathbf{E}) = \dim(H_i \cap \bar{\mathbf{E}}) = 1$.
b) *There are no linear transformations ϕ of $\mathbb{R}^n$ such that $\phi(H_i) = H_i$ for $i = 1, \ldots, k+2$ and $\phi(\mathbf{E}) = \bar{\mathbf{E}}$.*

Notice: Rationality of H_i is very important.

Proof. $\mathbf{E}$ is spanned by $2k$ vectors $a_i + b_i\sqrt{D}$ where a_i and b_i are rational vectors. By Proposition 1.1 the $2k$ vectors a_i, b_i are linear independent. Let H_i for $i = 1, \ldots, k$ is spanned by a_i and b_i, H_{k+1} by $\sum_{i=1}^k a_i$ and $\sum_{i=1}^k b_i$, and finally H_{k+2} by $a_1 + b_2$ and $Da_2 + b_1$. Then obviously *a*) is fulfilled. Suppose ϕ is a linear transformation satisfying the conditions of b). Then $\phi(H_i \cap \mathbf{E}) = H_i \cap \bar{\mathbf{E}}$. The set $H_i \cap \mathbf{E}$ is the line spanned by $a_i + b_i\sqrt{D}$ while $H_i \cap \bar{\mathbf{E}}$ is the line spanned by $a_i - b_i\sqrt{D}$. After rescaling we may assume that $\phi(a_1 + b_1\sqrt{D}) = (a_1 - b_1\sqrt{D})$. Let for $i = 2, \ldots, k$ $\phi(a_i + b_i\sqrt{D}) = \lambda_i(a_i - b_i\sqrt{D})$ where λ_i are real numbers. Then

$$\phi\Big(\sum_{i=1}^k (a_i + b_i\sqrt{D})\Big) = a_1 - b_1\sqrt{D} + \sum_{i=2}^k (a_i - b_i\sqrt{D}).$$

This vector must be colinear with $\sum_{i=1}^k (a_i - b_i\sqrt{D})$. It follows that $\lambda_2 = \ldots = \lambda_k = 1$. But in this case $\phi(H_{k+2} \cap \mathbf{E})$ is not $H_{k+2} \cap \bar{\mathbf{E}}$. □

A system of 2-planes $H_1, \ldots, H_m$ which contains a subset satisfying the conditions a), b) of this proposition is called a *complete system* for $\mathbf{E}$ and $\bar{\mathbf{E}}$.

Let F_i be the 2-plane spanned by $e_i = \mathbf{p}(\varepsilon_i)$ and $e_i' = \mathbf{p}'(\varepsilon_i)$ for $i = 1, 2, \ldots, k$. Then $\dim(F_i \cap \mathbf{E}) = \dim(F_i \cap \bar{\mathbf{E}}) = 1$, and each F_i is a prism. The system $F_1, F_2, \ldots, F_{2k}$ may be complete or not.

Remark. The completeness of this system is equivalent to the existence of weak local rules in the sense of Levitov (cf. [15]). Note that the "completeness property" is "open" in the sense that if $H_1, \ldots, H_{k+2}$ are complete and $H_1', \ldots, H_{k+2}'$ are 2-planes satisfying a) and very close to $H_1, \ldots, H_{k+2}$ then the system $H_1', \ldots, H_{k+2}'$ is also complete.

If the system $F_1, \ldots, F_{2k}$ is not complete, by Proposition 3.2 we can add some extra rational 2-planes $F_{2k+1}, \ldots, F_m$ such that there are $k+2$ 2-planes from $F_1, \ldots, F_m$ which form a complete system. We can choose $F_{2k+1}, \ldots, F_m$ such that for each $J = (j_1, \ldots, j_{k-1})$ with $1 \leqslant j_1 < \cdots < j_{k-1} \leqslant m$ all the 2-planes $F_{j_1}, \ldots, F_{J_{k-1}}$ are in generic position, i.e., $\dim F_{j_1} + \cdots + F_{J_{k-1}} = 2k-2$. Fix such $F_{2k+1}, \ldots, F_m$. Let $\tilde{M}$ be the set of multi-indices $(j_1, \ldots, j_{k-1})$ such that $1 \leqslant j_1 < j_2 < \ldots < j_{k-1} \leqslant m$. For each $J = (j_1, \ldots, j_{k-1}) \in \tilde{M}$ let H_J be the plane spanned by $F_{j_1}, \ldots, F_{j_{k-1}}$. The dimension of H_J is $2k-2$. Denote $h_J = \mathbf{p}(H_J)$, $h_J' = \mathbf{p}'(H_J)$. It is easy to see that each H_J is a prism and $H_J = h_J + h_J'$.

