ON THE AJ CONJECTURE FOR KNOTS

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Abstract. We confirm the AJ conjecture [Ga2] that relates the A-polynomial and the colored Jones polynomial for hyperbolic knots satisfying certain conditions. In particular, we show that the conjecture holds true for some classes of two-bridge knots and pretzel knots. This extends the result of the first author in [Le2] where he established the AJ conjecture for a large class of two-bridge knots, including all twist knots. Along the way, we explicitly calculate the universal SL_2-character ring of the knot group of the $(-2,3,2n+1)$-pretzel knot and show that it is reduced for all integers n.

0. Introduction

0.1. **The AJ conjecture.** For a knot K in S^3, let $J_K(n) \in \mathbb{Z}[t^{\pm 1}]$ be the colored Jones polynomial of K colored by the (unique) n-dimensional simple representation of sl_2 [Jo, RT], normalized so that for the unknot U,

$$J_U(n) = [n] := \frac{t^{2n} - t^{-2n}}{t^2 - t^{-2}}.$$

The color n can be assumed to take negative integer values by setting $J_K(-n) = -J_K(n)$. In particular, $J_K(0) = 0$. It is known that $J_K(1) = 1$, and $J_K(2)$ is the ordinary Jones polynomial.

Define two linear operators L, M acting on the set of discrete functions $f : \mathbb{Z} \to \mathcal{R} := \mathbb{C}[t^{\pm 1}]$ by

$$(Lf)(n) := f(n+1), \qquad (Mf)(n) := t^{2n} f(n).$$

It is easy to see that $LM = t^2 ML$. The inverse operators L^{-1}, M^{-1} are well-defined. One can consider L, M as elements of the quantum torus

$$\mathcal{T} := \mathcal{R}\langle L^{\pm 1}, M^{\pm 1}\rangle / (LM - t^2 ML),$$

which is not commutative, but almost commutative.

Let

$$\mathcal{A}_K := \{P \in \mathcal{T} \mid P J_K = 0\}.$$

It is a left ideal of $\mathcal{T}$, called the *recurrence ideal* of K. It was proved in [GL] that for every knot K, the recurrence ideal $\mathcal{A}_K$ is non-zero. Partial results were obtained earlier by Frohman, Gelca and Lofaro through their theory of non-commutative A-ideals [FGL, Ge]. An element in $\mathcal{A}_K$ is called a recurrence relation for the colored Jones polynomial of K.

The ring $\mathcal{T}$ is not a principal left ideal domain, i.e. not every left ideal of $\mathcal{T}$ is generated by one element. By adding all inverses of polynomials in t, M to $\mathcal{T}$ one gets a principal

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left ideal domain $\tilde{\mathcal{T}}$, cf. [Ga2]. Denote the generator of the extension $\tilde{\mathcal{A}}_K := \mathcal{A}_K \cdot \tilde{\mathcal{T}}$ by α_K. The element α_K can be presented in the form

$$\alpha_K(t;M,L) = \sum_{j=0}^{d} \alpha_{K,j}(t,M)\, L^j,$$

where the degree in L is assumed to be minimal and all the coefficients $\alpha_{K,j}(t,M) \in \mathbb{Z}[t^{\pm1}, M]$ are assumed to be co-prime. The polynomial α_K is defined up to a polynomial in $\mathbb{Z}[t^{\pm1}, M]$. We call α_K the *recurrence polynomial* of K.

Garoufalidis [Ga2] formulated the following conjecture (see also [FGL, Ge]).

Conjecture 1. **(AJ conjecture)** *For every knot K, $\alpha_K|_{t=-1}$ is equal to the A-polynomial, up to a factor depending on M only.*

In the definition of the A-polynomial [CCGLS], we also allow the abelian component of the character variety, see Section 2.

0.2. **Main results.** Conjecture 1 was established for a large class of two-bridge knots, including all twist knots, by the first author [Le2] using skein theory. In this paper we generalize his result as follows.

Theorem 1. *Suppose K is a knot satisfying all the following conditions:*
(i) K is hyperbolic,
(ii) the SL_2-character variety of $\pi_1(S^3 \setminus K)$ consists of two irreducible components (one abelian and one non-abelian),
(iii) the universal SL_2-character ring of $\pi_1(S^3 \setminus K)$ is reduced,
(iv) the localized skein module $\bar{\mathcal{S}}$ of $S^3 \setminus K$ is finitely generated, and
(v) the recurrence polynomial of K has L-degree greater than 1.
Then the AJ conjecture holds true for K.

For the definition of the localized skein module $\bar{\mathcal{S}}$ of $S^3 \setminus K$ in the condition (iv) of Theorem 1, see Section 3.

Theorem 2. *The following knots satisfy all the conditions (i)–(v) of Theorem 1 and hence the AJ conjecture holds true for them.*
(a) All pretzel knots of type $(-2, 3, 6n \pm 1)$, $n \in \mathbb{Z}$.
(b) All two-bridge knots for which the SL_2-character variety has exactly two irreducible components; these include

- *all double twist knots of the form $J(k,l)$ (see Figure 1) with $k \neq l$,*
- *all two-bridge knots $\mathfrak{b}(p,m)$ with $m = 3$, and*
- *all two-bridge knots $\mathfrak{b}(p,m)$ with p prime and $\gcd(\frac{p-1}{2}, \frac{m-1}{2}) = 1$.*

Here we use the notation $\mathfrak{b}(p,m)$ for two bridge knots from [BZ].

Remark 0.1. Besides the infinitely many cases of two-bridge knots listed in Theorem 2, explicit calculations seem to confirm that "most two-bridge knots" satisfy the conditions of Theorem 1 and hence AJ conjecture holds for them. In fact, among 155 $\mathfrak{b}(p,m)$ with $p < 45$, only 9 hyperbolic knots $\mathfrak{b}(15,11)$, $\mathfrak{b}(21,13)$, $\mathfrak{b}(27,5)$, $\mathfrak{b}(27,17)$, $\mathfrak{b}(27,19)$, $\mathfrak{b}(33,5)$, $\mathfrak{b}(33,13)$, $\mathfrak{b}(33,23)$, and $\mathfrak{b}(35,29)$ do not satisfy the condition (ii) of Theorem 1. Thus, the AJ conjecture holds for all two-bridge knots $\mathfrak{b}(p,m)$ with $p < 45$ except for these 9 knots.

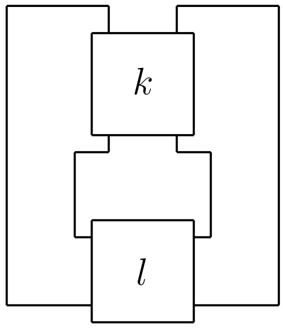

FIGURE 1. The double twist knot $J(k,l)$. Here k and l denote the numbers of half twists in the boxes. Positive (resp. negative) numbers correspond to right-handed (resp. left-handed) twists .

0.3. **Other results.** In our proof of Theorem 2, it is important to know whether the universal SL_2-character ring of a knot group is reduced, i.e. whether its nil-radical is zero. Although it is difficult to construct a group whose universal SL_2-character ring is not reduced (see [LM]), so far there are a few groups for which the universal SL_2-character ring is known to be reduced: free groups [Si], surface groups [CM, Si], two-bridge knot groups [Le2, PS], torus knot groups [Mar], two-bridge link groups [LT]. In the present paper, we show that the universal SL_2-character ring of the $(-2,3,2n+1)$-pretzel knot is reduced for all integers n.

0.4. **Plan of the paper.** We review skein modules and their relation with the colored Jones polynomial in Section 1. In Section 2 we prove some properties of the SL_2-character variety and the A-polynomial a knot. We discuss the role of localized skein modules in our approach to the AJ conjecture and give proofs of Theorems 1 and 2 in Section 3. In Section 4, we prove the reducedness of the universal SL_2-character ring of the $(-2,3,2n+1)$-pretzel knot. In Section 5, we prove that the localized skein module $\bar{\mathcal{S}}$ of the $(-2,3,2n+1)$-pretzel knot is finitely generated. Finally we study the irreducibility of non-abelian SL_2-character varieties of two-bridge knots in the appendix.

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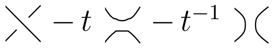
1. Skein Modules and the colored Jones polynomial

In this section we will review skein modules and their relation with the colored Jones polynomial. The theory of the Kauffman bracket skein module (KBSM) was introduced by Przytycki [Pr] and Turaev [Tu] as a generalization of the Kauffman bracket [Kau] in S^3 to an arbitrary 3-manifold. The KBSM of a knot complement contains a lot, if not all, of information about its colored Jones polynomial.

1.1. **Skein modules.** Recall that $\mathcal{R} = \mathbb{C}[t^{\pm 1}]$. A *framed link* in an oriented 3-manifold Y is a disjoint union of embedded circles, equipped with a non-zero normal vector field. Framed links are considered up to isotopy. Let $\mathcal{L}$ be the set of isotopy classes of framed links in the manifold Y, including the empty link. Consider the free $\mathcal{R}$-module with basis $\mathcal{L}$, and factor it by the smallest submodule containing all expressions of the form

 and $\bigcirc + (t^2 + t^{-2})\emptyset$, where the links in each expression are identical except in a ball in which they look like depicted. This quotient is denoted by $\mathcal{S}(Y)$ and is called the Kauffman bracket skein module, or just skein module, of Y.

For an oriented surface Σ we define $\mathcal{S}(\Sigma) := \mathcal{S}(Y)$, where $Y = \Sigma \times [0, 1]$ is the cylinder over Σ. The skein module $\mathcal{S}(\Sigma)$ has an algebra structure induced by the operation of gluing one cylinder on top of the other. The operation of gluing the cylinder over ∂Y to Y induces a $\mathcal{S}(\partial Y)$-left module structure on $\mathcal{S}(Y)$.

1.2. **The skein module of S^3 and the colored Jones polynomial.** When $Y = S^3$, the skein module $\mathcal{S}(Y)$ is free over $\mathcal{R}$ of rank one, and is spanned by the empty link. Thus if ℓ is a framed link in S^3, then its value in the skein module $\mathcal{S}(S^3)$ is $\langle \ell \rangle$ times the empty link, where $\langle \ell \rangle \in \mathcal{R}$ is the Kauffman bracket of ℓ [Kau] which is the Jones polynomial of the *framed link* ℓ in a suitable normalization.

Let $S_n(z)$'s be the Chebychev polynomials defined by $S_0(z) = 1$, $S_1(z) = z$ and $S_{n+1}(z) = zS_n(z) - S_{n-1}(z)$ for all $n \in \mathbb{Z}$. For a framed knot K in S^3 and an integer $n \geq 0$, we define the n-th power K^n as the link consisting of n parallel copies of K (this is a 0-framing cabling operation). Using these powers of a knot, $S_n(K)$ is defined as an element of $\mathcal{S}(S^3)$. We define the colored Jones polynomial $J_K(n)$ by the equation

$$J_K(n + 1) := (-1)^n \times \langle S_n(K) \rangle.$$

The $(-1)^n$ sign is added so that for the unknot U, $J_U(n) = [n]$. Then $J_K(1) = 1$, $J_K(2) = -\langle K \rangle$. We extend the definition for all integers n by $J_K(-n) = -J_K(n)$ and $J_K(0) = 0$. In the framework of quantum invariants, $J_K(n)$ is the sl_2-quantum invariant of K colored by the n-dimensional simple representation of sl_2.

1.3. **The skein module of the torus.** Let $\mathbb{T}^2$ be the torus with a fixed pair (μ, λ) of simple closed curves intersecting at exactly one point. For co-prime integers k and l, let $\lambda_{k,l}$ be a simple closed curve on the torus homologically equal to $k\mu + l\lambda$. It is not difficult to show that the skein algebra $\mathcal{S}(\mathbb{T}^2)$ of the torus is generated, as an $\mathcal{R}$-algebra, by all $\lambda_{k,l}$'s. In fact, Bullock and Przytycki [BP] showed that $\mathcal{S}(\mathbb{T}^2)$ is generated over $\mathcal{R}$ by 3 elements μ, λ and $\lambda_{1,1}$, subject to some explicit relations.

Recall that $\mathcal{T} = \mathcal{R}\langle M^{\pm 1}, L^{\pm 1} \rangle / (LM - t^2 ML)$ is the quantum torus. Let $\sigma : \mathcal{T} \to \mathcal{T}$ be the involution defined by $\sigma(M^k L^l) := M^{-k} L^{-l}$. Frohman and Gelca [FG] showed that there is an algebra isomorphism $\Upsilon : \mathcal{S}(\mathbb{T}^2) \to \mathcal{T}^\sigma$ given by

$$\Upsilon(\lambda_{k,l}) := (-1)^{k+l} t^{kl} (M^k L^l + M^{-k} L^{-l}).$$

The fact that $\mathcal{S}(\mathbb{T}^2)$ and $\mathcal{T}^\sigma$ are isomorphic algebras was also proved by Sallenave [Sa].

1.4. **The orthogonal and peripheral ideals.** Let $N(K)$ be a tubular neighborhood of an oriented knot K in S^3, and X the closure of $S^3 \setminus N(K)$. Then $\partial(N(K)) = \partial(X) = \mathbb{T}^2$. There is a standard choice of a meridian μ and a longitude λ on $\mathbb{T}^2$ such that the linking number between the longitude and the knot is zero. We use this pair (μ, λ) and the map Υ in the previous subsection to identify $\mathcal{S}(\mathbb{T}^2)$ with $\mathcal{T}^\sigma$.

The torus $\mathbb{T}^2 = \partial(N(K))$ cut S^3 into two parts: $N(K)$ and X. We can consider $\mathcal{S}(X)$ as a left $\mathcal{S}(\mathbb{T}^2)$-module and $\mathcal{S}(N(K))$ as a right $\mathcal{S}(\mathbb{T}^2)$-module. There is a bilinear bracket

$$\langle \cdot, \cdot \rangle : \mathcal{S}(N(K)) \otimes_{\mathcal{S}(\mathbb{T}^2)} \mathcal{S}(X) \to \mathcal{S}(S^3) \equiv \mathcal{R}$$

given by $\langle \ell', \ell'' \rangle := \langle \ell' \cup \ell'' \rangle$, where ℓ' and ℓ'' are links in respectively $N(K)$ and X. Note that if $\ell \in \mathcal{S}(\mathbb{T}^2)$, then

$$\langle \ell' \cdot \ell, \ell'' \rangle = \langle \ell', \ell \cdot \ell'' \rangle.$$

In general $\mathcal{S}(X)$ does not have an algebra structure, but it has the identity element–the empty link. The map

$$\Theta : \mathcal{S}(\mathbb{T}^2) \to \mathcal{S}(X), \quad \Theta(\ell) := \ell \cdot \emptyset$$

is $\mathcal{S}(\mathbb{T}^2)$-linear. Its kernel $\mathcal{P} := \ker \Theta$ is called the *quantum peripheral ideal*, first introduced in [FGL]. In [FGL, Ge], it was proved that every element in $\mathcal{P}$ gives rise to a recurrence relation for the colored Jones polynomial.

The *orthogonal ideal* $\mathcal{O}$ in [FGL] is defined by

$$\mathcal{O} := \{\ell \in \mathcal{S}(\partial X) \quad | \quad \langle \ell', \Theta(\ell) \rangle = 0 \quad \text{for every } \ell' \in \mathcal{S}(N(K))\}.$$

It is clear that $\mathcal{O}$ is a left ideal of $\mathcal{S}(\partial X) \equiv \mathcal{T}^\sigma$ and $\mathcal{P} \subset \mathcal{O}$. In [FGL], $\mathcal{O}$ was called the formal ideal. According to [Le2], if $\mathcal{P} = \mathcal{O}$ for all knots then the colored Jones polynomial distinguish the unknot from other knots.

1.5. **Relation between the recurrence and orthogonal ideals.** As mentioned above, the skein algebra of the torus $\mathcal{S}(\mathbb{T}^2)$ can be identified with $\mathcal{T}^\sigma$ via the $\mathcal{R}$-algebra isomorphism Υ sending μ, λ and $\lambda_{1,1}$ to respectively $-(M + M^{-1}), -(L + L^{-1})$ and $t(ML + M^{-1}L^{-1})$.

Proposition 1.1. *One has*

$$(-1)^n \langle S_{n-1}(\lambda), \Theta(\ell) \rangle = \Upsilon(\ell) J_K(n)$$

for all $\ell \in \mathcal{S}(\mathbb{T}^2)$.