Proposition 3.3. a) *If $\mathbf{p}'(\xi) \in \mathbf{p}'(H_J)$ where $\xi \in \mathbb{Z}^n$ then $\xi \in H_J$.*

b) *If $\mathbf{p}'(\xi + H_J) = \mathbf{p}'(\eta + H_J)$ for $\xi, \eta \in \mathbb{Z}^n$ then $\xi + H_J = \eta + H_J$.*

c) *If the projections of $H_{J_1} + \xi_1, \ldots, H_{J_p} + \xi_p$ on $\bar{\mathbf{E}}$ (for $\xi_i \in \mathbb{Z}^n$) have non-empty intersection then the planes $H_{J_1} + \xi_1, \ldots, H_{J_p} + \xi_p$ also have non-empty intersection.*

Proof. a) First note that if v is a rational vector then $\mathbf{p}(v), \mathbf{p}'(v)$ are quadratic and $\mathbf{p}'(v) = \overline{\mathbf{p}(v)}$. So that $h_J' = \overline{h_J}$. If $\mathbf{p}'(\xi) \in h_J'$ then $\mathbf{p}(\xi) \in h_J$, and $\xi = \mathbf{p}'(\xi) + \mathbf{p}(\xi) \in (h_J + h_J') = H_J$.

b) This is a corollary of the previous.

c) We need only to prove that if A and B are two planes, each is a prism, $A = X + v$, $B = Y + w$ where X, Y are rational planes and v, w are rational vectors then $\mathbf{p}'(A \cap B) = \mathbf{p}'(A) \cap \mathbf{p}'(B)$. Suppose $\mathbf{p}'(A) \cap \mathbf{p}'(B) \neq \emptyset$. Since A, B are prisms we have

$$A \cap B = [\mathbf{p}(A) \cap \mathbf{p}(B)] + [\mathbf{p}'(A) \cap \mathbf{p}'(B)].$$

It suffices to prove that $\mathbf{p}(A) \cap \mathbf{p}(B) \neq \emptyset$.

$$\mathbf{p}'(A) = \mathbf{p}'(X) + \mathbf{p}'(v), \;\; \mathbf{p}'(B) = \mathbf{p}'(Y) + \mathbf{p}'(w).$$

Two planes $\mathbf{p}'(A)$ and $\mathbf{p}'(B)$ have non-empty intersection if and only if $\mathbf{p}'(v) - \mathbf{p}'(w)$ belongs to the planes $\mathbf{p}'(X) + \mathbf{p}'(Y)$. But in that case by conjugation (changing $\sqrt{D}$ to $-\sqrt{D}$ and vice-versa) we see that $\mathbf{p}(v) - \mathbf{p}(w)$ belongs to $\mathbf{p}(X) + \mathbf{p}(Y)$, i.e., $\mathbf{p}(A)$ and $\mathbf{p}(B)$ have non-empty intersection. □

3.2. *Bootstrapped property*

Definition. Three sets X, Y, Z are called *bootstrapped* if $X \cap Y = Y \cap Z = Z \cap X$.

Suppose H_1, H_2, H_3 are three rational $(2k-2)$-planes in $\mathbb{R}^n$ satisfying:

a) $\dim(H_i \cap \mathbf{E}) = \dim(H_i \cap \bar{\mathbf{E}}) = k-1$, in particular, each H_i is a prism, $i = 1, 2, 3$.

b) H_1, H_2, H_3 are bootstrapped and $H = H_1 \cap H_2$ is a $(2k-4)$-plane.

Let $H^\perp$ be the orthogonal complement of H, and Γ be the projection of $\mathbb{Z}^n$ on $H^\perp$ (taken along H). Then $H^\perp$ is a rational 4-plane and Γ is a 4-dimensional lattice in $H^\perp$.

Put

$$\Gamma_i = [\Gamma + (H_j \cap H^\perp)] \cap [\Gamma + (H_l \cap H^\perp)],$$

where (i,j,l) is a permutation of $(1,2,3)$. Then each Γ_i is a discrete lattice in $H^\perp$, and all its elements are rational. Let $\mathcal{H}_i = H_i + \Gamma_i$. This is a locally discrete family of $(2k-2)$-planes in $\mathbb{R}^n$. Here local discreteness means that every compact in $\mathbb{R}^n$ meets only a finite number of planes from this family. This follows from rationality of H_i and Γ_i. The set $\mathcal{H}_i$ obviously contains the lattice $\mathbb{Z}^n$.

Proposition 3.4. *Three sets $\mathcal{H}_i, i = 1,2,3$ are bootstrapped.*

Proof. It suffices to prove that three sets $\mathcal{H}_i \cap H^\perp$ are bootstrapped. This is the case $n = 4,\ k = 2$ which was considered in [13] and we refer the reader to that paper. □

4. Existence of local rule

4.1. *The construction of the refinement*

Consider $(2k-2)$-planes H_J. For each $J \in \tilde{M}$ there are many pairs (J_2, J_3), both are in $\tilde{M}$, such that the triple $H_1 = H_J, H_2 = H_{J_2}, H_3 = H_{J_3}$ satisfy the conditions a) and b) of Section 3.2. By the construction there we can get rational discrete groups $\Gamma_1, \Gamma_2, \Gamma_3$. Let

$$\mathcal{H}_J = \bigcup_{J_2, J_3} (H_J + \Gamma_1)$$

where J_2, J_3 belong to $\tilde{M}$ such that the triple $H_1 = H_J, H_2 = H_{J_2}, H_3 = H_{J_3}$ satisfy the conditions a) and b) of Section 3.2 and Γ_1 is the group constructed there. Put $\mathcal{H} = \bigcup_J \mathcal{H}_J$, $\Phi = \mathbf{p}'(\mathcal{H})$, $\Phi_J = \mathbf{p}'(\mathcal{H}_J)$. Each $\mathcal{H}_J$ is a locally discrete family of parallel $(2k-2)$-planes in $\mathbb{R}^n$, while Φ_J is a dense family of $(k-1)$-planes in $\bar{\mathbf{E}}$.

Now we return to the family $\mathcal{O}$ and its prisms. If $C_{I,\xi}$ is a prism of $\mathcal{O}$ where $I \in M_k$ and $\xi \in \mathbb{Z}^n$ then $\mathbf{p}'(C_{I,\xi})$ is a polyhedron in $\bar{\mathbf{E}}$. Facets of this polyhedron are $(k-1)$-dimensional polyhedra, each facet lies on some $(k-1)$-plane $\mathbf{p}'(H)$ where H is a $(2k-2)$ -plane from $\bigcup_{J \in M_{k-1}} (H_J + \mathbb{Z}^n)$. For a fixed $J \in M_{k-1}$ there are only a finite number of prisms C from $\mathcal{O}$, up to translations from $\mathbb{Z}^n$, such that one of the facets of $\mathbf{p}'(C)$ is lying on h'_J $(= \mathbf{p}'(H_J))$. Suppose $C_1, \ldots, C_p$ are representatives of these prisms: $\partial(\mathbf{p}'(C_i)) \cap h'_J$ is a $(k-1)$-dimensional polyhedron for $i = 1, 2, \ldots, p$ and if C is a prism from $\mathcal{O}$ and $\partial(\mathbf{p}'(C)) \cap h'_J$ is a $(k-1)$-dimensional polyhedron then $C = C_i + v$ for some $i \in \{1, 2, \ldots, p\}$ and $v \in (H_J \cap \mathbb{Z}^n)$. Let d_J be the maximal Hausdorff distance between $\mathbf{p}(C_i)$ and $\mathbf{p}(H_J)$ for $i = 1, 2, \ldots, p$, and $d = \max_{J \in M_{k-1}} d_J$. Let U be the ball in $\mathbf{E}$ with center at $\mathbf{0}$ and radius d. The following proposition is obvious:

Proposition 4.1. *Suppose C is a prism from $\mathcal{O}$ such that one of the facets of the projection of C on $\bar{\mathbf{E}}$ is lying in the projection of $H_J + \xi$ where $J \in \tilde{M}$ and $\xi \in \mathbb{Z}^n$*

then the distance between the projections of C and $H_J + \xi$ on $\mathbf{E}$ is less than d. Other words $\mathbf{p}(C) \cap [\mathbf{p}(H_J) + U] \neq \emptyset$. $\square$

A set $W = H + U$ where H is a $(2k-2)$-plane from $\mathcal{H}$ is called *a big wall.* Let $\mathcal{W} = \mathcal{H} + U$, this can be considered as the family of all big walls. This family of big walls is locally discrete. A big wall $W = H + U$ is contained in a unique $(2k-1)$-plane, it is $H + E$ $(= W + E)$.

Now we construct a refinement of the $\mathcal{O}$ as follow. For a fixed $I \in M_k$ the prism C_I meets only a finite number of big walls $W_1, W_2, \ldots, W_p$. The projections $\mathbf{p}'(W_i), i = 1, 2, \ldots, p$ are $(k-1)$-planes in $\bar{\mathbf{E}}$, they partition the polyhedron $\mathbf{p}'(C_I) = P_I^c$ into smaller polyhedra $P_I^{c,j}, j = 1, 2, \ldots, j(I)$. Using this partition we define a refinement $\tilde{\mathcal{O}}$ of the $\mathcal{O}$ and the corresponding local rule as in Section 2.2. The parallel boundary of the $\tilde{\mathcal{O}}$ is denoted by $\tilde{\mathbf{B}}$. A direct consequence of the construction is that all the big walls are contained in $\tilde{\mathbf{B}}$. We shall prove that the local rule thus defined always enforces quasiperiodicity. Due to Theorem 2.4 it suffices to prove that every section Ω not meeting $\tilde{\mathbf{B}}$ defines a quasiperiodic tiling.

4.2. *Orientation of hyperplanes from $\bar{\Phi}$*

From now on we fix a section Ω in $\mathbb{R}^n$ not meeting $\tilde{\mathbf{B}}$. Let $\Omega(x)$ for $x \in \mathbf{E}$ be the point of Ω lying upon x, $\Omega(x) = \Omega \cap \mathbf{p}^{-1}(x)$, and $\rho(x) = \mathbf{p}'(\Omega(x))$. We can regard Ω as the graph of the map $\rho : \mathbf{E} \to \bar{\mathbf{E}}$.

A $(k-1)$-plane h in $\bar{\mathbf{E}}$ is called oriented if one open half-space of $\bar{\mathbf{E}}$ separated by h is marked, called the positive half-space of this oriented hyperplane h. We say that a point $x \in \bar{\mathbf{E}}$ is greater than this oriented hyperplane ($x > h$) if x belongs to the positive half-space. The notion $x \geqslant h$ means $x > h$ or $x \in h$. For a set X the notion $X > h$ means $x > h$ for every $x \in X$. A family of oriented hyperplane in $\bar{\mathbf{E}}$ is *compatible* if the intersection of their positive half-spaces is not empty. A family of parallel oriented hyperplanes is said to have the same direction if the intersection of the positive half-spaces of any two of them is a half-space.