Proof. We know from the properties of the Jones-Wenzl idempotent (see e.g. [Oh]) that

$$\begin{aligned}
\langle S_{n-1}(\lambda) \cdot \mu, \emptyset \rangle &= (t^{2n} + t^{-2n}) \langle S_{n-1}(\lambda), \emptyset \rangle \\
\langle S_{n-1}(\lambda) \cdot \lambda, \emptyset \rangle &= \langle S_n(\lambda) + S_{n-2}(\lambda), \emptyset \rangle \\
\langle S_{n-1}(\lambda) \cdot \lambda_{1,1}, \emptyset \rangle &= -\langle t^{2n+1} S_n(\lambda) + t^{-2n+1} S_{n-2}(\lambda), \emptyset \rangle.
\end{aligned}$$

By definition $J_K(n) = (-1)^{n-1} \langle S_{n-1}(\lambda), \emptyset \rangle$ and $(MJ_K)(n) = t^{2n} J_K(n)$, $(LJ_K)(n) = J_K(n+1)$. Hence the above equations can be rewritten as

$$\begin{aligned}
(-1)^n \langle S_{n-1}(\lambda), \Theta(\mu) \rangle &= -(M + M^{-1}) J_K(n) = \Upsilon(\mu) J_K(n), \\
(-1)^n \langle S_{n-1}(\lambda), \Theta(\lambda) \rangle &= -(L + L^{-1}) J_K(n) = \Upsilon(\lambda) J_K(n), \\
(-1)^n \langle S_{n-1}(\lambda), \Theta(\lambda_{1,1}) \rangle &= t(ML + M^{-1}L^{-1}) J_K(n) = \Upsilon(\lambda_{1,1}) J(n).
\end{aligned}$$

Since $\mathcal{S}(\mathbb{T}^2)$ is generated by μ, λ and $\lambda_{1,1}$, we conclude that

$$(-1)^n \langle S_{n-1}(\lambda), \Theta(\ell) \rangle = \Upsilon(\ell) J_K(n)$$

for all $\ell \in \mathcal{S}(\mathbb{T}^2)$. □

Corollary 1.2. *One has* $\mathcal{O} = \mathcal{A}_K \cap \mathcal{T}^\sigma$.

Proof. Since $\{S_n(\lambda)\}_n$ generates the skein module $\mathcal{S}(N(K))$, Proposition 1.1 implies that

$$\begin{aligned}
\mathcal{O} &= \{\ell \in \mathcal{S}(\partial X) \ | \ \langle \ell', \Theta(\ell) \rangle = 0 \quad \text{for every } \ell' \in \mathcal{S}(N(K))\} \\
&= \{\ell \in \mathcal{S}(\partial X) \ | \ \langle S_n(\lambda), \Theta(\ell) \rangle = 0 \quad \text{for all integers } n\} \\
&= \{\ell \in \mathcal{S}(\partial X) \ | \ \Upsilon(\ell) J_K(n) = 0 \quad \text{for all integers } n\}.
\end{aligned}$$

Hence $\mathcal{O} = \mathcal{A}_K \cap \mathcal{T}^\sigma$. □

Remark 1.3. Corollary 1.2 was already obtained in [Ga1] by another method. Our proof uses the properties of the Jones-Wenzl idempotent only.

2. Character varieties and the A-polynomial

For non-zero $f, g \in \mathbb{C}[M, L]$, we say that f is *M-essentially equal to* g, and write $f \stackrel{M}{=} g$, if the quotient f/g does not depend on L. We say that f is *M-essentially divisible by g* if f is M-essentially equal to a polynomial divisible by g.

2.1. **The character variety of a group.** The set of representations of a finitely presented group G into $SL_2(\mathbb{C})$ is an algebraic set defined over $\mathbb{C}$, on which $SL_2(\mathbb{C})$ acts by conjugation. The set-theoretic quotient of the representation space by that action does not have good topological properties, because two representations with the same character may belong to different orbits of that action. A better quotient, the algebro-geometric quotient denoted by $\chi(G)$ (see [CS1, LM]), has the structure of an algebraic set. There is a bijection between $\chi(G)$ and the set of all characters of representations of G into $SL_2(\mathbb{C})$, hence $\chi(G)$ is usually called the *character variety* of G. For a manifold Y we use $\chi(Y)$ also to denote $\chi(\pi_1(Y))$.

Suppose $G = \mathbb{Z}^2$, the free abelian group with two generators. Every pair of generators μ, λ will define an isomorphism between $\chi(G)$ and $(\mathbb{C}^*)^2/\tau$, where $(\mathbb{C}^*)^2$ is the set of non-zero complex pairs (L, M) and τ is the involution $\tau(M, L) := (M^{-1}, L^{-1})$, as follows: Every representation is conjugate to an upper diagonal one, with M and L being the upper left entry of μ and λ respectively. The isomorphism does not change if one replaces (μ, λ) with (μ^{-1}, λ^{-1}).

2.2. **The universal character ring.** For a finitely presented group G, the character variety $\chi(G)$ is determined by the traces of some fixed elements $g_1, \cdots, g_k$ in G. More precisely, one can find $g_1, \cdots, g_k$ in G such that for every element $g \in G$ there exists a polynomial $\mathbf{P}_g$ in k variables such that for any representation $r : G \to SL_2(\mathbb{C})$ one has $\operatorname{tr}(r(g)) = \mathbf{P}_g(x_1, \cdots, x_k)$ where $x_j := \operatorname{tr}(r(g_j))$. The universal character ring of G is then defined to be the quotient of the ring $\mathbb{C}[x_1, \cdots, x_k]$ by the ideal generated by all expressions of the form $\operatorname{tr}(r(v)) - \operatorname{tr}(r(w))$, where v and w are any two words in $g_1, \cdots, g_k$ which are equal in G. The universal character ring of G is actually independent of the choice of $g_1, \cdots, g_k$. The quotient of the universal character ring of G by its nil-radical is equal to the ring of regular functions on the character variety of G.

The universal character ring defined here is the skein algebra of G of [PS], where it is proved that it is $TH(G)$ of Brumfiel-Hilden's book [BH]. They prove that it is the universal character ring, which is defined as the coefficient algebra of the universal representation.

2.3. **The A-polynomial.** Let X be the closure of S^3 minus a tubular neighborhood $N(K)$ of a knot K. The boundary of X is a torus whose fundamental group is free abelian of rank two. An orientation of K will define a unique pair of an oriented meridian and an oriented longitude such that the linking number between the longitude and the knot is zero, as in Subsection 1.4. The pair provides an identification of $\chi(\partial X)$ and $(\mathbb{C}^*)^2/\tau$ which actually does not depend on the orientation of K.

The inclusion $\partial X \hookrightarrow X$ induces the restriction map

$$\rho : \chi(X) \longmapsto \chi(\partial X) \equiv (\mathbb{C}^*)^2/\tau$$

Let Z be the image of ρ and $\hat{Z} \subset (\mathbb{C}^*)^2$ the lift of Z under the projection $(\mathbb{C}^*)^2 \to (\mathbb{C}^*)^2/\tau$. The Zariski closure of $\hat{Z} \subset (\mathbb{C}^*)^2 \subset \mathbb{C}^2$ in $\mathbb{C}^2$ is an algebraic set consisting of components of dimension 0 or 1. The union of all the one-dimension components is defined by a single polynomial $A_K \in \mathbb{Z}[M, L]$, whose coefficients are co-prime. Note that A_K is defined up to ± 1. We call A_K the *A-polynomial* of K. By definition, A_K does not have repeated factors. It is known that A_K is always divisible by $L - 1$. The A-polynomial here is actually equal to $L - 1$ times the A-polynomial defined in [CCGLS].

2.4. **The B-polynomial.** It is also instructive to see the dual picture in the construction of the A-polynomial. For an algebraic set V (over $\mathbb{C}$) let $\mathbb{C}[V]$ denote the ring of regular functions on V. For example, $\mathbb{C}[(\mathbb{C}^*)^2/\tau] = \mathfrak{t}^\sigma$, the σ-invariant subspace of $\mathfrak{t} := \mathbb{C}[L^{\pm 1}, M^{\pm 1}]$, where $\sigma(M^k L^l) = M^{-k} L^{-l}$.

The map ρ in the previous subsection induces an algebra homomorphism

$$\theta : \mathbb{C}[\chi(\partial X)] \equiv \mathfrak{t}^\sigma \longrightarrow \mathbb{C}[\chi(X)].$$

We call the kernel $\mathfrak{p}$ of θ the *classical peripheral ideal*; it is an ideal of $\mathfrak{t}^\sigma$. We have the exact sequence

$$0 \to \mathfrak{p} \to \mathfrak{t}^\sigma \xrightarrow{\theta} \mathbb{C}[\chi(X)]. \tag{2.1}$$

The ring $\mathfrak{t}^\sigma \subset \mathfrak{t} = \mathbb{C}[M^{\pm 1}, L^{\pm 1}]$ embeds naturally into the principal ideal domain $\tilde{\mathfrak{t}} := \mathbb{C}(M)[L^{\pm 1}]$, where $\mathbb{C}(M)$ is the fractional field of $\mathbb{C}[M]$. The ideal extension $\tilde{\mathfrak{p}} := \tilde{\mathfrak{t}}\,\mathfrak{p}$ of $\mathfrak{p}$ in $\tilde{\mathfrak{t}}$ is thus generated by a single polynomial $B_K \in \mathbb{Z}[M, L]$ which has co-prime coefficients and is defined up to a factor $\pm M^k$ with $k \in \mathbb{Z}$. Again B_K can be chosen to have integer coefficients because everything can be defined over $\mathbb{Z}$. We call B_K the *B-polynomial* of K.

2.5. **Relation between the A-polynomial and B-polynomial.** From the definitions one has immediately that the polynomial B_K is M-essentially divisible by A_K. Moreover, their zero sets $\{B_K = 0\}$ and $\{A_K = 0\}$ are equal, up to some lines parallel to the L-axis in the LM-plane.

Lemma 2.1. *The field $\mathbb{C}(M)$ is a flat $\mathbb{C}[M^{\pm 1}]^\sigma$-algebra, and $\tilde{\mathfrak{t}} = \mathfrak{t}^\sigma \otimes_{\mathbb{C}[M^{\pm 1}]^\sigma} \mathbb{C}(M)$.*

Proof. The extension from $\mathbb{C}[M^{\pm 1}]^\sigma$ to $\mathbb{C}(M)$ can be done in two steps: The first one is from $\mathbb{C}[M^{\pm 1}]^\sigma$ to $\mathbb{C}[M^{\pm 1}]$ (note that $\mathbb{C}[M^{\pm 1}]$ is free over $\mathbb{C}[M^{\pm 1}]^\sigma$ since $\mathbb{C}[M^{\pm 1}] = \mathbb{C}[M^{\pm 1}]^\sigma \oplus M\mathbb{C}[M^{\pm 1}]^\sigma$); the second step is from $\mathbb{C}[M^{\pm 1}]$ to its field of fractions $\mathbb{C}(M)$. Each step is a flat extension, hence $\mathbb{C}(M)$ is flat over $\mathbb{C}[M^{\pm 1}]^\sigma$.

It follows that the extension $(\mathfrak{t}^\sigma \hookrightarrow \mathfrak{t}) \otimes \mathbb{C}(M)$ is still an injection, i.e.

$$\psi : \mathfrak{t}^\sigma \otimes_{\mathbb{C}[M^{\pm 1}]^\sigma} \mathbb{C}(M) \to \mathfrak{t} \otimes_{\mathbb{C}[M^{\pm 1}]} \mathbb{C}(M) = \tilde{\mathfrak{t}}, \quad \psi(x \otimes y) = xy,$$

is injective. Let us show that ψ is surjective. For every $n \in \mathbb{Z}$,

$$L^n \;=\; \psi\left((ML^n + M^{-1}L^{-n}) \otimes \frac{1}{M - M^{-1}} - (L^n + L^{-n}) \otimes \frac{M^{-1}}{M - M^{-1}} \right).$$

Since $\{L^n \mid n \in \mathbb{Z}\}$ generates $\tilde{\mathfrak{t}} = \mathbb{C}(M)[L^{\pm 1}]$, ψ is surjective. Thus ψ is an isomorphism. $\square$

Consider the exact sequence (2.1). The ring $\mathbb{C}[\chi(X)]$ has a $\mathfrak{t}^\sigma$-module structure via the algebra homomorphism $\theta : \mathbb{C}[\chi(\partial X)] \equiv \mathfrak{t}^\sigma \to \mathbb{C}[\chi(X)]$, hence a $\mathbb{C}[M^{\pm1}]^\sigma$-module structure since $\mathbb{C}[M^{\pm1}]^\sigma$ is a subring of $\mathfrak{t}^\sigma$. By Lemma 2.1, $\tilde{\mathfrak{t}} = \mathfrak{t}^\sigma \otimes_{\mathbb{C}[M^{\pm1}]^\sigma} \mathbb{C}(M)$. It follows that $\tilde{\mathfrak{p}} = \mathfrak{p} \otimes_{\mathbb{C}[M^{\pm1}]^\sigma} \mathbb{C}(M)$. Hence by taking the tensor product over $\mathbb{C}[M^{\pm1}]^\sigma$ of the exact sequence (2.1) with $\mathbb{C}(M)$, we get the exact sequence

$$0 \to \tilde{\mathfrak{p}} \to \tilde{\mathfrak{t}} \xrightarrow{\tilde{\theta}} \widetilde{\mathbb{C}[\chi(X)]}, \tag{2.2}$$

where $\widetilde{\mathbb{C}[\chi(X)]} := \mathbb{C}[\chi(X)] \otimes_{\mathbb{C}[M^{\pm1}]^\sigma} \mathbb{C}(M)$.

Proposition 2.2. *The B-polynomial B_K does not have repeated factors.*

Proof. We want to show that $\tilde{\mathfrak{p}}$ is radical, i.e. $\sqrt{\tilde{\mathfrak{p}}} = \tilde{\mathfrak{p}}$. Here $\sqrt{\tilde{\mathfrak{p}}}$ denotes the radical of $\tilde{\mathfrak{p}}$.

Let $x := M + M^{-1}$ and

$$\underline{\mathfrak{t}} := \mathfrak{t}^\sigma \otimes_{\mathbb{C}[M^{\pm1}]^\sigma} \mathbb{C}(x), \quad \underline{\mathfrak{p}} := \mathfrak{p} \otimes_{\mathbb{C}[M^{\pm1}]^\sigma} \mathbb{C}(x).$$

Note that $\mathfrak{p}$, the kernel of $\theta : \mathfrak{t}^\sigma \to \mathbb{C}[\chi(X)]$, is radical since the ring $\mathbb{C}[\chi(X)]$ is reduced. We claim that $\underline{\mathfrak{p}}$ is also radical. Indeed, suppose $\gamma \in \underline{\mathfrak{t}}$ and $\gamma^2 \in \underline{\mathfrak{p}}$. Then $\gamma^2 = \delta/f$ for some $\delta \in \mathfrak{p}$ and $f \in \mathbb{C}[x]$. It follows that $(f\gamma)^2 = f\delta$ is in $\mathfrak{p}$. Hence $f\gamma \in \sqrt{\mathfrak{p}} = \mathfrak{p}$ which means $\gamma \in \underline{\mathfrak{p}}$.

Since $\underline{\mathfrak{t}} = \mathbb{C}(x)[L^{\pm1}]$ is a principal ideal domain, the radical ideal $\underline{\mathfrak{p}}$ can be generated by one element, say $\gamma(L) \in \mathbb{C}(x)[L^{\pm1}]$, which does not have repeated factors. Note that the polynomial $\gamma(L)$ and $\delta(L) := \gamma'(L)$, the derivative of $\gamma(L)$ with respect to L, are co-prime. Since $\mathbb{C}(x)[L^{\pm1}]$ is an Euclidean domain, there are $f, g \in \mathbb{C}(x)$ such that $f\gamma + g\delta = 1$. It follows that $\gamma(L)$ and $\delta(L)$ are also co-prime in $\mathbb{C}(M)[L^{\pm1}]$. Hence the ideal $\tilde{\mathfrak{p}} = \underline{\mathfrak{p}} \otimes_{\mathbb{C}(x)} \mathbb{C}(M)$ in $\mathbb{C}(M)[L^{\pm1}]$ is radical. This means that the B-polynomial B_K does not have repeated factors. □

Corollary 2.3. *For every knot K one has*

$$B_K = \frac{A_K}{\text{its } M\text{-factor}}.$$

Here the M-factor of A_K is the maximal factor of A_K depending on M only; it is defined up to a non-zero complex number.

2.6. **Small knots.** A knot K is called *small* if its complement X does not contain closed essential surfaces. It is known that all two-bridge knots and all three-tangle pretzel knots are small [HT, Oe].

Proposition 2.4. *Suppose K is a small knot. Then the A-polynomial A_K has trivial M-factor. Hence the A-polynomial and B-polynomial of a small knot are equal.*

Proof. The A-polynomial A_K always contains the factor $L - 1$ coming from characters of abelian representations [CCGLS]. Hence we write $A_K = (L-1)A_{nab}$ where A_{nab} is a polynomial in $\mathbb{C}[M, L]$.

Suppose the polynomial A_{nab} of a knot has non-trivial M-factor, then the Newton polygon of A_{nab} has the slope infinity. It is known that every slope of the Newton polygon of A_{nab} is a boundary slope of the knot complement [CCGLS]. Hence the knot complement has boundary slope infinity. The complement of a small knot in S^3 does not have boundary slope infinity (this fact follows easily from [CGLS, Theorem 2.0.3]), hence its polynomial A_{nab} has trivial M-factor. □

Remark 2.5. By [IMS], according to a calculation by Culler, there exists a non-small knot whose A-polynomial has non-trivial M-factor; it is the knot 9_{38} in the Rolfsen table.

3. Skein modules and the AJ conjecture

Our proofs of the main theorems are based on the ideology that the KBSM is a quantization of the $SL_2(\mathbb{C})$-character variety [Bul, PS] which has been exploited in the work of Frohman, Gelca and Lofaro [FGL] where they defined the non-commutative A-ideal. In this section we will discuss the role of localized skein modules in our approach to the AJ conjecture, and then prove Theorems 1 and 2.

3.1. **Ring extensions.** Suppose R_1 is a ring (with unit), $\mathcal{C}$ is an R_1-complex, and R_2 is an R_1-algebra. We will say that $R_2 \otimes_{R_1} \mathcal{C}$ is obtained from $\mathcal{C}$ by a change of ground ring.