Suppose $W = H + U$ is a big wall, where H is a $(2k-2)$-plane from $\mathcal{H}$ with $h = \mathbf{p}(H), h' = \mathbf{p}'(H)$. Then $\mathbf{p}'(W) = h'$ and $\mathbf{p}(W) = h + U$. The set $\rho(h + U) = \bigcup_{x \in (h+U)} \rho(x)$ is a connected set lying in $\bar{\mathbf{E}}$.

Proposition 4.2. *The set $\rho(\mathbf{p}(W))$ does not meet the hyperplane $h' = \mathbf{p}'(W)$ for every big wall W.*

Proof. Suppose $y \in \rho(h + U) \cap h'$. Let $y = \rho(x)$, where $x \in (h + U)$. Then $x + y$ belongs to $(h + U) + h' = W$. From the other hand, $x + y = x + \rho(x)$ belongs to Ω. This is a contradiction because Ω does not meet any big wall. $\square$

The set $\rho(\mathbf{p}(W))$ is a connected set, hence it lies in one half-space of $\bar{\mathbf{E}}$ separated by $h' = \mathbf{p}'(W)$. We orient every hyperplane $h' = \mathbf{p}'(W)$ (for every big wall W) such that $\rho(\mathbf{p}(W)) > h'$. Note that by Proposition 3.3b two different big walls have different projections on $\bar{\mathbf{E}}$. By this way we orient all the hyperplanes from $\Phi = \mathbf{p}'(\mathcal{H})$.

4.3. *Boundedness of* $\rho(\mathbf{E})$

The following proposition is a generalization of an (unpublished) result of Levitov. The proof is also a generalization of Levitov's proof.

Proposition 4.3. *The map ρ is bounded: the set $\rho(\mathbf{E})$ is a bounded set in $\bar{\mathbf{E}}$.*

Proof. First note that if the distance between x and y (both are points in $\mathbf{E}$) is less than 1 then the distance between $\rho(x)$ and $\rho(y)$ is less than a constant c_1, not depending on x and y. This follows from the fact that Ω is lying inside the union of all prisms from $\mathcal{L}_T$.

Lemma 4.4. *Suppose $J \in \tilde{M}$. There is a constant c such that for every $(k-1)$-plane h in $\mathbf{E}$ parallel to h_J the set $\rho(h)$ lies in the c-neighborhood of a $(k-1)$-plane in $\bar{\mathbf{E}}$ parallel to h'_J.*

Proof. Let V be the set $\mathbf{p}'[(H_J+\mathbb{Z}^n)\cap \mathbf{p}^{-1}(h+U_{1/2}]$ where $U_{1/2}$ is the ball in $\mathbf{E}$ with center at $\mathbf{0}$ and radius $1/2$. The set V is a discrete family of parallel $(k-1)$-planes in $\bar{\mathbf{E}}$, parallel to h'_J while the set $\mathbf{p}(H_J+\mathbb{Z}^n)$ is a dense family of parallel $(k-1)$-planes in $\mathbf{E}$. It is easy to see that the distance between two neighboring hyperplanes from V is less than a constant c_2. The set $X = \rho(x+U_{1/2})$ where x is a point of h is bounded, and the diameter of X is less than $2c_1$. If $H = H_J + \xi$ where $\xi \in \mathbb{Z}^n$ such that $\mathbf{p}(H)$ meets $(x+U)$ then there is a point of X greater than $\mathbf{p}'(H)$. So that if in addition X does not meet $\mathbf{p}'(H)$ then $X > \mathbf{p}'(H)$. Let h'_1, h'_2 be two hyperplanes from V such that X lies between these hyperplanes and every hyperplane from V lying between them must meet X. The distance between h'_1 and h'_2 is less than $2c_1 + c_2$. Suppose that H_1, H_2 are two $(2k-2)$-planes from $H_J+\mathbb{Z}^n$ such that $h'_j = \mathbf{p}'(H_j)$ for j=1, 2. Let $h_j = \mathbf{p}(H_j)$. By definition $\mathbf{p}'(\rho(h_j)) > h'_j$. Since the distance between h and h_j (for $j = 1,2$) is less than 1, $\mathbf{p}'(\rho(h))$ lies between two $(k-1)$-planes l_1 and l_2 where l_1 (resp. l_2) is the $(k-1)$-plane in $\bar{\mathbf{E}}$ parallel to h'_J, lying in the unmarked half-space of h'_1 (resp. h'_2) and having distance to h'_1 (resp. h'_2) equal to c (see Fig. 1, l_1, l_2 are figured as punctured lines). The distance between l_1 and l_2 is less than $3c_1 + 2c_2$.

Proof of Proposition 4.3. From the family $F_1, F_2, \dots, F_m$ we choose $k+2$ planes which form a complete system. For simplicity assume that they are $F_1, F_2, \dots, F_{k+2}$. Let $x_i \in F_i$ be rational points, $x_i \neq 0$, $i = 1,2,\dots,k+2$. Put $v_i = \mathbf{p}(x_i)$. Then $\bar{v}_i = \mathbf{p}'(x_i)$ and F_i is spanned by $v_i, \bar{v}_i$. There must be k vectors, say $v_1, \dots, v_k$ from $\{v_1, \dots, v_{k+2}\}$ which span $\mathbf{E}$, because otherwise $F_1, \dots, F_{k+2}$ would not be complete. Suppose $v_{k+1} = \sum_{j=1}^{k} \lambda_j v_j$. Then λ_j must be quadratic numbers. By changing scale of the type $v_j \to \lambda_j v_j$ if $\lambda_j \neq 0$ and after a permutation we may assume that $v_{k+1} =$

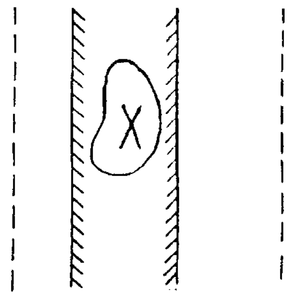

Figure 1.

$v_1+\cdots+v_p$ for some p, $1 \leqslant p \leqslant k$. In this new base suppose $v_{k+2} = \sum_{j=1}^{k} \mu_j v_j$, $\mu_j \in \mathbb{R}$. Then $\bar{v}_{k+1} = \bar{v}_1 + \cdots + \bar{v}_p$ and $\bar{v}_{k+2} = \sum_{j=1}^{k} \bar{\mu}_j \bar{v}_j$.

Completeness means that $(\mu_1 : \mu_2 : \cdots : \mu_p) \neq (\bar{\mu_1} : \bar{\mu_2} : \cdots : \bar{\mu_p})$. In fact, on the contrary, if $(\mu_1 : \mu_2 : \cdots, \mu_p) = (\bar{\mu_1} : \bar{\mu_2} : \cdots : \bar{\mu_p})$ the linear transformation ϕ defined by

$$\phi(v_j) = \begin{cases} \bar{v}_j & \text{for } j \leqslant p \text{ or } \mu_j = 0, \\ (\bar{\mu_j} \bar{v}_j)/\mu_j & \text{in other cases,} \end{cases}$$

satisfies $\phi(F_i) = F_i$ and $\phi(\mathbf{E}) = \bar{\mathbf{E}}$, a contradiction.

We suppose, for example, that $(\mu_1 : \mu_2) \neq (\bar{\mu_1} : \bar{\mu_2})$. We use $(v_1, \ldots, v_k)$ as the base of $\mathbf{E}$ and $(\bar{v_1}, \ldots, \bar{v_k})$ as the base of $\bar{\mathbf{E}}$. Every point $x \in \mathbb{R}^n$ has coordinates $(a_1, \ldots, a_k, b_1, \ldots, b_k)$. The map ρ in this coordinate systems can be written by k functions

$$\rho(a_1, \ldots, a_k) = (b_1(a_1, \ldots, a_k), \ldots, b_k(a_1, \ldots, a_k)).$$

The $(k-1)$-plane spanned by $v_2, v_3, \ldots, v_k$ is defined by the equation $a_1 = 0$, and the $(k-1)$-plane spanned by $\bar{v_2}, \bar{v_3}, \ldots, \bar{v_k}$ is defined by the equation $b_1 = 0$. By applying Lemma 4.4 to $J = (2, 3, \ldots, k)$ we see that there is a function f_1 on $\mathbf{E}$, depending only on a_1 such that $b_1(a_1, \ldots, a_k) \equiv f_1(a_1)$. Here the notation $f \equiv g$ means that $|f - g| <$ const for functions f, g on $\mathbf{E}$. Similarly for some function f_2 we have $b_2(a_1, \ldots, a_k) \equiv f_2(a_2)$.

The $(k-1)$-plane going through $v_3, v_4, \ldots, v_k, v_{k+1}$ is defined by the equation $a_1 - a_2 = 0$. By applying the lemma to $J = (3, \ldots, k+1)$ we have

$$f_1(a_1) - f_2(a_2) \equiv f(a_1 - a_2)$$

for some function f. Put $a_2 = 0$ we get $f_1(a_1) \equiv f(a_1)$, similarly $f_2(a_2) \equiv -f(-a_2)$, and so $f(a_1) + f(-a_2) \equiv f(a_1 - a_2)$ or

$$f(x+y) \equiv f(x) + f(y), \quad \text{for } x, y \in \mathbb{R}.$$

Besides if $|x - y| < 1$ then $|f(x) - f(y)| <$ const. From this and the above property it is easy to prove that for a fixed constant c we have $cf(x) \equiv f(cx)$. It follows that

$f_1(x) \equiv f(x) \equiv f_2(x)$.

The $(k-1)$-plane going through $v_3, v_4, \ldots, v_k, v_{k+2}$ is defined by the equation $\mu_2 a_1 - \mu_1 a_2 = 0$. The $(k-1)$-plane going through $\bar{v}_3, \bar{v}_4, \ldots, \bar{v}_k, \bar{v}_{k+2}$ is defined by the equation $\bar{\mu}_2 a_1 - \bar{\mu}_1 a_2 = 0$. Applying the lemma to $J = (3, 4, \ldots, k, k+2)$ we have :

$$\bar{\mu}_2 f(a_1) - \bar{\mu}_1 f(a_2) \equiv g(\mu_2 a_1 - \mu_1 a_2)$$

for some function g. Put $a_2 = 0$ we get $\bar{\mu}_2 f(a_1) \equiv g(\mu_2 a_1)$ or $(\bar{\mu}_2/\mu_2) f(\mu_2 a_1) \equiv g(\mu_2 a_1)$. Put $a_1 = 0$ we get $(\bar{\mu}_1/\mu_1) f(x) \equiv g(x)$. Hence

$$(\bar{\mu}_2/\mu_2) f(x) \equiv (\bar{\mu}_1/\mu_1) f(x).$$

Since $(\mu_1 : \mu_2) \neq (\bar{\mu}_1 : \bar{\mu}_2)$ we have $f(x) \equiv 0$ or the function f is bounded. This also means ρ is bounded. $\square$