Recall that $\mathcal{R} = \mathbb{C}[t^{\pm 1}]$. We often consider $\mathbb{C}$ as an $\mathcal{R}$-algebra by $\mathbb{C} \equiv \mathcal{R}/((1+t))$. In this case, we use the notation $\varepsilon(\mathcal{C}) := \mathcal{C} \otimes_{\mathcal{R}} \mathbb{C}$, where $\mathcal{C}$ is an $\mathcal{R}$-complex or an $\mathcal{R}$-module. Thus if $\mathcal{M}$ is an $\mathcal{R}$-module, then

$$\varepsilon(\mathcal{M}) = \mathcal{M}/((1+t)\mathcal{M}).$$

If $\hat{\mathcal{R}}$ is an $\mathcal{R}$-algebra and $\mathcal{M}$ is an $\hat{\mathcal{R}}$-module, then one can easily see that

$$\varepsilon(\mathcal{M}) = \mathcal{M} \otimes_{\hat{\mathcal{R}}} \varepsilon(\hat{\mathcal{R}}).$$

3.2. **Skein modules as quantizations of character varieties.** An important result [Bul, PS] in the theory of skein modules is that $\mathfrak{s}(Y) := \varepsilon(\mathcal{S}(Y))$, the skein module at $t = -1$, has a natural $\mathbb{C}$-algebra structure and is isomorphic to the universal SL_2-character algebra $\mathbb{C}^{\rm univ}[\chi(Y)]$ of $\pi_1(Y)$. The product of two links in $\mathfrak{s}(Y)$ is their disjoint union, which is well-defined when $t = -1$. The isomorphism between $\mathfrak{s}(Y)$ and the universal SL_2-character algebra of $\pi_1(Y)$ is given by $K(r) = -\operatorname{tr} r(K)$, where K is a knot in Y representing an element of $\pi_1(Y)$, and $r : \pi_1(Y) \to SL_2(\mathbb{C})$ is a representation of $\pi_1(Y)$. The quotient of $\mathfrak{s}(Y)$ by its nilradical is canonically isomorphic to $\mathbb{C}[\chi(Y)]$, the ring of regular functions on the SL_2-character variety of $\pi_1(Y)$. For the case when $\mathcal{S} = \mathcal{S}(X)$, where X is the knot complement, we have

$$\varepsilon(\mathcal{T}^\sigma \xrightarrow{\Theta} \mathcal{S}) = (\mathfrak{t}^\sigma \xrightarrow{\theta} \mathfrak{s}), \tag{3.1}$$

where $\mathfrak{s} = \mathfrak{s}(X) = \mathbb{C}^{\rm univ}[\chi(X)]$.

In many cases $\mathfrak{s}(Y)$ is reduced, i.e. its nilradical is zero, and hence $\mathfrak{s}(Y)$ is exactly the ring of regular functions on the SL_2-character variety of $\pi_1(Y)$. For example, this is the case when Y is a torus, or when Y is the complement of a two-bridge knot/link [Le2, PS, LT], or when Y is the complement of the $(-2, 3, 2n+1)$-pretzel knot for any integer n (see Section 4 below). We conjecture that

Conjecture 2. *For every knot K in S^3, the universal SL_2-character ring of $\pi_1(S^3 \setminus K)$ is reduced.*

3.3. **The non-reduced kernel.** Extending the right hand side of (3.1) from ground ring $\mathbb{C}[M^{\pm 1}]^\sigma$ to $\mathbb{C}(M)$, we get

$$(\bar{\mathfrak{t}} \xrightarrow{\bar{\theta}} \bar{\mathfrak{s}}) := (\mathfrak{t}^\sigma \xrightarrow{\theta} \mathfrak{s}) \otimes_{\mathbb{C}[M^{\pm 1}]^\sigma} \mathbb{C}(M). \tag{3.2}$$

We call the $\mathbb{C}(M)$-vector space $\bar{\mathfrak{s}}$ the *localized universal character ring* of $\pi_1(X)$.

Let $\bar{\mathfrak{p}} := \ker \bar{\theta} \subset \bar{\mathfrak{t}} = \mathbb{C}(M)[L^{\pm 1}]$. We have the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bar{\mathfrak{p}} & \longrightarrow & \bar{\mathfrak{t}} & \xrightarrow{\bar{\theta}} & \bar{\mathfrak{s}} \\ & & \downarrow & & \| & & \downarrow q \\ 0 & \longrightarrow & \tilde{\mathfrak{p}} & \longrightarrow & \tilde{\mathfrak{t}} & \xrightarrow{\tilde{\theta}} & \bar{\mathfrak{s}}/\sqrt{0} \end{array}$$

where q is the quotient map. Note that the second row of the above diagram is exactly the sequence (2.2).

Both $\bar{\mathfrak{p}}$ and $\tilde{\mathfrak{p}}$ are ideals in the principal ideal domain $\bar{\mathfrak{t}} = \tilde{\mathfrak{t}}$. Recall that B_K is a generator of $\tilde{\mathfrak{p}}$. Let $\bar{B}_K$ be a generator of $\bar{\mathfrak{p}}$.

Lemma 3.1. *One has $B_K \mid \bar{B}_K \mid (B_K)^l$ for some positive integer l. Consequently, $\bar{\mathfrak{t}}/\bar{\mathfrak{p}}$ is a finite dimensional $\mathbb{C}(M)$-vector space.*

Proof. Since $\bar{\theta}(\bar{B}_K) = 0$, one has $\tilde{\theta}(\bar{B}_K) = 0$. This implies $\bar{B}_K \in \tilde{\mathfrak{p}}$, and hence $B_K \mid \bar{B}_K$.

Since $\tilde{\theta}(B_K) = 0$, one has $\bar{\theta}(B_K) \in \sqrt{0}$. It follows that $(B_K)^l \in \bar{\mathfrak{p}}$ for some positive integer l, and hence $\bar{B}_K \mid (B_K)^l$.

Note that $B_K \neq 0$ (since $A_K \neq 0$), hence we also have $\bar{B}_K \neq 0$. If $\bar{B}_K = L^d + \sum_{j=0}^{d-1} a_j(M) L^j$, with $a_j(M) \in \mathbb{C}(M)$ and $d \geq 0$, then the dimension of the $\mathbb{C}(M)$-vector space $\bar{\mathfrak{t}}/\bar{\mathfrak{p}}$ is d. □

3.4. **Localization.** Let $D := \mathcal{R}[M^{\pm 1}] = \mathbb{C}[t^{\pm 1}, M^{\pm 1}]$ and $\bar{D}$ be its localization at $(1+t)$:

$$\bar{D} := \left\{ \frac{f}{g} \mid f, g \in D,\ g \notin (1+t)D \right\},$$

which is a discrete valuation ring and is flat over D.

The ring $D = \mathcal{R}[M^{\pm 1}]$ is flat over $D^\sigma = \mathcal{R}[M^{\pm 1}]^\sigma$, where $\sigma(M) = M^{-1}$, since it is free over $\mathcal{R}[M^{\pm 1}]^\sigma$:

$$\mathcal{R}[M^{\pm 1}] = \mathcal{R}[M^{\pm 1}]^\sigma \oplus M\, \mathcal{R}[M^{\pm 1}]^\sigma.$$

The quantum torus $\mathcal{T}$ is a D^σ-algebra. Let $\bar{\mathcal{T}} := \mathcal{T} \otimes_{D^\sigma} \bar{D}$. Similar to Lemma 2.1, we have

$$\bar{\mathcal{T}} = \left\{ \sum_{j \in \mathbb{Z}} a_j(M) L^j \mid a_j(M) \in \bar{D},\ a_j = 0 \text{ almost everywhere} \right\},$$

with commutation rule: $a(M)L^k \cdot b(M)L^l = a(M)b(t^{2k}M)L^{k+l}$.

3.5. **The localized skein module.** Let $\mathcal{S} := \mathcal{S}(X)$ be the skein module of the knot complement X.

Definition 1. The *localized skein module* of the knot complement X is the $\bar{D}$-module $\bar{\mathcal{S}} := \mathcal{S} \otimes_{D^\sigma} \bar{D}$. We say $\bar{\mathcal{S}}$ is finitely generated if it is finitely generated as a $\bar{D}$-module.

Recall from Subsection 1.4 that we have the map $\Theta : \mathcal{T}^\sigma \to \mathcal{S}$, which is considered as a D^σ-morphism. Let $\bar\Theta := \Theta \otimes_{D^\sigma} \bar D$, i.e.

$$(\bar{\mathcal{T}} \xrightarrow{\bar\Theta} \bar{\mathcal{S}}) \quad = \quad \big(\mathcal{T}^\sigma \xrightarrow{\Theta} \mathcal{S}\big) \otimes_{D^\sigma} \bar D.$$

Lemma 3.2. *One has*

$$\varepsilon(\bar{\mathcal{T}} \xrightarrow{\bar\Theta} \bar{\mathcal{S}}) = (\bar{\mathfrak{t}} \xrightarrow{\bar\theta} \bar{\mathfrak{s}}).$$

Proof. Recall $D^\sigma = \mathbb{C}[t^{\pm 1}, M^{\pm 1}]^\sigma$, and $\mathbb{C}(M)$ is a D^σ-algebra by the composition of two maps

$$D^\sigma \hookrightarrow \bar D \to \varepsilon(\bar D) = \mathbb{C}(M).$$

Hence $\otimes_{D^\sigma}\mathbb{C}(M)$ is the composition of two tensor products

$$(\mathcal{T}^\sigma \xrightarrow{\Theta} \mathcal{S}) \otimes_{D^\sigma} \mathbb{C}(M) = \varepsilon\big((\mathcal{T}^\sigma \xrightarrow{\Theta} \mathcal{S}) \otimes_{D^\sigma} \bar D\big) = \varepsilon(\bar{\mathcal{T}} \xrightarrow{\bar\Theta} \bar{\mathcal{S}}). \tag{3.3}$$

The same D^σ-algebra structure of $\mathbb{C}(M)$ can also be obtained by the composition of

$$D^\sigma \to \varepsilon(D^\sigma) = \mathbb{C}[M^{\pm 1}]^\sigma \hookrightarrow \mathbb{C}(M).$$

Hence the left hand side of (3.3) can be written as

$$(\mathcal{T}^\sigma \xrightarrow{\Theta} \mathcal{S}) \otimes_{D^\sigma} \mathbb{C}(M) = \big(\varepsilon(\mathcal{T}^\sigma \xrightarrow{\Theta} \mathcal{S})\big) \otimes_{\mathbb{C}[M^{\pm1}]^\sigma} \mathbb{C}(M) = (\bar{\mathfrak{t}} \xrightarrow{\bar\theta} \bar{\mathfrak{s}}), \tag{3.4}$$

where the last identity follows from the definitions (3.1) and (3.2).

The lemma follows from (3.3) and (3.4). □

3.6. **Left ideals of $\bar{\mathcal{T}}$.** Recall that $\bar{\mathcal{T}} = \bar D[L^{\pm 1}]$ is the set of all Laurent polynomials

$$\sum_{j\in\mathbb{Z}} a_j(M) L^j, \quad a_j(M) \in \bar D \text{ and } a_j = 0 \text{ for almost every } j \in \mathbb{Z}.$$

Let $\bar{\mathcal{T}}_+$ be the subring of $\bar{\mathcal{T}}$ consisting of all polynomials in L, i.e. polynomials like the above with $a_j(M) = 0$ if $j < 0$. For f, g in $\bar{\mathcal{T}}_+$, we say that f is divisible by g and write $g \mid f$ if there exists $h \in \bar{\mathcal{T}}_+$ such that $f = hg$.

Although the ring $\bar{\mathcal{T}}$ is not a left PID, we have the following description of its ideals.

Proposition 3.3. *Suppose $I \subset \bar{\mathcal{T}}$ is a non-zero left ideal. There are $h_0, \dots, h_{m-1} \in \bar{\mathcal{T}}_+$ with leading coefficients 1 and $\gamma \in \bar{\mathcal{T}}_+$ such that I is generated by $\{h_0\gamma, (1+t)h_1\gamma, \dots, (1+t)^{m-1}h_{m-1}\gamma, (1+t)^m\gamma\}$. Besides, $1 \le \deg_L(h_{j+1}) \le \deg_L(h_j)$ for $j = 0, \dots, m-2$; and γ is the generator of the principal left ideal $\tilde I = I \cdot \tilde{\mathcal{T}} \subset \tilde{\mathcal{T}}$.*

Proof. Note that any ideal of $\bar D$ is a power of the prime ideal $(1+t)$. Suppose $\mathcal{M}$ is a $\bar D$-module. We say that $u \in \mathcal{M}$ has *height* $k \in \mathbb{Z}_{\ge 0}$ if

$$u \in (1+t)^k \mathcal{M} \setminus (1+t)^{k+1}\mathcal{M}.$$

We have the following *weak division algorithm*: Suppose $f, g \in \bar{\mathcal{T}}_+$, with $g \neq 0$. Assume that $\deg(f) \ge \deg(g)$, and the height of the leading coefficient of f is greater than or equal to that of the leading coefficient of g. Then there are $q, r \in \bar D[L]$ such that

$$f = qg + r,$$

where $\deg(r) < \deg(f)$. Here we use the convention that $\deg(0) = -\infty$. In fact, one can take $q = a\,L^k$, where a is the quotient of the leading of f by that of g, and k is the difference between the degree of f and that of g.

We will frequently use the weak division algorithm with $f, g \in I$. Then the remainder r is also in I.

Let $I_+ = I \cap \bar{\mathcal{T}}_+$, and I_n be the set of all elements in $I_+ \setminus \{0\}$ whose leading coefficient has height n. By definition

$$I_+ \setminus \{0\} = \sqcup_{n=0}^{\infty} I_n.$$

Choose the smallest integer $N \geq 0$ such that $I_N \neq \emptyset$.

Claim 1: $I_n \neq \emptyset$ for every $n \geq N$.

Proof. (of Claim 1) Since $(1+t)^{n-N} I_N \subset I_n$ for every $n \geq N$, each $I_n \neq \emptyset$. □

Suppose d_n is the least L-degree of elements of I_n, and choose $f_n \in I_n$ such that the degree of f_n is d_n. The choice of f_n guarantees that if $f \in I_n$ than one can divide f by f_n using the weak division algorithm.

Since $(1+t)I_n \subset I_{n+1}$, we have $d_N \geq d_{N+1} \geq d_{N+2} \geq \dots$. Hence the sequence of decreasing non-negative integers $d_N, d_{N+1}, \dots$ eventually stabilizes. Let $m \geq 0$ be the smallest integer such that $d_{N+m} = d_{N+m+j}$ for every $j = 0, 1, 2, \dots$.

Claim 2: If $f \in I_+ \setminus \{0\}$ has degree $< d_j$, then $f \in I_{j'}$ for some $j' > j$.

Proof. (of Claim 2) Since $I_+ \setminus \{0\} = \sqcup_{n=0}^{\infty} I_n$, $f \in I_{j'}$ for some j'. If $j' \leq j$ then $\deg(f) \geq d_{j'} \geq d_j$, a contradiction. Hence $j' > j$. □

Claim 3: If $f \in I_+ \setminus \{0\}$ has degree $\leq d_{N+m}$, then f is divisible by f_{N+m}.

Proof. (of Claim 3) Suppose $f \in I_+ \setminus \{0\}$ has degree $\leq d_{N+m}$. Since $\deg(f) < d_{N+m-1}$, by Claim 2, $f \in I_j$ for some $j \geq N+m$. Note that $\deg(f) \geq d_j = d_{N+m}$. Dividing f by f_{N+m} using the weak division algorithm, the remainder r has degree $< \deg(f) = d_{N+m}$. Since there are no elements in $I_+ \setminus \{0\}$ of degree $< d_{N+m}$, we must have $r = 0$, which implies that f is divisible by f_{N+m}. □

For $0 \leq j \leq m$ let $I^{(j)}$ be the left ideal of $\bar{\mathcal{T}}_+$ generated by $f_{N+j}, f_{N+j+1}, \cdots, f_{N+m}$.

Claim 4: Suppose $f \in I_+ \setminus \{0\}$ has degree $< d_{N+j}$, where $0 \leq j < m$, then $f \in I^{(j+1)}$.

Proof. (of Claim 4) We use induction on the degree of f. Suppose $f \in I_+ \setminus \{0\}$ has degree $< d_{N+j}$ for $0 \leq j < m$. Then, by Claim 2, $f \in I_{j'}$ for some $j' \geq j+1$. Dividing f by $f_{N+j'}$ using the weak division algorithm, the remainder has degree $< \deg(f) < d_{N+j}$, hence, by induction, it belongs to $I^{(j+1)}$.

If $j' > m$ then, by Claim 3, $f_{j'}$ is divisible by f_{N+m}. Otherwise, i.e. if $j' \leq m$, then $f_{j'}$ belongs to $I^{(j+1)}$. Hence we always have $f_{j'} \in I^{(j+1)}$. It follows that $f \in I^{(j+1)}$. □

Claim 5: $I_+ = I^{(0)}$, i.e. I_+ is generated by $\{f_j \mid j = N, \dots, N+m\}$.

Proof. (of Claim 5) We use induction on the degree of $f \in I_+$. If the degree of $f < d_N$, then $f \in I^{(1)}$ by Claim 4. Suppose the degree of f is $\geq d_N$. Dividing f by f_N using the weak division algorithm, the remainder has degree less than that of f and hence belongs to $I^{(0)}$ by induction hypothesis. Thus $f \in I^{(0)}$. □

Claim 6: For every $0 \leq j \leq m$, f_{N+m} divides $(1+t)^{m-j} f_{N+j}$.