4.4. *System Φ of hyperplanes is compatible*

Proposition 4.5. *Suppose that $h'_j = \mathbf{p}'(H_j)$ for $j = 1, 2, \ldots, p$, where H_j are $(2k-2)$-planes from $\mathcal{H}$, have common points. Then they are compatible.*

Proof. By Proposition 3.3 there is a point y belonging to all H_j. Let $x = \rho(y)$, then by definition the point $\rho(x)$ is greater than all h'_j. $\square$

Proposition 4.6. *Suppose H_1, H_2 are two $(2k-2)$-planes from $\mathcal{H}_J$ for some fixed $J \in \tilde{M}$. Then $h'_1 = \mathbf{p}'(H_1)$ and $h'_2 = \mathbf{p}'(H_2)$ are compatible.*

Proof. We consider two cases: a) H_1 is contained in $H_J + \mathbb{Z}^n$ and b) H_1 is not contained in $H_J + \mathbb{Z}^n$.

a) Suppose $H_1 = H_J + \xi$, $\xi \in \mathbb{Z}^n$, and suppose that $h'_1 = \mathbf{p}'(H_1)$ and $h'_2 = \mathbf{p}'(H_2)$ are not compatible.

By definition there are two indices $J_2, J_3 \in \tilde{M}$ such that $H_{J_1} \cap H_{J_2} = H_{J_1} \cap H_{J_3} = H_{J_2} \cap H_{J_3} = H$, here $J_1 = J$ and if Γ_1 is constructed as in Section 3.2 then there is $\eta \in \Gamma_1$ such that $H_2 = H_J + \eta$ (see Section 3.2).The groups $\Gamma_1, \Gamma_2, \Gamma_3$ are constructed as in Section 3.2. Note that since $\mathbb{Z}^n$ is in $H_J + \Gamma_1$, H_1 also is in $H_J + \Gamma_1$. From the bootstrapped property of $(H_{J_1} + \Gamma_1), (H_{J_2} + \Gamma_2), (H_{J_3} + \Gamma_3)$ it follows that there are $(k-1)$-planes from $\mathbf{p}'(H_{J_2} + \Gamma_2), \mathbf{p}'(H_{J_3} + \Gamma_3)$ located just as on Fig. 2.

This figure is written in the orthogonal complement $(\mathbf{p}'(H))^{\perp}$ of $\mathbf{p}'(H)$ in $\bar{\mathbf{E}}$ (the orthogonal plane is 2-dimensional). Here two shadowed lines are h'_1, h'_2, or more precisely the intersections of h'_1, h'_2 with the orthogonal complement $(\mathbf{p}'(H))^{\perp}$. The shadow indicates the orientation: the shadowed half-space is positive. All the other lines of this figure are intersections of $(k-1)$-planes from $\mathbf{p}'(H_{J_2} + \Gamma_2), \mathbf{p}'(H_{J_3} + \Gamma_3)$ with the orthogonal complement $(\mathbf{p}'(H))^{\perp}$. All the intersection points here are triple, this is the bootstrapped property. Through point A_0 there is a line from $\mathbf{p}'(H_{J_2} + \Gamma_2)$, and there are two possibilities of its orientation as indicated in Fig. 3.

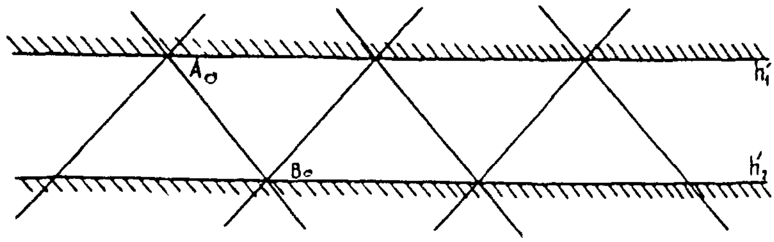

Figure 2.

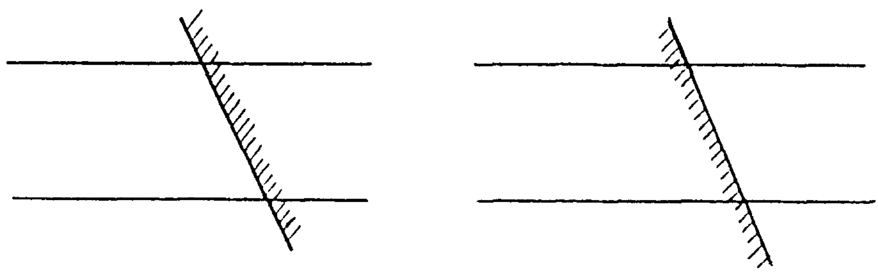

Figure 3.