Proof. (of Claim 6) We use induction, beginning with the case $j = m$ which is obvious. Suppose $j \leq m-1$. Dividing $(1+t)f_{N+j}$ by f_{N+j+1} using the weak division algorithm, the remainder r is an element in I_+ of degree $< d_{N+j}$. By Claim 4, r is an element in $I^{(j+1)} = (f_{N+j+1}, \cdots, f_{N+m}) \subset I_+$. It follows that $(1+t)f_{N+j}$ is an element in $I^{(j+1)}$.

By induction hypothesis, every element in $(1+t)^{m-j-1}I^{(j+1)}$ is divisible by f_{N+m}. In particular, $(1+t)^{m-j}f_{N+j}$ is divisible by f_{N+m}. $\square$

End of Proof of Proposition 3.3. By Claim 6, $(1+t)^m f_N = h_0 f_{N+m}$, for some $h_0 \in \bar{\mathcal{T}}_+$. Comparing the leading coefficients, we see that the leading coefficient of h_0 is 1. From $(1+t)^m f_N = h_0 f_{N+m}$, with h_0 having leading coefficient 1, one can easily show that f_{N+m} is divisible by $(1+t)^m$. Hence $f_{N+m} = (1+t)^m \gamma$, where $\gamma \in \bar{\mathcal{T}}_+$.

By Claim 6, for each $0 \le j \le m-1$ there is $h_j \in \bar{\mathcal{T}}_+$ (whose leading coefficient is 1) such that $(1+t)^{m-j} f_{N+j} = h_j f_{N+m} = (1+t)^m h_j \gamma$, i.e. $f_{N+j} = (1+t)^j h_j \gamma$. $\square$

3.7. **Assumption $\bar{\theta}$ is surjective and $\bar{\mathcal{S}}$ is finitely generated.**

Lemma 3.4. *Suppose $\bar{\theta}$ is surjective and $\bar{\mathcal{S}}$ is finitely generated. Then $\bar{\Theta}$ is surjective.*

Proof. From Lemma 3.2, one has the commutative diagram

$$\begin{array}{ccc} \bar{\mathcal{T}} & \xrightarrow{\bar{\Theta}} & \bar{\mathcal{S}} \\ \varepsilon\downarrow & & \varepsilon\downarrow \\ \bar{\mathfrak{t}} & \xrightarrow{\bar{\theta}} & \bar{\mathfrak{s}} \end{array} \tag{3.5}$$

Suppose $\{x_1, \dots, x_d\}$ is a basis of the $\mathbb{C}(M)$-vector space $\bar{\mathfrak{s}}$. Let $\bar{x}_j \in \bar{\mathcal{S}}$ be a lift of x_j. By Nakayama's Lemma, $\{\bar{x}_1, \dots, \bar{x}_d\}$ spans $\bar{\mathcal{S}}$ over $\bar{D}$. Since $\bar{\theta}$ and ε in diagram (3.5) are surjective, each $\bar{x}_j$ is in the image of $\bar{\Theta}$. This proves that $\bar{\Theta}$ is surjective. $\square$

Proposition 3.5. *Suppose $\bar{\theta}$ is surjective and $\bar{\mathcal{S}}$ is finitely generated. Then $\varepsilon(\beta_K) \mid \bar{B}_K$.*

Proof. Recall that $\bar{\mathfrak{p}} = \ker \bar{\theta}$. By Lemma 3.4, $\bar{\Theta}$ is surjective. Diagram (3.5) can be extended to the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bar{\mathcal{P}} & \xrightarrow{\iota} & \bar{\mathcal{T}} & \xrightarrow{\bar{\Theta}} & \bar{\mathcal{S}} & \longrightarrow & 0 \\ & & h\downarrow & & \varepsilon\downarrow & & \varepsilon\downarrow & & \\ 0 & \longrightarrow & \bar{\mathfrak{p}} & \longrightarrow & \bar{\mathfrak{t}} & \xrightarrow{\bar{\theta}} & \bar{\mathfrak{s}} & \longrightarrow & 0 \end{array}$$

Taking the tensor product of the first row, which is an exact sequence of $\bar{D}$-modules, with the $\bar{D}$-algebra $\mathbb{C}(M)$, we get the exact sequence

$$\bar{\mathcal{P}} \otimes_{\bar{D}} \mathbb{C}(M) \xrightarrow{\varepsilon(\iota)} \bar{\mathfrak{t}} \xrightarrow{\bar{\theta}} \bar{\mathfrak{s}} \to 0.$$

It follows that $\bar{\mathfrak{p}} = \ker(\bar{\theta}) = \mathrm{Im}(\varepsilon(\iota)) = \mathrm{Im}(\varepsilon \circ \iota) = h(\bar{\mathcal{P}})$.

Suppose $\{g_j := (1+t)^j h_j \gamma, j = 0, 1, \dots, m\}$ (with $h_m = 1$) be a set of generators of $I = \bar{\mathcal{P}}$ as described in Proposition 3.3. Then $h(g_j) = \varepsilon(g_j) = 0$ except possibly for $j = 0$. It follows that $\bar{\mathfrak{p}} = h(\bar{\mathcal{P}})$ is the principal ideal generated by $\varepsilon(g_0)$. Hence $\varepsilon(g_0) = \bar{B}_K \neq 0$. On the other hand, it is clear that $\beta_K = \gamma$. Hence $\varepsilon(\beta_K) \mid \varepsilon(g_0) = \bar{B}_K$. $\square$

Note that if the localized universal character ring $\bar{\mathfrak{s}}$ is reduced, then $\bar{B}_K = B_K$. Since $\alpha_K \mid \beta_K$, Proposition 3.5 implies the following.

Corollary 3.6. *Suppose $\bar{\theta}$ is surjective, $\bar{\mathcal{S}}$ is finitely generated, and the localized universal character ring $\bar{\mathfrak{s}}$ is reduced. Then $\varepsilon(\alpha_K) \mid B_K$ in $\tilde{\mathfrak{t}} = \mathbb{C}(M)[L^{\pm 1}]$.*

3.8. **Proofs of Theorems 1 and 2.**

3.8.1. *Proof of Theorem 1.* Since K is a hyperbolic knot, it has discrete faithful $SL_2(\mathbb{C})$-representations. Let χ_0 be an irreducible component of $\chi(X)$ containing the character of a discrete faithful $SL_2(\mathbb{C})$-representation. Since X has one boundary component, by a result of Thurston [Th] χ_0 has dimension 1.

Lemma 3.7. *Suppose R is a $\mathbb{C}$-algebra which is an integral domain, and the transcendence degree of the fractional field $F(R)$ of R over $\mathbb{C}$ is 1. Suppose $x \in R$ is transcendental over $\mathbb{C}$. Then the natural map $R \otimes_{\mathbb{C}[x]} \mathbb{C}(x) \to F(R)$ is an isomorphism.*

Proof. Note that $\mathbb{C}(x)$ is flat over $\mathbb{C}[x]$ and $R \otimes_{\mathbb{C}[x]} \mathbb{C}(x) \subset F(R)$. Hence we only need to show that $R \otimes_{\mathbb{C}[x]} \mathbb{C}(x)$ is a field, or that every $0 \neq y \in R$ is invertible in $R \otimes_{\mathbb{C}[x]} \mathbb{C}(x)$. Fix $y \in R$, $y \neq 0$. Since $x \in R$ is transcendental over $\mathbb{C}$, the transcendence degree of the field $\mathbb{C}(x)$ over $\mathbb{C}$ is 1. The field $\mathbb{C}(x)$ is contained in the fractional field $F(R)$ of R whose transcendence degree over $\mathbb{C}$ is also 1, hence $F(R)$ is algebraic over $\mathbb{C}(x)$. In particular, y is algebraic over $\mathbb{C}(x)$. Since $\mathbb{C}(x)[y]$ is a subfield of $F(R)$, $y^{-1} \in \mathbb{C}(x)[y]$. Clearly $\mathbb{C}(x)[y] \subset R \otimes_{\mathbb{C}[x]} \mathbb{C}(x)$. Hence $y^{-1} \in R \otimes_{\mathbb{C}[x]} \mathbb{C}(x)$. □

Recall that the inclusion $\partial X \hookrightarrow X$ induces the restriction map $\rho : \chi(X) \to \chi(\partial X)$. Let Y_0 be the Zariski closure of $\rho(\chi_0) \subset \chi(\partial X)$. Then Y_0 is irreducible and has dimension 1. Since $\rho|_{\chi_0} : \chi_0 \to Y_0$ has dense image, the pullback map $(\rho|_{\chi_0})^* : \mathbb{C}[Y_0] \to \mathbb{C}[\chi_0]$ is an embedding. Both $\mathbb{C}[Y_0]$ and $\mathbb{C}[\chi_0]$ are integral domains. [Du, Theorem 3.1] says that $(\rho|_{\chi_0})^*$ induces an isomorphism, also denoted by $(\rho|_{\chi_0})^*$, $\mathbb{C}(Y_0) \to \mathbb{C}(\chi_0)$, where $\mathbb{C}(Y_0)$ (resp. $\mathbb{C}(\chi_0)$) is the fractional field of $\mathbb{C}[Y_0]$ (resp. $\mathbb{C}[\chi_0]$).

Let $x \in \mathbb{C}[Y_0] \subset \mathbb{C}[\chi_0]$ be defined by $x(\rho) = \mathrm{tr}(\rho(\mu)) = M + M^{-1}$. By [CS2], x is not a constant function on χ_0. It follows that x is transcendental over $\mathbb{C}$. Hence Lemma 3.7 implies that $(\rho|_{\chi_0})^* : \mathbb{C}[Y_0] \otimes_{\mathbb{C}[x]} \mathbb{C}(x) \to \mathbb{C}[\chi_0] \otimes_{\mathbb{C}[x]} \mathbb{C}(x)$ is an isomorphism.

By assumption, the character variety $\chi(X)$ consists of two irreducible components: the abelian component, denoted by χ_{ab}, and χ_0. It is known that χ_{ab} has dimension 1 and the restriction map $\rho|_{\chi_{ab}} : \chi_{ab} \to \chi(\partial X)$ is a birational isomorphism onto its image. It follows that $(\rho|_{\chi_{ab}})^* : \mathbb{C}[Y_{ab}] \otimes_{\mathbb{C}[x]} \mathbb{C}(x) \to \mathbb{C}[\chi_{ab}] \otimes_{\mathbb{C}[x]} \mathbb{C}(x)$ is an isomorphism, where Y_{ab} is the Zariski closure of $\rho(\chi_{ab}) \subset \chi(\partial X)$.

Lemma 3.8. *One has*

$$\mathbb{C}[\chi(X)] \otimes_{\mathbb{C}[x]} \mathbb{C}(x) \cong \left(\mathbb{C}[\chi_0] \otimes_{\mathbb{C}[x]} \mathbb{C}(x)\right) \times \left(\mathbb{C}[\chi_{ab}] \otimes_{\mathbb{C}[x]} \mathbb{C}(x)\right). \tag{3.6}$$

Proof. From Subsection 2.2, we see that $\mathbb{C}[\chi(X)]$ is the quotient of the polynomial ring $R := \mathbb{C}[x, y_1, \cdots, y_m]$ by an ideal $I \subset R$, where $x = M + M^{-1}$ is the trace of the meridian and $y_1, \dots, y_m$ are traces of some fix elements in the knot group.

Suppose $\mathbb{C}[\chi_0] = R/I_0$ and $\mathbb{C}[\chi_{ab}] = R/I_{ab}$ where I_0, I_{ab} are ideals in R. Since $\chi(X)$ consists of two irreducible components χ_{ab} and χ_0, $I = I_0 \cap I_{ab}$, and hence $\mathbb{C}[\chi(X)] = R/(I_0 \cap I_{ab})$. Consider the following sequence of $\mathbb{C}[x]$-modules

$$0 \to R/(I_0 \cap I_{ab}) \xrightarrow{\eta} R/I_0 \times R/I_{ab} \xrightarrow{\xi} R/(I_0 + I_{ab}) \to 0. \tag{3.7}$$

where $\eta(y) = (y, y)$ and $\xi(y, z) = y - z$. It is easy to check that the sequence (3.7) is exact.

Consider the $\mathbb{C}[x]$-module $R/(I_0 + I_{ab})$. The zero set of the ideal $(I_0 + I_{ab}) \subset R$ is precisely the intersection of the two algebraic varieties χ_0 and χ_{ab}, hence it is a set consisting of a finite number of points. It follows that $R/(I_0 + I_{ab}) \cong \mathbb{C}^k$ for some k. The exact sequence (3.7) can be rewritten as

$$0 \to R/(I_0 \cap I_{ab}) \xrightarrow{\eta} R/I_0 \times R/I_{ab} \xrightarrow{\xi} \mathbb{C}^k \to 0. \tag{3.8}$$

Since $\mathbb{C}(x)$ is flat over $\mathbb{C}[x]$ and $\mathbb{C}^k \otimes_{\mathbb{C}[x]} \mathbb{C}(x) = 0$, the following sequence which is obtained from (3.8) by tensoring it with $\mathbb{C}(x)$ over $\mathbb{C}[x]$ is exact

$$0 \to R/(I_0 \cap I_{ab}) \otimes_{\mathbb{C}[x]} \mathbb{C}(x) \overset{\eta \otimes_{\mathbb{C}[x]} \mathbb{C}(x)}{\longrightarrow} (R/I_0 \times R/I_{ab}) \otimes_{\mathbb{C}[x]} \mathbb{C}(x) \overset{\xi \otimes_{\mathbb{C}[x]} \mathbb{C}(x)}{\longrightarrow} 0.$$

The lemma follows. □

Let $Y \subset \chi(\partial X)$ be the algebraic set consisting of Y_0 and Y_{ab}. By similar arguments as in the proof of Lemma 3.8, we have

$$\mathbb{C}[Y] \otimes_{\mathbb{C}[x]} \mathbb{C}(x) \cong \big(\mathbb{C}[Y_0] \otimes_{\mathbb{C}[x]} \mathbb{C}(x)\big) \times \big(\mathbb{C}[Y_{ab}] \otimes_{\mathbb{C}[x]} \mathbb{C}(x)\big). \tag{3.9}$$

We have

$$\begin{aligned}(\rho|_{\chi_0})^* \times (\rho|_{\chi_{ab}})^* : \big(\mathbb{C}[Y_0] \otimes_{\mathbb{C}[x]} \mathbb{C}(x)\big) \times \big(\big(\mathbb{C}[Y_{ab}] \otimes_{\mathbb{C}[x]} \mathbb{C}(x)\big)\big) \\ \longrightarrow \big(\mathbb{C}[\chi_0] \otimes_{\mathbb{C}[x]} \mathbb{C}(x)\big) \times \big(\mathbb{C}[\chi_{ab}] \otimes_{\mathbb{C}[x]} \mathbb{C}(x)\big)\end{aligned}$$

is an isomorphism. Equations (3.6) and (3.9) then imply that $(\rho|_{\chi_0})^* \times (\rho|_{\chi_{ab}})^* : \mathbb{C}[Y] \otimes_{\mathbb{C}[x]} \mathbb{C}(x) \to \mathbb{C}[\chi(X)] \otimes_{\mathbb{C}[x]} \mathbb{C}(x)$ is an isomorphism. Since Y is an algebraic set in $\chi(\partial X)$, $\mathbb{C}[Y]$ is a quotient of $\mathbb{C}[\chi(\partial X)] \equiv \mathfrak{t}^\sigma$. Hence the map $\mathfrak{t}^\sigma \otimes_{\mathbb{C}[x]} \mathbb{C}(x) \to \mathbb{C}[\chi(X)] \otimes_{\mathbb{C}[x]} \mathbb{C}(x)$, induced by $\rho : \chi(X) \to \chi(\partial X)$, is surjective. Taking the tensor product of this map with $\mathbb{C}(M)$ over $\mathbb{C}(x)$, we get the map

$$\tilde{\theta} : \tilde{\mathfrak{t}} = \mathfrak{t}^\sigma \otimes_{\mathbb{C}[x]} \mathbb{C}(M) \to \widetilde{\mathbb{C}[\chi(X)]} = \mathbb{C}[\chi(X)] \otimes_{\mathbb{C}[x]} \mathbb{C}(M),$$

which is also surjective. (Note that $\mathbb{C}(M)$ is flat over $\mathbb{C}(x)$.)

Note that $\tilde{\mathfrak{t}} = \mathbb{C}(M)[L^{\pm 1}]$. Hence the dimension of the $\mathbb{C}(M)$-vector space $\widetilde{\mathbb{C}[\chi(X)]}$ is equal to the L-degree of B_K, the generator of the kernel $\tilde{\mathfrak{p}}$ of $\tilde{\theta}$, and is finite. Since the universal character ring $\mathfrak{s}(X)$ is reduced and $\bar{\mathcal{S}}$ is a finitely generated $\bar{D}$-module, Corollary 3.6 implies that B_K is M-essentially divisible by $\varepsilon(\alpha_K)$. Since A_K is M-essentially equal to B_K by Corollary 2.3, it must be M-essentially divisible by $\varepsilon(\alpha_K)$.

It is known that A_K always contains the factor $L-1$ coming from characters of abelian representations [CCGLS]), and $\varepsilon(\alpha_K)$ is also divisible by $L-1$ [Le2, Proposition 2.3]. Hence $\frac{A_K}{L-1}$ is M-essentially divisible by $\frac{\varepsilon(\alpha_K)}{L-1}$.

Since the character variety $\chi(X)$ consists of two irreducible components, A_K has exactly two irreducible factors. One factor is $L-1$, hence the other one, $\frac{A_K}{L-1}$, is irreducible. Since $\frac{A_K}{L-1}$ is M-essentially divisible by $\frac{\varepsilon(\alpha_K)}{L-1}$, it follows that

$$\frac{\varepsilon(\alpha_K)}{L-1} \overset{M}{=} 1, \quad \text{or} \quad \frac{\varepsilon(\alpha_K)}{L-1} \overset{M}{=} \frac{A_K}{L-1}.$$

If $\frac{\varepsilon(\alpha_K)}{L-1} \overset{M}{=} 1$, then, by Lemma 3.9 below, the recurrence polynomial α_K has L-degree 1. This contradicts condition (iii) of Theorem 1, hence we must have $\frac{\varepsilon(\alpha_K)}{L-1} \overset{M}{=} \frac{A_K}{L-1}$. In other words, the AJ conjecture holds true for K.

Lemma 3.9. *The polynomial $\varepsilon(\alpha_K)$ is M-essentially equal to $L-1$ if and only if the L-degree of the recurrence polynomial α_K is 1.*

Proof. The backward direction is obvious since $\varepsilon(\alpha_K)$ is always divisible by $L-1$.

Now suppose the polynomial $\varepsilon(\alpha_K)$ is M-essentially equal to $L-1$, i.e. $\varepsilon(\alpha_K) = g(M)(L-1)$ for some non-zero $g(M) \in \mathbb{C}[M^{\pm 1}]$. Then

$$\alpha_K = g(M)(L-1) + (1+t)\sum_{j=0}^{d} a_j(M)L^j \tag{3.10}$$

where $a_j(M)$'s are Laurent polynomials in $\mathcal{R}[M^{\pm 1}]$ and d is the L-degree of α_K.

By a result in [Ga1], the recurrence ideal $\mathcal{A}_K$ is invariant under the involution σ. Hence $\sigma(\alpha_K)$ is an element in $\mathcal{A}_K$. Since α_K is the generator of $\widetilde{\mathcal{A}}_K$, it follows that $\alpha_K = h(M)\sigma(\alpha_K)L^d$ for some $h(M) \in \mathcal{R}(M)$. Equation (3.10) implies that

$$g(M)(L-1) + (1+t)\sum_{j=0}^{d} a_j(M)L^j$$

$$= h(M)g(M^{-1})(L^{-1}-1)L^d + (1+t)\sum_{j=0}^{d} h(M)a_j(M^{-1})L^{d-j}.$$

If $d > 1$ then by comparing the coefficients of L^0 in both sides of the above equation, we get $-g(M) + (1+t)a_0(M) = (1+t)h(M)a_d(M^{-1})$, i.e.

$$g(M) = (1+t)\left(a_0(M) - h(M)a_d(M^{-1})\right) \tag{3.11}$$

Since $g(M)$ is a Laurent polynomial in M with coefficients in $\mathbb{C}$, equation (3.11) implies that $g(M)$ must be equal to 0. This is a contradiction. Hence we must have $d = 1$. □

Remark 3.10. In the proof of Theorem 1, instead of condition *(iii)* we actually use the following weakened version

(iii') the localized universal SL_2-character ring $\bar{\mathfrak{s}}$ of $\pi_1(S^3 \setminus K)$ is reduced.

(Recall that $\bar{\mathfrak{s}} = \mathfrak{s} \otimes_{\mathbb{C}[M+M^{-1}]} \mathbb{C}(M)$ is the localization of the universal SL_2-character ring.) Thus Theorem 1 holds if the condition *(iii)* is replaced by *(iii')* above.

3.8.2. *Proof of Theorem 2.* It is known that two-bridge knots and $(-2,3,2n+1)$-pretzel knots, excluding torus knots, are hyperbolic. (Note that the AJ conjecture holds true for torus knots by [Hi, Tr]). Their universal character rings are reduced by [Le2, Corollary 5.8] and Theorem 4.6 respectively. Their localized skein modules are finitely generated over the local ring $\bar{D}$ by [Le2, Theorem 2] and Proposition 5.1 respectively.

The L-degree of the recurrence polynomial of a two-bridge knot is > 1 according to [Le2, Proposition 2.2]. Also by [Le2, Proposition 2.2], for any knot if the L-degree of its recurrence polynomial is 1 then the breadth of its colored Jones polynomial is a linear function (in the color). For the $(-2,3,2n+1)$-pretzel knot, by [Ga3, Section 4.7] the breadth of its colored Jones polynomial is not a linear function, hence the L-degree of its recurrence polynomial is > 1.

Double twist knots of the form $J(k,l)$ with $k \neq l$, two-bridge knots of the form $\mathfrak{b}(p,m)$ with $m = 3$ or "p prime and $\gcd(\frac{p-1}{2}, \frac{m-1}{2}) = 1$", and $(-2,3,6n \pm 1)$-pretzel knots satisfy condition (ii) of Theorem 1 by [MPL], Theorem A.5 and [Bur], and [Mat] respectively. Hence the theorem follows.

4. The universal character ring of the $(-2, 3, 2n+1)$-pretzel knot

In this section we explicitly calculate the universal character ring of the $(-2, 3, 2n+1)$-pretzel knot and prove its reducedness for all integers n.

4.1. The character variety. For the $(-2, 3, 2n+1)$-pretzel knot K_{2n+1}, we have

$$\pi_1(X) = \langle a, b, c \mid cacb = acba,\ ba(cb)^n = a(cb)^n c\rangle,$$

where $X = S^3 \setminus K_{2n+1}$ and a, b, c are meridians depicted in Figure 1.

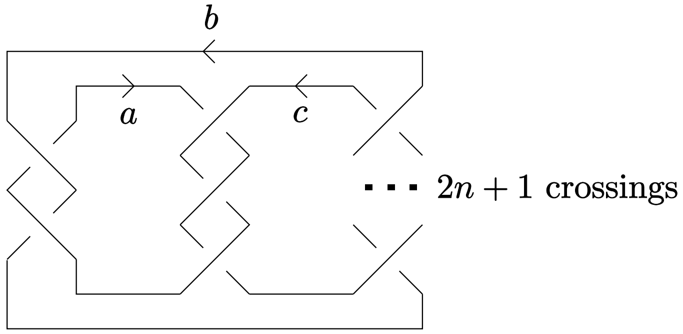


Figure 2. The $(-2, 3, 2n+1)$-pretzel knot

Let $w = cb$ then the first relation of $\pi_1(X)$ becomes $caw = awa$. It implies that $c = awaw^{-1}a^{-1}$ and $b = c^{-1}w = awa^{-1}w^{-1}a^{-1}w$. The second relation then has the form

$$awa^{-1}w^{-1}a^{-1}waw^n = aw^n awaw^{-1}a^{-1}$$

i.e.

$$w^n awa^{-1}w^{-1}a^{-1} = a^{-1}w^{-1}awaw^{-1}w^n.$$

Hence we obtain a presentation of $\pi_1(X)$ with two generators and one relation

$$\pi_1(X) = \langle a, w \mid w^n E = F w^n\rangle$$

where $E := awa^{-1}w^{-1}a^{-1}$ and $F := a^{-1}w^{-1}awaw^{-1}$.

The character variety of the free group $F_2 = \langle a, w\rangle$ in 2 letters a and w is isomorphic to $\mathbb{C}^3$ by the Fricke-Klein-Vogt theorem, see [LM]. For every element $\omega \in F_2$ there is a unique polynomial $\mathbf{P}_\omega$ in 3 variables such that for any representation $r : F_2 \to SL_2(\mathbb{C})$ we have $\mathrm{tr}(r(\omega)) = \mathbf{P}_\omega(x, y, z)$ where $x := \mathrm{tr}(r(a))$, $y := \mathrm{tr}(r(w))$ and $z := \mathrm{tr}(r(aw))$. The polynomial $\mathbf{P}_\omega$ can be calculated inductively using the following identities for traces of matrices $A, B \in SL_2(\mathbb{C})$:

$$\mathrm{tr}(A) = \mathrm{tr}(A^{-1}), \quad \mathrm{tr}(AB) + \mathrm{tr}(AB^{-1}) = \mathrm{tr}(A)\,\mathrm{tr}(B). \tag{4.1}$$

Thus for every representation $r : \pi_1(X) \to SL_2(\mathbb{C})$, we consider x, y, and z as functions of r. The character variety of $\pi_1(X)$ is the zero locus of an ideal in $\mathbb{C}[x, y, z]$, which we describe explicitly in the next theorem.

Theorem 4.1. *The character variety of the pretzel knot K_{2n+1} is the zero locus of 2 polynomials $P := \mathbf{P}_E - \mathbf{P}_F$ and $Q_n := \mathbf{P}_{w^n Ea} - \mathbf{P}_{Fw^n a}$. Explicitly,*

$$P = x - xy + (-3 + x^2 + y^2)z - xyz^2 + z^3, \tag{4.2}$$

$$\begin{aligned} Q_n = {} & S_{n-2}(y) + S_{n-3}(y) - S_{n-4}(y) - S_{n-5}(y) - S_{n-2}(y)\,x^2 \\ & + \big(S_{n-1}(y) + S_{n-3}(y) + S_{n-4}(y)\big)\,xz - \big(S_{n-2}(y) + S_{n-3}(y)\big)\,z^2 \end{aligned} \tag{4.3}$$

where $S_k(y)$'s are the Chebychev polynomials defined by $S_0(y) = 1$, $S_1(y) = y$ and $S_{k+1}(y) = yS_k(y) - S_{k-1}(y)$ for all integers k.

Proof. The explicit formulas (4.2) and (4.3) follow from easy calculations of the trace polynomials using (4.1).

Because E and F are conjugate (by w^n) and $w^nEa = Fw^na$ in $\pi_1(X)$, we have $P = Q_n = 0$ for every representation $r : \pi_1(X) \to SL_2(\mathbb{C})$.

We will prove the converse: fix a solution (x, y, z) of $P = Q_n = 0$, we will find a representation $r : \pi_1(X) \to SL_2(\mathbb{C})$ such that $x = \mathrm{tr}(r(a))$, $y = \mathrm{tr}(r(w))$ and $z = \mathrm{tr}(r(aw))$.

We consider the following 3 cases:

Case 1: $y^2 \neq 4$. Then there exist $s, u, v \in \mathbb{C}$ such that $s+s^{-1} = y$, $u+v = x$, $su+s^{-1}v = z$. Since $S_k(y) = \frac{s^{k+1}-s^{-k-1}}{s-s^{-1}}$ for all integers k, we have

$$\begin{aligned} P &= s^{-3}(s-1)P', \\ Q_n &= s^{-3-n}\big((s^{2n}u - sv)P' - (1+s)(-1+uv)Q_n'\big), \end{aligned}$$

where

$$\begin{aligned} P' &= s^3u - s^4u - s^5u + v + sv - s^2v - s^2u^2v - s^3u^2v + s^4u^2v + s^5u^2v \\ &\quad - uv^2 - suv^2 + s^2uv^2 + s^3uv^2, \\ Q_n' &= s^5 + s^{2n} - s^{2+2n}u^2 + s^{4+2n}u^2 + s^3uv - s^5uv - s^{2n}uv + s^{2+2n}uv + sv^2 - s^3v^2. \end{aligned}$$

Since $s \neq \pm 1$, $P = Q_n = 0$ is equivalent to $P' = (-1+uv)Q_n' = 0$. We consider the following 2 subcases:

Subcase 1.1: $Q_n' = 0$. Choose $r(a) = \begin{pmatrix} u & 1 \\ uv-1 & v \end{pmatrix}$ and $r(w) = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}$. It is easy to check $x = \mathrm{tr}(r(a))$, $y = \mathrm{tr}(r(w))$, $z = \mathrm{tr}(r(aw))$ and the calculations in the following 2 lemmas.

Lemma 4.2. *One has*

$$r(E) = \begin{pmatrix} s^{-2}H_{11} & -s^{-2}H_{12} \\ s^{-2}(-1+uv)H_{21} & -s^{-2}H_{22} \end{pmatrix}, \quad r(F) = \begin{pmatrix} -s^{-3}H_{22} & -s^{-1}H_{21} \\ s^{-3}(-1+uv)H_{12} & s^{-1}H_{11} \end{pmatrix}$$

where

$$\begin{aligned} H_{11} &= s^2u - s^4u + v - s^2u^2v + s^4u^2v - uv^2 + s^2uv^2, \\ H_{12} &= 1 - s^2u^2 + s^4u^2 - uv + s^2uv, \\ H_{21} &= -s^4 - s^2uv + s^4uv - v^2 + s^2v^2, \\ H_{22} &= -s^4u + v - s^2v - s^2u^2v + s^4u^2v - uv^2 + s^2uv^2. \end{aligned}$$

Lemma 4.3. *One has*

$$r(w^nE - Fw^n) = \begin{pmatrix} s^{-3+n}P' & -s^{-2-n}Q_n' \\ -s^{-3-n}(-1+uv)Q_n' & -s^{-2-n}P' \end{pmatrix}.$$

Since $P' = Q'_n = 0$, Lemma 4.3 implies that $r(w^nE - Fw^n) = 0$, i.e. $r(w^nE) = r(Fw^n)$.

Subcase 1.2: $-1 + uv = 0$ then $v = u^{-1}$. In this case the equation $P' = 0$ becomes $s^2u^{-1}(s - u^2) = 0$ i.e. $s = u^2$. Let

$$r(a) = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \quad r(w) = \begin{pmatrix} u^2 & 0 \\ 0 & u^{-2} \end{pmatrix}.$$

Then it is easy to check that $x = \mathrm{tr}(r(a)),\ y = \mathrm{tr}(r(w)),\ z = \mathrm{tr}(r(aw))$ and $r(Ew^n) = r(w^nF)$. (Note that $r(a)$ and $r(w)$ commute in this case).

Case 2: $y = 2$. Then $S_k(y) = k$ for all integers k. Hence

$$\begin{aligned} P &= (x - z)(-1 + xz - z^2), \\ Q_n &= 4 - (n-1)x^2 + (3n-5)xz - (2n-3)z^2. \end{aligned}$$

It follows that $(x, z) = (-2, -2), (2, 2)$ or ($x = z + z^{-1}$ and $1 - n + (1+n)z^2 - z^4 = 0$).

If $x = z = 2$ we choose

$$r(a) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad r(w) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If $x = z = -2$ we choose

$$r(a) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad r(w) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If $x = z + z^{-1}$ and $1 - n + (1+n)z^2 - z^4 = 0$ we choose

$$r(a) = \begin{pmatrix} z & 0 \\ -z^{-1} & z^{-1} \end{pmatrix}, \quad r(w) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Lemma 4.4. *One has*

$$r(w^nE - Fw^n) = \begin{pmatrix} 0 & z^{-1}(-1 + n - (1+n)z^2 + z^4) \\ 0 & 0 \end{pmatrix}$$

Proof. By direct calculations we have $r(E) = \begin{pmatrix} z & -2z + z^3 \\ 0 & z^{-1} \end{pmatrix}$, $r(F) = \begin{pmatrix} z & z^{-1} - z \\ 0 & z^{-1} \end{pmatrix}$ and $r(w^n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. The lemma follows. □

Hence $x = \mathrm{tr}(r(a)),\ y = \mathrm{tr}(r(w)),\ z = \mathrm{tr}(r(aw))$ and $r(w^nE) = r(Fw^n)$.

Case 3: $y = -2$. Then $S_k(y) = (-1)^k k$ for all integers k. Hence

$$\begin{aligned} P &= 3x + z + x^2z + 2xz^2 + z^3, \\ Q_n &= (-1)^n\big(xP - (x+z)Q''_n\big)/2, \end{aligned}$$

where $Q''_n = x + 2nx + 2z + x^2z + xz^2$. It follows that the system $P = Q_n = 0$ is equivalent to $P = (x+z)Q''_n = 0$. We consider the following 2 subcases:

Subcase 3.1: $x + z = 0$. Then it is easy to see that $P = 0$ is equivalent to $x = z = 0$. In this case we choose

$$r(a) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad r(w) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

where i is the imaginary number.

Subcase 3.2: $x+z\neq 0$. Then $Q''_n=0$. Choose

$$r(a)=\begin{pmatrix} x/2 & (1-x^2/4)/(x+z) \\ -x-z & x/2 \end{pmatrix}, \quad r(w)=\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}.$$

Lemma 4.5. *One has*

$$r(w^nE-Fw^n)=(-1)^n\begin{pmatrix} nP-Q''_n & Q''_n/2 \\ 0 & -(n-1)P+Q''_n \end{pmatrix}.$$

Proof. By direct calculations, we have $r(w^n)=(-1)^n\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ and

$$\begin{aligned} r(E) &= \begin{pmatrix} -(x+2z+x^2z+xz^2)/2 & -\frac{4+3x^2+4xz+x^3z+x^2z^2}{4(x+z)} \\ (x+z)(1+xz+z^2) & (3x+2z+x^2z+xz^2)/2 \end{pmatrix}, \\ r(F) &= \begin{pmatrix} (x+2z+x^2z+xz^2)/2 & -\frac{4+5x^2+10xz+3x^3z+4z^2+5x^2z^2+2xz^3}{4(x+z)} \\ (x+z)(1+xz+z^2) & (-5x-4z-3x^2z-5xz^2-2z^3)/2. \end{pmatrix} \end{aligned}$$

The lemma follows. □

Hence $x=\mathrm{tr}(r(a))$, $y=\mathrm{tr}(r(w))$, $z=\mathrm{tr}(r(aw))$ and $r(w^nE)=r(Fw^n)$ in all cases. It follows that the character variety of the pretzel knot K_{2n+1} is exactly equal to the algebraic set $\{P=Q_n=0\}$. □

4.2. **The universal character ring.** In this subsection, we will prove the following

Theorem 4.6. *The universal character ring of K_{2n+1} is reduced and is equal to the ring $\mathbb{C}[x,y,z]/(P,Q_n)$.*

Proof. Suppose we have shown that the ring $\mathbb{C}[x,y,z]/(P,Q_n)$ is reduced, then it is exactly the character ring $\mathbb{C}[\chi(X)]$ of K_{2n+1}.