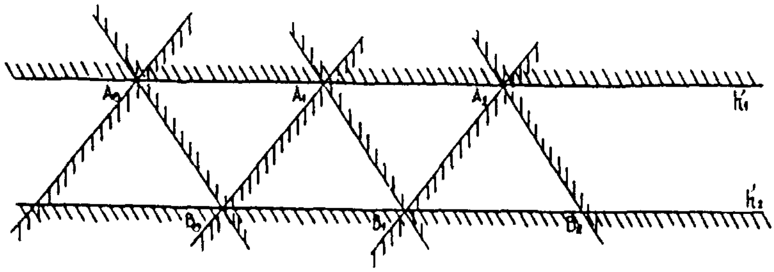

Figure 4.

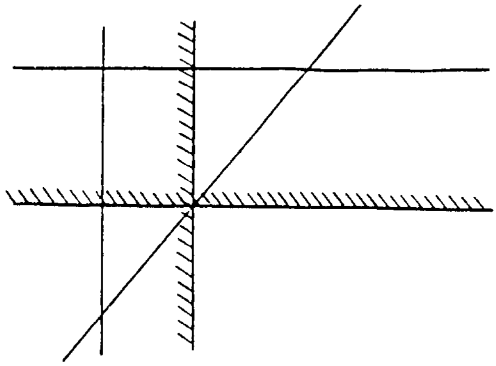

Figure 5.

We consider only the first case because the second is quite similar. In this case by Proposition 4.5 the orientation of the line from $\mathbf{p}'(H_{J_3} + \Gamma_3)$ can be found and must look lie in Fig. 4, because the three lines going through A_0 have to be compatible.

By using this proposition again one sees easily that all the lines going through A_iB_i have the same direction. This would contradict to the fact that $\rho(\mathbf{E})$ is bounded.

b) We have proved that any pair of hyperplanes from $\mathbf{p}'(H_J + \mathbb{Z}^n)$ are compatible. Note that the projection $\mathbf{p}'(H_J + \mathbb{Z}^n)$ is a dense family of parallel $(k-1)$-planes. If h_1' and h_2' are not compatible then we choose a $(2k-2)$-plane from $H_J + \mathbb{Z}^n$ such that $\mathbf{p}'(H_3)$ lies between h_1' and h_2'. Then one of two pairs $(\mathbf{p}'(H_3), \mathbf{p}'(H_1)), (\mathbf{p}'(H_3), \mathbf{p}'(H_2))$ are not compatible, that contradicts the part a). □

Corollary 4.7. *For each $J \in \tilde{M}$ there is a unique $(k-1)$plane l_J in $\bar{\mathbf{E}}$ such that $l_J \geqslant \mathbf{p}'(H)$ for every $(2k-2)$-plane H from $\mathcal{H}_J$.*

Proof. Since $\rho(\mathbf{E})$ is bounded there are at least two $(k-1)$-planes from $\mathbf{p}'(\mathcal{H}_J)$ which have not the same direction. From this and the previous proposition one gets the corollary immediately. □

Proposition 4.8. *There is a point $\alpha \in \bar{\mathbf{E}}$ greater than all the $(k-1)$-planes from $\mathbf{p}'(\mathcal{H})$.*

Proof. We can choose $J_1, J_2, \ldots, J_k$ such that $h'_{J_1}, \ldots, h'_{J_k}$ lie in general position. Then k planes $l_{J_1}, l_{J_2}, \ldots, l_{J_k}$ intersect at a point α.

We shall prove that $\alpha \geqslant \mathbf{p}'(H)$ for every $(2k-2)$-plane H from $\mathcal{H}$. It suffices to consider the case when H is in $H_J + \mathbb{Z}^n$ for some $J \in \tilde{M}$ (as in the proof of Proposition 4.6b). Suppose $H = H_J + \xi$ for $\xi \in \mathbb{Z}^n$ and $J \in \tilde{M}$. Let $l_j, j = 1, \ldots, k$ be hyperplanes going through α and parallel respectively to l_{J_j}. The hyperplanes $h' = \mathbf{p}'(H), l_1, l_2, \ldots, l_k$ bound a k-simplex s where α is one of the vertices (on Fig. 5, where $k = 2$, the simplex s is a triangle).

The $(k-1)$-facet s of this simplex, opposite to the vertex α, is some $(k-1)$-dimensional simplex. Since $\mathbf{p}'(\mathbb{Z}^n)$ is dense in h' there exists a point x lying in s (see Fig. 5). Through the point x go k hyperplanes from respectively $\Phi_{J_1}, \ldots, \Phi_{J_k}$. Through x also goes h'. By applying Proposition 4.5 to these $k+1$ hyperplanes we see that the point α is greater than h'. □

4.5. *Main Theorem*

For a prism D of the refinement $\tilde{\mathcal{O}}$ its parallel boundary is $\partial \mathbf{p}'(D) + \mathbf{p}(D)$. The first term is the union of all the facets of the polyhedron $Q = \mathbf{p}'(D)$. We shall call the sum of a $(k-1)$-dimensional facet of Q and $\mathbf{p}(D)$ a *small wall* of D. Then $\tilde{\mathbf{B}}$ is the union of all the small walls. The projection of a small wall on $\bar{\mathbf{E}}$ must lie in the projection of some big wall.

Proposition 4.9. *If a small wall w of a prism D has a projection on $\bar{\mathbf{E}}$ lying in the projection of a big wall W then w and W have a non-empty intersection.*

Proof. D is a part of a prism C of the original family $\mathcal{O}$. Then $\mathbf{p}'(D)$ is a polyhedron lying in the bigger polyhedron $\mathbf{p}'(C)$, and $\mathbf{p}'(w)$ is a facet of $\mathbf{p}'(D)$. There are two cases:

a) $\mathbf{p}'(w)$ does not lie in any facet of $\mathbf{p}'(C)$. Then by the construction of the refinement this means that the big wall W must meet the prism C, and it follows that w and W have a non-empty intersection because $\mathbf{p}(w) = \mathbf{p}(C)$ has a non-empty intersection with $\mathbf{p}(W)$.

b) $\mathbf{p}'(w)$ lies in a facet s of $\mathbf{p}'(C)$. By definition of U, the sets $\mathbf{p}(C)$ and $\mathbf{p}(W)$ have a non-empty intersection. So do $\mathbf{p}(D)$ and $\mathbf{p}(W)$, because $\mathbf{p}(C) = \mathbf{p}(D)$. We conclude that $w = \mathbf{p}'(w) + \mathbf{p}(C)$ must intersect $W = \mathbf{p}'(W) + \mathbf{p}(W)$. □

Proposition 4.10. *Suppose that D is a prism of the lift $\mathcal{L}_T$. Then the polyhedron $\mathbf{p}'(D)$ contains the point α of Proposition* 4.8.

Proof. $\mathbf{p}'(D)$ is a polyhedron whose facets are lying in hyperplanes from Φ. It suffices to prove that if s is a facet of $\mathbf{p}'(D)$ and s lies in $h' = \mathbf{p}'(H)$ where H is a $(2k-2)$-plane from $\mathcal{H}$ then $\mathbf{p}'(D) \geqslant h'$. In this case $s + \mathbf{p}(D)$ is a small wall. By the previous proposition it has non-empty intersection with the big wall $H + U$. Let y be a point belonging to both $s + \mathbf{p}'(D)$ and $H + U$, and $x = \mathbf{p}(y)$. Then $\rho(x)$ is contained in the interior of $\mathbf{p}'(D)$ and $\rho(x) \geqslant h'$ by Proposition 4.2. So that $\mathbf{p}'(D) \geqslant h'$. □

Now we are ready to prove the following:

Theorem 4.11. *If Ω is a section not meeting the parallel boundary $\tilde{\mathbf{B}}$ of the refinement then the tiling T_Ω defined by Ω is a quasiperiodic associate with* $\mathbf{E}$.

Proof. If the point α of Proposition 4.8 is regular then by Proposition 4.10 the tiling T_Ω must coincide with T_α, and we are done.

Suppose α is not regular. By Proposition 2.3 we need to prove that if $D_1, D_2, \ldots, D_p$ are prisms from the lift $\mathcal{L}_T$ then the polyhedra $\mathbf{p}'(D_1)$, $\mathbf{p}'(D_2), \ldots$, $\mathbf{p}'(D_p)$ have a common interior point. By the previous proposition α belongs to all these polyhedra. Suppose α lies on the boundary of each of $\mathbf{p}'(D_j)$ for $j = 1, 2, \ldots, q$ and α is interior point of $\mathbf{p}'(D_j)$ for $j = q+1, \ldots, p$. There are facets of $\mathbf{p}'(D_1), \ldots, \mathbf{p}'(D_q)$ on which lies α. These facets lies on hyperplanes from Φ going through α. By Proposition 4.5 these hyperplanes are compatible. The intersection of all the positive half-spaces of these hyperplanes is an open set in $\bar{\mathbf{E}}$ which contains points arbitrarily close to α (the intersection is a "corner" with vertex α). We choose a point x very close to α and belonging to this intersection. Then x is an interior point of all the polyhedra $\mathbf{p}'(D_1), \ldots, \mathbf{p}'(D_p)$. $\square$

From Theorems 2.4 and 4.11 we obtain the main result:

Theorem 4.12. *Suppose* $\mathbf{E}$ *is a totally irrational, quadratic k-dimensional plane in the Euclidean space* $\mathbb{R}^{2k}$, *equipped with a standard base, and suppose the conditions* $(*)(**)$ *are fulfilled. Then there is a colored local rule* $\mathcal{A}$ *such that*

i) *Every quasiperiodic tiling* T *associated with* $\mathbf{E}$, *that is* $T \in \bar{\mathcal{T}}_E$, *can be colored in such a way that the resulting colored tiling satisfies the local rule* $\mathcal{A}$.

ii) *Conversely, every colored tiling* T *satisfying local rule* $\mathcal{A}$ *is a quasiperiodic tiling associated with* $\mathbf{E}$ *when ignoring the color,* $T \in \bar{\mathcal{T}}_E$.

Remark. The theorem holds true without the conditions $(*)(**)$ but the proof in this case is technically more complicated.

An example is the case of 3-dimensional Penrose tilings with icosahedral symmetry (cf. [9]). In this case $k = 3$, $n = 6$, $\mathbf{E}$ is totally irrational and quadratic. Moreover, $\mathbf{E}' = \mathbf{E}^{\perp}$ and conditions $(*)(**)$ are fulfilled. Hence the class of all 3-dimensional Penrose tilings with icosahedral symmetry admits local rule (with decoration) of vertex type.

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