Recall that $\pi_1(X)=\langle a,w \mid w^nE=Fw^n\rangle$, and $F_2=\langle a,w\rangle$ is the free group on two generators a,w. It is known that the universal character ring of F_2 is the ring $\mathbb{C}[x,y,z]$ where $x=\mathrm{tr}(r(a))$, $y=\mathrm{tr}(r(w))$ and $z=\mathrm{tr}(r(aw))$ as above. The quotient map $h: F_2\to\pi_1(X)$ induces the epimorphism $h_*:\mathbb{C}[x,y,z]\to\varepsilon(\mathcal{S}(X))$. Since P,Q_n come from traces, they are contained in $\ker h_*$.

Since $\mathbb{C}[\chi(X))]$ is the quotient of $\varepsilon(\mathcal{S}(X))$ by its nilradical, we have the quotient homomorphism $\phi:\varepsilon(\mathcal{S}(X))\to\mathbb{C}[\chi(X))]=\mathbb{C}[x,y,z]/(P,Q_n)$. Then

$$\phi\circ h_*:\mathbb{C}[x,y,z]\to\varepsilon(\mathcal{S}(X))\to\mathbb{C}[\chi(\pi)]=\mathbb{C}[x,y,z]/(P,Q_n)$$

is a homomorphism. It follows that $\ker h_*\subseteq(P,Q_n)$. Hence we must have $\ker h_*=(P,Q_n)$, which implies $\varepsilon(\mathcal{S}(X))\cong\mathbb{C}[x,y,z]/(P,Q_n)\equiv\mathbb{C}[\chi(X)]$.

In the remaining part of this section we will show that the ring $\mathbb{C}[x,y,z]/(P,Q_n)$ is reduced, i.e. the ideal $I_n:=(P,Q_n)$ is radical. The proof of this fact will be divided into several steps.

4.2.1. $\mathbb{C}[x,y,z]/I_n$ *is free over* $\mathbb{C}[x]$.

Lemma 4.7. *For every $x_0\neq 0,\pm 2$, the polynomial $P\mid_{x=x_0}$ is irreducible in $\mathbb{C}[y,z]$.*

Proof. Assume that $P\mid_{x=x_0}$ can be decomposed as

$$z^3 - x_0yz^2 + (y^2 + x_0^2 - 3)z + x_0(1-y) = \big(z + f_1\big)\big(z^2 - (x_0y + f_1)z + f_2\big), \tag{4.4}$$

where $f_j \in \mathbb{C}[y]$. Equation (4.4) implies that $f_2 - f_1(x_0y + f_1) = y^2 + x_0^2 - 3$ and $f_1f_2 = x_0(1-y)$.

If f_1 is a constant then $f_2 = x_0(1-y)/f_1$ has y-degree 1. Hence $f_2 - f_1(x_0y + f_1)$ has y-degree 1 also. It follows that $f_2 - f_1(x_0y + f_1) \neq y^2 + x_0^2 - 3$.

If f_2 is a constant then $f_1 = x_0(1-y)/f_2$. Hence

$$\begin{aligned} f_2 - f_1(x_0y + f_1) &= f_2 - \big(\frac{x_0}{f_2} - \frac{x_0}{f_2}y\big)\big(\frac{x_0}{f_2} - \frac{x_0}{f_2}y + x_0y\big) \\ &= \frac{x_0^2}{f_2}\big(1 - \frac{1}{f_2}\big)y^2 - \frac{x_0^2}{f_2}\big(1 - \frac{2}{f_2}\big)y + \big(f_2 - \frac{x_0^2}{f_2^2}\big). \end{aligned}$$

Then since $f_2 - f_1(x_0y + f_1) = y^2 + x_0^2 - 3$, we have $\frac{x_0^2}{f_2}\big(1 - \frac{1}{f_2}\big) = 1$, $\frac{x_0^2}{f_2}\big(1 - \frac{2}{f_2}\big) = 0$, and $f_2 - \frac{x_0^2}{f_2^2} = x_0^2 - 3$. This implies $x_0 = 0$ or $x_0 = \pm 2$. □

Lemma 4.8. *For every x_0, the polynomials $P\mid_{x=x_0}$ and $Q_n\mid_{x=x_0}$ are co-prime in $\mathbb{C}[y,z]$.*

Proof. If $x_0 \neq 0, \pm 2$ then, by Lemma 4.7, $P\mid_{x=x_0}$ is irreducible in $\mathbb{C}[y,z]$. Lemma 4.8 then follows since $P\mid_{x=x_0}$ and $Q_n\mid_{x=x_0}$ have z-degrees 3 and 2 respectively.

At $x_0 = 0$, we have $P = z(-3 + y^2 + z^2)$ and $Q_n = a_n + b_nz^2$ where

$$\begin{aligned} a_n &= S_{n-2}(y) + S_{n-3}(y) - S_{n-4}(y) - S_{n-5}(y), \\ b_n &= -S_{n-2}(y) - S_{n-3}(y). \end{aligned}$$

In this case, it suffices to show that $Q_n\mid_{z^2=3-y^2} = a_n + b_n(3 - y^2) \neq 0$. This is true by Lemma 4.13 below.

At $x_0 = 2$, we have $P = \big(z + 1 - y\big)\big(z^2 - (1+y)z + 2\big)$ and $Q_n = a_n' + b_n'z + c_n'z^2$ where $a_n', b_n', c_n' \in \mathbb{C}[y]$. When $z = y - 1$, we have $Q_0 = 1$ and $Q_1 = y - 1$ and $Q_{n+1} = yQ_n - Q_{n-1}$ for all integers n. It follows that $Q_n\mid_{z=y-1} = S_n(y) - S_{n-1}(y)$ is a polynomial of y-degree n if $n \geq 0$ and $-(n+1)$ if $n \leq -1$, with leading coefficient 1. Hence $Q_n\mid_{z=y-1}$ is not identically 0. It remains to show that $Q_n = a_n' + b_n'z + c_n'z^2 \neq c_n'(z^2 - (1+y)z + 2)$. It suffices to show that $b_n'\mid_{y=-1} \neq 0$. Indeed, when $x_0 = 2$ and $y = -1$ we have $b_n' = 2\big(S_{n-1}(-1) + S_{n-3}(-1) + S_{n-4}(-1)\big)$. It is easy to check that $S_k(-1) = 1$ if $k = 0 \pmod 3$, $S_k(-1) = -1$ if $k = 1 \pmod 3$ and $S_k(-1) = 0$ otherwise. Hence $b_n' = 2\big(S_{n-1}(-1) + S_{n-3}(-1) + S_{n-4}(-1)\big) \neq 0$.

The case $x_0 = -2$ is similar. □

Proposition 4.9. *$\mathbb{C}[x,y,z]/I_n$ is a torsion-free $\mathbb{C}[x]$-module.*

Proof. Suppose $S \in \mathbb{C}[x,y,z]$ and $(x - x_0)S \in I_n$ for some $x_0 \in \mathbb{C}$. We will show that $S \in I_n$. Indeed, we have $(x - x_0)S = fP - gQ_n$ for some $f, g \in \mathbb{C}[x,y,z]$. Hence $(fP)\mid_{x=x_0} = (gQ_n)\mid_{x=x_0}$ which implies that $f\mid_{x=x_0}$ is divisible by $Q_n\mid_{x=x_0}$, since $P\mid_{x=x_0}$ and $Q_n\mid_{x=x_0}$ are co-prime in the UFD $\mathbb{C}[y,z]$ by Lemma 4.8. Hence $f_{x=x_0} = h\,Q_n\mid_{x=x_0}$ for some $h \in \mathbb{C}[y,z]$. From this, we may write $f = h\,Q_n + (x - x_0)Q$ for some $Q \in \mathbb{C}[x,y,z]$. Then we have

$$(x - x_0)S = fP - gQ_n = \big(hQ_n + (x - x_0)Q\big)P - gQ_n = (x - x_0)QP + (hP - g)Q_n$$

which implies that $hP - g$ is divisible by $x - x_0$ and $S = QP + \frac{hP-g}{x-x_0}Q_n \in I_n$. □

Proposition 4.10. *$\mathbb{C}[x,y,z]/I_n$ is a finitely generated $\mathbb{C}[x]$-module.*

Proof. We want to show that y and z, considered as elements of $\mathbb{C}[x,y,z]/I_n$, are integral over $\mathbb{C}[x]$. Indeed, the resultant of P and Q_n with respect to z is

$$\mathfrak{r} = \begin{vmatrix} P_0 & P_1 & P_2 & P_3 & 0 \\ 0 & P_0 & P_1 & P_2 & P_3 \\ Q_{n,0} & Q_{n,1} & Q_{n,2} & 0 & 0 \\ 0 & Q_{n,0} & Q_{n,1} & Q_{n,2} & 0 \\ 0 & 0 & Q_{n,0} & Q_{n,1} & Q_{n,2} \end{vmatrix}$$

where $P_0 = x - xy,\ P_1 = -3 + x^2 + y^2,\ P_2 = -xy,\ P_3 = 1$ and

$$\begin{aligned} Q_{n,0} &= S_{n-2}(y) + S_{n-3}(y) - S_{n-4}(y) - S_{n-5}(y) - S_{n-2}(y)x^2, \\ Q_{n,1} &= (S_{n-1}(y) + S_{n-3}(y) + S_{n-4}(y))x, \\ Q_{n,2} &= -(S_{n-2}(y) + S_{n-3}(y)). \end{aligned}$$

Write $y = s + s^{-1}$ then $S_k(y) = \frac{s^{k+1} - s^{-k-1}}{s - s^{-1}}$ for all integers k. By a direct calculation

$$\begin{aligned} \mathfrak{r} \;=\; & \frac{s + s^{-1} + 2 - x^2}{(s - s^{-1})^2(s + s^{-1} + 2)}\big(s^{3n} + s^{-3n} + 3s^{3n-1} + 3s^{1-3n} + 3s^{3n-2} + 3s^{2-3n} \\ & + s^{3n-3} + s^{3-3n} + s^{n+5} + s^{-n-5} + 3s^{n+4} + 3s^{-n-4} + 3s^{n+3} + 3s^{-n-3} + s^{n+2} + s^{-n-2} \\ & - 2s^{n-1} - 2s^{1-n} - 6s^{n-2} - 6s^{2-n} - 6s^{n-3} - 6s^{3-n} - 2s^{n-4} - 2s^{4-n} \\ & + x^2(-s^{3n-1} - s^{1-3n} - s^{3n-2} - s^{2-3n} - 2s^{n+3} - 2s^{-n-3} - 3s^{n+2} - 3s^{-n-2} \\ & - s^{n+1} - s^{-n-1} - 5s^{n} - 5s^{-n} - 2s^{n-1} - 2s^{1-n} + 8s^{n-2} + 8s^{2-n} + 6s^{n-3} + 6s^{3-n} \\ & + s^{n-4} + s^{4-n}) + x^4(s^{n+1} + s^{-n-1} + 2s^{n} + 2s^{-n} - 2s^{n-2} - 2s^{2-n} - s^{n-3} - s^{3-n})\big). \end{aligned}$$

Let $T_k(y) = s^k + s^{-k}$ for all intergers k. Then we have

$$\begin{aligned} \mathfrak{r} \;=\; & \frac{y + 2 - x^2}{(y^2 - 4)(y + 2)}\big(T_{3n}(y) + 3T_{3n-1}(y) + 3T_{3n-2}(y) + T_{3n-3}(y) + T_{n+5}(y) \\ & + 3T_{n+4}(y) + 3T_{n+3}(y) + T_{n+2}(y) - 2T_{n-1}(y) - 6T_{n-2}(y) - 6T_{n-3}(y) - 2T_{n-4}(y) \\ & + x^2(-T_{3n-1}(y) - T_{3n-2}(y) - 2T_{n+3}(y) - 3T_{n+2}(y) - T_{n+1}(y) - 5T_{n}(y) - 2T_{n-1}(y) \\ & + 8T_{n-2}(y) + 6T_{n-3}(y) + T_{n-4}(y)) + x^4(T_{n+1}(y) + 2T_{n}(y) - 2T_{n-2}(y) - T_{n-3}(y))\big) \end{aligned}$$

Note that $T_k(y)$ has y-degree $|k|$ with leading coefficient 1. If $n \geq 4$ then it is easy to see that $\mathfrak{r}$ has y-degree $3n-2$; moreover the coefficient of y^{3n-2} is 1. Similarly, if $n \leq -5$ then $\mathfrak{r}$ has y-degree $1-3n$; moreover the coefficient of y^{1-3n} is 1. If $-4 \leq n \leq 3$ then by direct calculations, one can check that the coefficient of the highest power of y in $\mathfrak{r}$ is 1. Hence the coefficient of the highest power of y in $\mathfrak{r}$ is 1 for all integers n. It follows that y, considered as an element of $\mathbb{C}[x,y,z]/I_n$, is integral over $\mathbb{C}[x]$.

Since z, considered as an element of $\mathbb{C}[x,y,z]/I_n$, satisfies the equation $P = x - xy + (-3 + x^2 + y^2)z^2 - xyz^2 + z^3 = 0$ with 1 being the coefficient of the highest power of z, it is also integral over $\mathbb{C}[x]$. Therefore $\mathbb{C}[x,y,z]/I_n$ is a finitely generated $\mathbb{C}[x]$-module. ☐

Since $\mathbb{C}[x]$ is a PID, Propositions 4.9 and 4.10 imply that

Proposition 4.11. *$\mathbb{C}[x,y,z]/I_n$ is a free $\mathbb{C}[x]$-module.*

4.2.2. *Reduction to a special case.* For a $\mathbb{C}[x]$-module J, let $J\,|_{x=x_0} := J \otimes_{\mathbb{C}[x]} \mathbb{C}$, where $\mathbb{C}$ is considered as an $\mathbb{C}[x]$-module by reducing $x = x_0$.

Proposition 4.12. *I_n is radical if $I_n\,|_{x=x_0}$ is radical for some $x_0 \in \mathbb{C}$.*

Proof. Let $R = \mathbb{C}[x, y, z]$. Consider the exact sequence of $\mathbb{C}[x]$-modules

$$0 \to \sqrt{I_n}/I_n \to R/\sqrt{I_n} \to R/I_n \to 0.$$

By Proposition 4.11, R/I_n is free, hence the sequence splits and $\sqrt{I_n}/I_n$ is projective. Since $\mathbb{C}[x]$ is a PID, $\sqrt{I_n}/I_n$ is free. Let k be the rank of the $\mathbb{C}[x]$-module $\sqrt{I_n}/I_n$ then the rank of the $\mathbb{C}$-module $(\sqrt{I_n}/I_n)\,|_{x=x_0}$ is always k for every $x_0 \in \mathbb{C}$. Hence if $I_n\,|_{x=x_0}$ is radical for some $x_0 \in \mathbb{C}$ then $k = 0$ which implies that $\sqrt{I_n} = I_n$. □

4.2.3. $I_n\big|_{x=0}$ *is radical.* By Lemma 4.8, $P\big|_{x=0}$ and $Q_n\big|_{x=0}$ are co-prime in $\mathbb{C}[y, z]$. This means $I_n\big|_{x=0}$ is a zero-dimensional ideal of $\mathbb{C}[y, z]$. By Seidenberg's Lemma (see [KL, Proposition 3.7.15]), if there exist two non-zero free-square polynomials in $I_n\big|_{x=0} \cap \mathbb{C}[y]$ and $I_n\big|_{x=0} \cap \mathbb{C}[z]$ respectively, then $I_n\big|_{x=0}$ is radical.

From now on we fix $x = 0$. Then $P = z(-3 + y^2 + z^2)$ and $Q_n = a_n + b_n z^2$ where

$$\begin{aligned} a_n &= S_{n-2}(y) + S_{n-3}(y) - S_{n-4}(y) - S_{n-5}(y), \\ b_n &= -S_{n-2}(y) - S_{n-3}(y). \end{aligned}$$

Let $U_n = a_n + b_n(3 - y^2)$. Then $U_0 = 1, U_1 = y + 1$ and $U_{n+1} = yU_n - U_{n-1}$. Hence

$$U_n = S_n(y) + S_{n-1}(y).$$

We first consider the case $n \geq 3$. Then U_n and a_n have y-degrees n and $n-2$ respectively; moreover their leading coefficients are equal to 1.

Lemma 4.13. *One has*

$$U_n = \prod_{j=1}^{n} (y - 2\cos\frac{j2\pi}{2n+1}).$$

Proof. It is easy to see that U_n is a polynomial of degree n in y. Note that if $y = s + s^{-1} \neq \pm 2$ then $S_k(y) = \frac{s^{k+1} - s^{-k-1}}{s - s^{-1}}$. We now take $y = e^{i\frac{j2\pi}{2n+1}} + e^{-i\frac{j2\pi}{2n+1}} = 2\cos\frac{j2\pi}{2n+1}$ where $1 \leq j \leq n$. Then

$$S_n(y) = \frac{\sin((n+1)\frac{j2\pi}{2n+1})}{\sin(\frac{j2\pi}{2n+1})} = -\frac{\sin(n\frac{j2\pi}{2n+1})}{\sin(\frac{j2\pi}{2n+1})} = -S_{n-1}(y).$$

The lemma follows. □

Lemma 4.14. *One has*

$$a_n = \prod_{k=0}^{n-3} (y - 2\cos\frac{(2k+1)\pi}{2n-5}).$$

Proof. The proof is similar to that of the previous lemma. □

Note that

$$b_n^2\, z\, P = b_n\, z^2 \left((-3 + y^2)b_n + b_n z^2\right) = (Q_n - a_n)(Q_n - U_n).$$

Hence $a_n U_n = a_n Q_n - Q_n^2 + Q_n U_n + b_n^2 z P$ is contained in $I_n |_{x=0}$. But $a_n U_n$ is a polynomial in y, hence it is actually contained in $I_n |_{x=0} \cap \mathbb{C}[y]$. It is easy to see that $a_n U_n$ is square-free, i.e. does not have repeated factors.

Let

$$V_n = z \prod_{j=1}^{n} (-3 + 4\cos^2 \frac{j2\pi}{2n+1} + z^2) \prod_{k=0}^{n-3} (-3 + 4\cos^2 \frac{(2k+1)\pi}{2n-5} + z^2).$$

Then it is easy to show that $V_n \in \mathbb{C}[z]$ is square-free. Moreover, since

$$\begin{aligned} V_n &= z \prod_{j=1}^{n} (-3 + y^2 + z^2 + (4\cos^2 \frac{j2\pi}{2n+1} - y^2)) \\ &\quad \times \prod_{k=0}^{n-3} (-3 + y^2 + z^2 + (4\cos^2 \frac{(2k+1)\pi}{2n-5} - y^2)) \\ &\equiv z \prod_{j=1}^{n} (4\cos^2 \frac{j2\pi}{2n+1} - y^2) \prod_{k=0}^{n-3} (4\cos^2 \frac{(2k+1)\pi}{2n-5} - y^2) \pmod{P}, \\ &\equiv 0 \pmod{(P, a_n U_n)} \end{aligned}$$

it is contained in $I_n |_{x=0}$. Hence V_n is in $I_n |_{x=0} \cap \mathbb{C}[z]$ and is square-free.

Since both $a_n U_n \in I_n |_{x=0} \cap \mathbb{C}[y]$ and $V_n \in I_n |_{x=0} \cap \mathbb{C}[z]$ are square-free, $I_n |_{x=0}$ is a radical ideal by Seidenberg's Lemma. Hence by Proposition 4.12, I_n is also radical. It follows that R/I_n is reduced. Hence the ring $\mathbb{C}[x, y, z]/(P, Q_n)$ is reduced and is equal to the universal character ring of K_{2n+1}. This proves Theorem 4.6 for the case $n \geq 3$. The case $n \leq -1$ is similar (in this case U_n and a_n have y-degrees $3 - n$ and $-(n + 1)$ respectively; moreover their leading coefficients are equal to 1 and -1 respectively). If $0 \leq n \leq 2$ then by direct calculations one can check that $I_n |_{x=0}$ is reduced. This completes the proof of Theorem 4.6 for all integers n. □

5. The skein module of the $(-2, 3, 2n + 1)$-pretzel knot

Let K be the $(-2, 3, 2n + 1)$-pretzel knot and $\mathcal{S}$ be the skein module of $S^3 \setminus K$. Recall from Section 3 that $\mathcal{R} = \mathbb{C}[t^{\pm 1}]$, $\bar{D}$ is the localization of $D = \mathcal{R}[M^{\pm 1}]$ at the ideal $(1 + t)$, and $\bar{\mathcal{S}} = \mathcal{S} \otimes_{D^\sigma} \bar{D}$. The goal of this section is to prove the following.

Proposition 5.1. *The localized skein module $\bar{\mathcal{S}}$ is a finitely generated $\bar{D}$-module.*

For $n = 0, 1$ or 2, K is a torus knot and hence its skein module $\mathcal{S}$ has been understood in [Mar]. Hence we consider $n \geq 3$ or $n \leq -1$ only.

5.1. **Knot complement.** Consider the genus 2 handlebody H_2, which is presented as the cylinder $D^2 \times [0, 1]$ minus two vertical tubes as in Figure 3.

Let x (resp. y) be a small loop on ∂H_2 circling the top of the left (resp. right) hand side tube of ∂H_2, and let z is a small loop on ∂H_2 circling the top of the left hand side tube and the top of the right hand side tube of ∂H_2 as in Figure 4.

For $n \geq 3$ (resp. $n \leq -1$) we consider the closed curve C on ∂H_2 as in Figure 5 (resp. Figure 6), where the part of C along the right hand side tube of ∂H_2 is drawn in the right hand side part of Figure 5 (resp. Figure 6). We also choose the point *pt* on C as in Figure 5 (or Figure 6) and consider it as the based point in H_2.

Let X be the 3-manifold obtained by attaching a 2-handle along C to H_2.

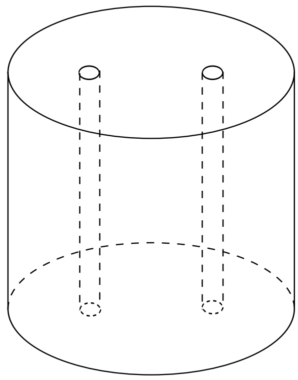


FIGURE 3. The genus 2 handlebody H_2

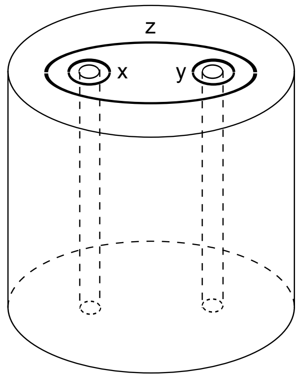


FIGURE 4. The loops x, y and z on ∂H_2

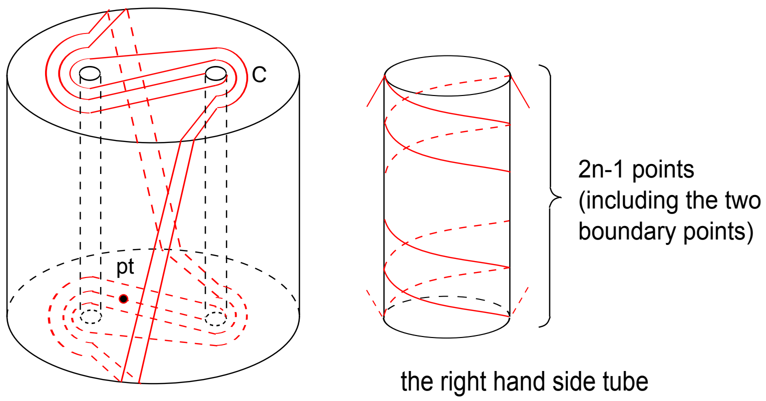


FIGURE 5. The closed curve C on ∂H_2 and the based point pt, for $n \geq 3$.

Lemma 5.2. *There is a homeomorphism between X and the complement of the $(-2, 3, 2n+1)$-pretzel knot K in S^3 under which x is mapped to a meridian of K.*

Proof. Recall from Section 4 that

$$\pi_1(S^3 \setminus K) = \langle a, w \mid w^n awa^{-1}w^{-1}a^{-1} = a^{-1}w^{-1}awaw^{-1}w^n \rangle$$

where a is a meridian of $S^3 \setminus K$.

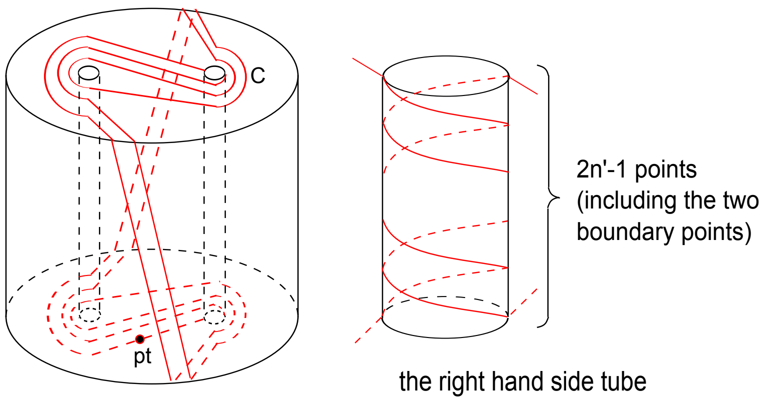


FIGURE 6. The closed curve C on ∂H_2 and the based point pt, for $n \leq -1$. Here $n' = 1 - n$.

We have $\pi_1(H_2) = \langle x, y \mid \rangle$. In $\pi_1(H_2)$, one has

$$C = y^{1-n}x^{-1}y^{-1}x^{-1}yxy^n xyx^{-1}y^{-1}x^{-1} \text{ if } n \geq 3$$

and

$$C = y^{1-n'}xyx^{-1}y^{-1}x^{-1}y^{n'}x^{-1}y^{-1}x^{-1}yx \text{ if } n \leq -1.$$

Here $n' = 1 - n$. Since X is obtained by attaching a 2-handle along C to H_2,

$$\pi_1(X) = \langle x, y \mid y^{1-n}x^{-1}y^{-1}x^{-1}yxy^n xyx^{-1}y^{-1}x^{-1} = 1\rangle.$$

which is isomorphic to $\pi_1(S^3 \setminus K)$ via the isomorphism sending x, y to a, w respectively.

Let H_2' be the 3-manifold obtained by attaching a 2-handle along x to H_2. Then H_2' is just a solid torus whose core is homotopic in H_2' to y. In $\pi_1(H_2')$, one has $x = 1$ and $C = y$. It implies that the 3-manifold obtained by attaching a 2-handle along C to H_2' is a 3-dimensional ball D^3.

Note that this 3-manifold is also obtained by attaching a 2-handle along x to X, since x and C are disjoint. Hence the 3-manifold obtained by attaching a 2-handle along x to X is D^3. Let $D'^3 = \overline{S^3 - D^3}$. Then D'^3 is another 3-dimensional ball in S^3, and $\partial D'^3 = \partial D^3$ is a 2-sphere. The complement of X in S^3 is the 3-manifold obtained by attaching a 2-handle along x to D'^3, and hence is a solid torus. It follows that X is the complement of a knot K' in S^3.

Since $S^3 \setminus K'$ and $S^3 \setminus K$ have isomorphic fundamental groups and K is a prime knot, a theorem of Gordon and Lueck [GoLu] implies that K' is equivalent to K. □

From now on, we identify X with the knot complement $S^3 \setminus K$. Note that x is a meridian of X.

The proofs of Proposition 5.1 in the cases $n \geq 3$ and $n \leq -1$ are similar. Hence in the remaining part of this section, without of loss of generality we assume that $n \geq 3$.

5.2. **Skein module.** We now describe the skein module $\mathcal{S}$ as a quotient of $\mathcal{R}[x, y, z]$ by a submodule.

A type 1 tangle in a 3-manifold Y (with boundary) is the disjoint union of a framed link and a framed arc in Y such that the parts of the arc near the two end points are on the boundary ∂Y, and the framing on these parts are given by vectors normal to ∂Y. Type 1 tangles are considered up to isotopy relative the endpoints.

Recall that X is obtained from H_2 by attaching a 2-handle along the closed curve C. Note that $\mathcal{S}(H_2)$ is isomorphic to the commutative algebra $\mathcal{R}[x, y, z]$ where x, y and z are as defined as above, see [Pr]. The embedding of H_2 into X gives rise to a linear map from $\mathcal{S}(H_2) \equiv \mathcal{R}[x, y, z]$ to $\mathcal{S}(X)$. It is known that the map is surjective, and its kernel N, see [Pr, BL], can be described through slides as follows.

Suppose $\mathfrak{a}$ is a type 1 tangle in H_2 whose 2 endpoints are on C such that outside a small neighborhood of the 2 endpoints $\mathfrak{a}$ is in the interior of H_2 and in a small neighborhood of the endpoints $\mathfrak{a}$ is on the boundary ∂H_2. The two end points of $\mathfrak{a}$ divide C into 2 arcs C_1 and C_2. Let $sl(\mathfrak{a})$ be $\mathfrak{a} \cdot C_1 - \mathfrak{a} \cdot C_2$, considered as an element of the skein module $\mathcal{S}(H_2)$. Here $\mathfrak{a} \cdot C_i$ is the framed link obtained by combining $\mathfrak{a}$ and C_i.

It is clear that as framed links in X, $\mathfrak{a} \cdot C_1$ is isotopic to $\mathfrak{a} \cdot C_2$, since one is obtained from the other by sliding over the 2-handle attached to the curve C. Hence we always have $sl(\mathfrak{a}) \in N$. It was known that N is spanned by all possible $sl(\mathfrak{a})$, where $\mathfrak{a}$ is chosen among all type 1 tangles in H_2 with endpoints on C.

There is a natural bilinear map $\mathcal{S}(\partial H_2) \otimes \mathcal{S}(H_2) \to \mathcal{S}(H_2)$, where $\ell \otimes \ell' \to \ell \star \ell'$, which is the disjoint union of ℓ and ℓ'. Hence N contains all $sl(\mathfrak{a}) \star \mathcal{S}(H_2)$, where $\mathfrak{a}$ is any type 1 tangles in $\partial H_2 \times [0, 1]$ with endpoints on C.

Consider the points u, v, u' and v' on the curve C as in Figure 7. Let $\tilde{P} = sl(\overline{uv})$ and $\tilde{Q} = sl(\overline{u'v'})$ be skein elements in $\mathcal{S}(\partial H_2)$, where $\overline{uv}$ (resp. $\overline{u'v'}$) is the straight segment on ∂H_2 connecting u, v (resp. u', v') whose interior is slightly pushed inside the interior of H_2 (to avoid intersections with other arcs on ∂H_2) and whose framing is given by vector normal to ∂H_2.

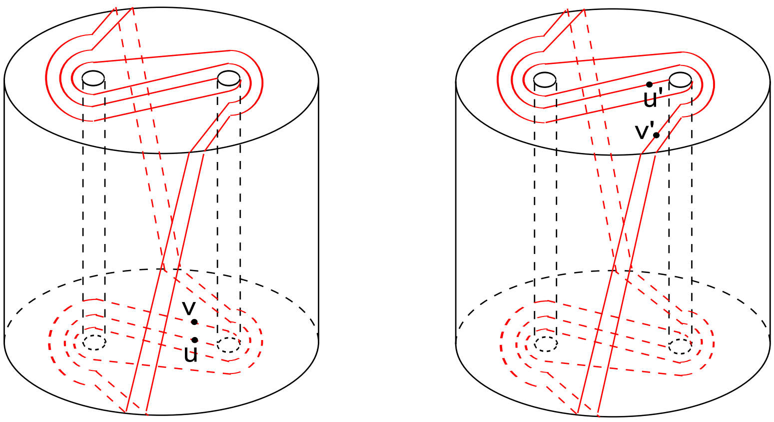


FIGURE 7. The points u, v, u' and v' on C

From the discussion above, one has

Proposition 5.3. *The skein module of the complement of the* $(-2, 3, 2n+1)$*-pretzel knot is* $\mathcal{R}[x, y, z]/N$*, where* N *is an* $\mathcal{R}[x]$*-submodule of* $\mathcal{R}[x, y, z]$ *containing all* $\tilde{P} * y^k z^l$ *and* $\tilde{Q} * y^k z^l$ *with* $k, l \geq 0$.

By isotopies in ∂H_2 one can check that $\tilde{P} = \tilde{P}_1 - \tilde{P}_2$ in $\mathcal{S}(\partial H_2)$, where $\tilde{P}_1$ and $\tilde{P}_2$ are curves on ∂H_2 depicted in Figure 8.

Similarly $\tilde{Q} = \tilde{Q}_1 - \tilde{Q}_2$ in $\mathcal{S}(\partial H_2)$, where $\tilde{Q}_1$ and $\tilde{Q}_2$ are curves on ∂H_2 depicted in Figure 9. Note that Figure 10 explains the notation used in Figure 9.

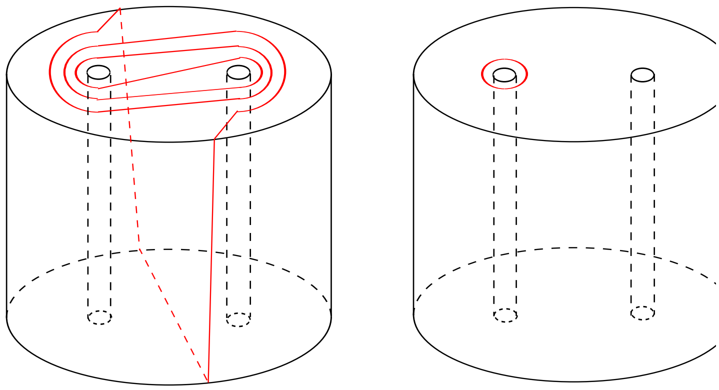


FIGURE 8. The curve $\tilde{P}_1$ on the left and the curve $\tilde{P}_2$ on the right

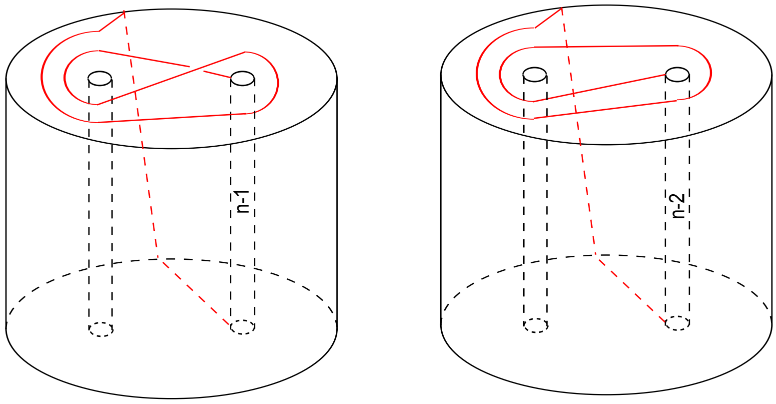


FIGURE 9. The curve $\tilde{Q}_1$ on the left and the curve $\tilde{Q}_2$ on the right

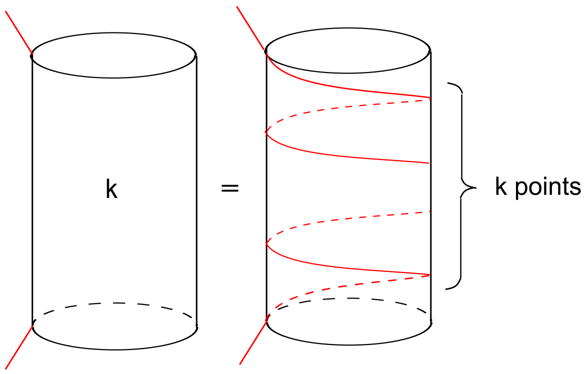


FIGURE 10

Note that $\tilde{P}$ and $\tilde{Q}$ respectively correspond to taking the SL_2-trace of the relations $w^n awa^{-1}w^{-1}a^{-1}w^{1-n} = a^{-1}w^{-1}awa$ and $w^n awa^{-1}w^{-1} = a^{-1}w^{-1}awaw^{n-1}a$ in the fundamental group $\pi_1(S^3 \setminus K)$. Hence $\varepsilon(\tilde{P} \star 1) = P, \varepsilon(\tilde{Q} \star 1) = Q$, where

$$\begin{aligned} P &= x - xy + (-3 + x^2 + y^2)z - xyz^2 + z^3, \\ Q &= S_{n-2}(y) + S_{n-3}(y) - S_{n-4}(y) - S_{n-5}(y) - S_{n-2}(y)\, x^2 \\ &\quad + \big(S_{n-1}(y) + S_{n-3}(y) + S_{n-4}(y)\big)\, xz - \big(S_{n-2}(y) + S_{n-3}(y)\big)\, z^2. \end{aligned}$$

are the defining equations for the character variety of the $(-2,3,2n+1)$-pretzel knot K as in Theorem 4.1.

5.3. **Degrees.** For a monomial $\mathbf{m} := y^k z^l$ define $\deg_y(\mathbf{m}) = k$, $\deg_z(\mathbf{m}) = l$ and $\deg_{yz}(\mathbf{m}) = k+l$. We linearly order the monomials $y^k z^l$ by the lexicographic order of the pair $(\deg_{yz}, \deg_z)$. Using this linear order, for a non-zero element in $\bar{D}[y,z]$ (or in $\mathbb{C}(M)[y,z]$) one can define its leading term, leading coefficient, and leading monomial.

In the discussion below, when talking about polynomials, we assume that the ground ring is either $\bar{D}$ or $\varepsilon(\bar{D}) = \mathbb{C}(M)$.

We say that two polynomials f and g has *equivalent leading term*, and write

$$f \overset{\text{lt}}{=} g,$$

if the leading term of f is a unit times the leading term of g. Here unit means an invertible of the ground ring. For the case when the ground ring is $\mathbb{C}(M)$, a unit is a non-zero element.

Lemma 5.4. *(a) There are polynomials $c_P, c_Q \in \mathbb{C}(M)[y,z]$ with $\deg_{yz}(c_P) \le 3n-3$ and $\deg_{yz}(c_Q) \le 2n$ such that*

$$c_P P + c_Q Q \overset{\text{lt}}{=} y^{3n-2}.$$

(b) There are polynomials $d_P, d_Q \in \mathbb{C}(M)[y,z]$ with $\deg_{yz}(d_P) \le 2n-4$ and $\deg_{yz}(d_Q) \le n-1$ such that

$$d_P P + d_Q Q \overset{\text{lt}}{=} y^{2n-2} z.$$

Proof. (a) In general, if $P = z^3 + az^2 + bz + c$ and $Q = dz^2 + ez + f$, then the resultant $\mathfrak{r}$ of P and Q with respect to z is

$$\mathfrak{r} = c_Q\, Q + c_P\, P,$$

where

$$c_P = (-d^3\, b - d\, e^2 + d^2\, a\, e + d^2\, f)\, z + 2\, e\, d\, f - d^2\, b\, e + d^3\, c + d\, a\, e^2 - a\, d^2\, f - e^3$$

$$\begin{aligned} c_Q =& (e^2 - d\, f + d^2\, b - d\, a\, e)\, z^2 + (a\, d^2\, b - d\, a^2\, e - f\, e - d^2\, c + a\, e^2)\, z \\ &+ d\, a^2\, f + f^2 + d^2\, b^2 - e\, a\, f + d\, e\, c + e^2\, b - 2\, d\, f\, b - d^2\, a\, c - d\, a\, e\, b. \end{aligned}$$

In our case, with explicit P and Q, one can check that c_P is a polynomial in y,z with $\deg_{yz}(c_P) = 3n-3$, c_Q is a polynomial in y,z with $\deg_{yz}(c_Q) = 2n$. By Proposition 4.10, $\mathfrak{r}$ is a polynomial in y with (equivalent) leading term y^{3n-2}.

(b) Let

$$T = P\,\mathrm{coeff}(Q, z^2) - zQ = (ad-e)z^2 + (bd-f)z + cd.$$

Then $\deg_z(T) = 2$. Let

$$\mathfrak{r}' = T\,\mathrm{coeff}(Q,z^2) - Q\,\mathrm{coeff}(T,z^2) = (d(bd-f) - e(ad-e))z + cd^2 - f(ad-e).$$

As in the proof of Proposition 4.10, one can show that $\mathfrak{r}'$ is a polynomial in y,z with (equivalent) leading term $y^{2n-2}z$.

Note that $\mathfrak{r}' = d^2 P - (dz + ad - e)Q$. Let $d_P = d^2$ and $d_Q = -(dz + ad - e)$. Then $\mathfrak{r}' = d_P P + d_Q Q$. With explicit P and Q, it is easy to see that $\deg_{yz}(d_P) = 2n-4$ and $\deg_{yz}(d_Q) = n-1$. This completes the proof of the lemma. $\square$

5.4. **Filtration on $\bar{\mathcal{S}}$.** Recall that $\mathcal{S} = \mathcal{R}[x][y,z]/N$, where N is the submodule defined using sliding relations, see Proposition 5.3. It follows that $\bar{\mathcal{S}} = \mathcal{S} \otimes_{\mathcal{R}[x]} \bar{D}$ is given by

$$\bar{\mathcal{S}} = \bar{D}[y,z]/\mathscr{N},$$

where $\mathscr{N} = N \otimes_{\mathcal{R}[x]} \bar{D}$.

Let $\mathscr{P}$ be the $\bar{D}$-span of $\tilde{P} \star y^k z^l, k, l \geq 0$, and $\mathscr{Q}$ be the $\bar{D}$-span of $\tilde{Q} \star y^k z^l, k, l \geq 0$. By Proposition 5.3,

$$\mathscr{P}, \mathscr{Q} \subset \mathscr{N}. \tag{5.1}$$

Let $\mathfrak{M}_1 = \{z\}$. For $k \geq 2$ let $\mathfrak{M}_k$ be the set $\{y^l z^{k-l}, l = 0, \dots, k-1\} \cup \{y^{k-2}\}$.

For $m \geq 1$ let

$$\mathfrak{M}_{\leq m} := \cup_{k=1}^m \mathfrak{M}_k,$$

and $\mathfrak{U}_m \subset \bar{D}[y,z]$ be the $\bar{D}$-span of $\mathfrak{M}_{\leq m}$, which is a free finitely generated $\bar{D}$-module. Then $\{\mathfrak{U}_m, m \geq 1\}$ forms a filtration of $\bar{D}[y,z]$:

$$\mathfrak{U}_1 \subset \mathfrak{U}_2 \subset \mathfrak{U}_3 \dots, \quad \bigcup \mathfrak{U}_m = \bar{D}[y,z].$$

This filtration induces a filtration on the quotient $\bar{\mathcal{S}} = \bar{D}[y,z]/\mathscr{N}$ as follows. Let $\mathscr{X}_m := \mathscr{X} \cap \mathfrak{U}_m$ for $\mathscr{X} = \mathscr{N}, \mathscr{P}, \mathscr{Q}$, and

$$\bar{\mathcal{S}}_m := \mathfrak{U}_m/\mathscr{N}_m.$$

Then $\bar{\mathcal{S}}_m$ is the set of elements of $\bar{\mathcal{S}}$ which can be represented by an element in $\mathfrak{U}_m$.

There is the natural embedding

$$j_m : \bar{\mathcal{S}}_m \hookrightarrow \bar{\mathcal{S}}_{m+1}.$$

and

$$\bar{\mathcal{S}} = \bigcup_{m=1}^{\infty} \bar{\mathcal{S}}_m.$$

We will show that if $m \geq 3n$, then j_m is an isomorphism. This implies that $\bar{\mathcal{S}}$ is a finitely generated $\bar{D}$-module.

Recall that for a $\bar{D}$-homomorphism $f : V \to V'$, $\varepsilon(f)$ is the map $(V \xrightarrow{f} V') \otimes_{\bar{D}} \varepsilon(\bar{D})$.

Lemma 5.5. *If $\varepsilon(j_m)$ is surjective, then j_m is surjective.*

Proof. Since $\bar{\mathcal{S}}_m$ is a finitely generated module over the local ring $\bar{D}$, surjectivity of j_m follows from Nakayama's lemma as in the proof of Lemma 3.4. □

5.5. **On $\varepsilon(\bar{\mathcal{S}}_m)$.** Recall that $\bar{\mathcal{S}}_m = \mathfrak{U}_m/\mathscr{N}_m$. Let $p : \bar{D}[y,z] \to \mathbb{C}(M)[y,z]$ be the algebra map which sends t to -1.

Let $\mathfrak{u}_m := p(\mathfrak{U}_m)$. Then $\mathfrak{u}_m$ is the $\mathbb{C}(M)$-vector space spanned by $\mathfrak{M}_{\leq m}$. As $\mathscr{N}_m$ is a subset of $\mathfrak{U}_m$, one has $p(\mathscr{N}_m) \subset \mathfrak{u}_m$.

Lemma 5.6. *One has a natural isomorphism*

$$\varepsilon(\bar{\mathcal{S}}_m) = \mathfrak{u}_m/p(\mathscr{N}_m). \tag{5.2}$$

Proof. Taking the tensor product of the exact sequence $\mathscr{N}_m \to \mathfrak{U}_m \to \bar{\mathcal{S}}_m \to 0$ with $\mathbb{C}(M)$ over $\bar{D}$, one gets the following commutative diagram with exact rows

$$\begin{array}{ccccccc} \mathscr{N}_m & \xrightarrow{\iota} & \mathfrak{U}_m & \longrightarrow & \bar{\mathcal{S}}_m & \longrightarrow & 0 \\ \downarrow q & & \downarrow p & & \downarrow & & \\ \varepsilon(\mathscr{N}_m) & \xrightarrow{\varepsilon(\iota)} & \varepsilon(\mathfrak{U}_m) & \longrightarrow & \varepsilon(\bar{\mathcal{S}}_m) & \longrightarrow & 0 \end{array} \tag{5.3}$$

where $q : \mathscr{N}_m \to \varepsilon(\mathscr{N}_m)$ is the natural map. Since q is surjective, one has

$$\operatorname{Im}(\varepsilon(\iota)) = \operatorname{Im}(\varepsilon(\iota) \circ q) = \operatorname{Im}(p \circ \iota) = p(\mathscr{N}_m).$$

The exactness of the second row of Diagram (5.3) shows that

$$\varepsilon(\bar{\mathcal{S}}_m) = \varepsilon(\mathfrak{U}_m)/\operatorname{Im}(\varepsilon(\iota)) = \varepsilon(\mathfrak{U}_m)/p(\mathscr{N}_m) = \mathfrak{u}_m/p(\mathscr{N}_m).$$

This completes the proof of the lemma. □

5.6. **On the filtrations $\mathscr{P}_m$ and $\mathscr{Q}_m$.**

Lemma 5.7. *Suppose* $f(y,z) \in \mathbb{C}(M)[y,z]$ *such that* $\deg_{yz} f \le m$. *Then*

$$\tilde{P} \star f \in \mathscr{P}_{m+3} \subset \mathscr{N}_{m+3},$$
$$\tilde{Q} \star f \in \mathscr{Q}_{m+n} \subset \mathscr{N}_{m+n}.$$

Proof. Recall that H_2 is the cylinder $D^2 \times [0,1]$ minus two vertical tubes. Let D_{**} be the top surface of H_2, which is the disk $D^2 \times 1$ minus 2 holes. Let a_0 and a_2 be the straight arcs on D_{**} depicted in Figure 11.

To prove Lemma 5.7 we apply the upper bound and parity of $\deg_{yz}$ using the intersection number of link diagrams (on D_{**}) of $\tilde{P} \star f$ and $\tilde{Q} \star f$ with the arc a_2, and an upper bound for $\deg_y$ using the intersection with a_0 as in [Le1, Lemma 5.1].

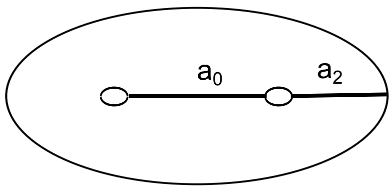


FIGURE 11. The straight segments a_0 and a_2 on D_{**}

We can assume that f is a monomial, i.e. $f = y^k z^l$ for some $k, l \ge 0$. We will only show that $\tilde{P} \star f \in \mathscr{P}_{k+l+3}$. (The proof that $\tilde{Q} \star f \in \mathscr{Q}_{k+l+n}$ is similar.)

We have $\tilde{P} \star y^k z^l = \tilde{P}_1 \star y^k z^l - \tilde{P}_2 \star y^k z^l$, where $\tilde{P}_1$ and $\tilde{P}_2$ are depicted in Figure 8. Let k_0 (resp. k_2) be the intersection number of the link diagram (on D_{**}) of $\tilde{P}_1 \star y^k z^l$ with a_0 (resp. a_2). It is easy to see that $k_0 = k+2$ and $k_2 = k+l+3$.

Suppose $c_{a,b}\, y^a z^b$ is a monomial of $\tilde{P}_1 \star y^k z^l \in \mathbb{C}[x][y,z]$, where $a, b \ge 0$ and $c_{a,b} \in \mathbb{C}[x]$. By [Le1, Lemma 5.1], one has

$$(\deg c_{a,b}) + a \le k_0 = k+2,$$
$$a + b \le k_2 = k+l+3, \quad \text{and} \quad a+b \equiv k+l+3 \pmod 2.$$

If $b > 0$, then since $a+b \le k+l+3$ one has $c_{a,b}\, y^a z^b \in \mathfrak{U}_{k+l+3}$.

If $b = 0$, then $a \le k+2 < k+l+3$. Since $a \equiv k+l+3 \pmod 2$, we must have $a \le k+l+1$ which implies that $c_{a,b}\, y^a \in \mathfrak{U}_{k+l+3}$.

Hence $\tilde{P}_1 \star y^k z^l \in \mathfrak{U}_{k+l+3}$. Similarly, $\tilde{P}_2 \star y^k z^l \in \mathfrak{U}_{k+l+3}$. It follows that $\tilde{P} \star y^k z^l = \tilde{P}_1 \star y^k z^l - \tilde{P}_2 \star y^k z^l$ is in $\mathfrak{U}_{k+l+3} \cap \mathscr{P} = \mathscr{P}_{k+l+3}$. □

5.7. **Surjectivity of $\varepsilon(j_m)$.**

Lemma 5.8. *If $m \ge 3n$, then $\varepsilon(j_{m-1})$ is surjective.*

Proof. By Lemma 5.6, $\varepsilon(\bar{\mathcal{S}}_m) \cong \mathfrak{u}_m / p(\mathscr{N}_m)$, and

$$\varepsilon(j_{m-1}) : \mathfrak{u}_{m-1}/p(\mathscr{N}_{m-1}) \to \mathfrak{u}_m / p(\mathscr{N}_m)$$

descends from the embedding $\mathfrak{u}_{m-1} \hookrightarrow \mathfrak{u}_m$. It follows that $\varepsilon(j_{m-1})$ is surjective if and only

$$\mathfrak{u}_m \subset \mathfrak{u}_{m-1} + p(\mathscr{N}_m),$$

which, because of (5.1), will follow from

$$\mathfrak{u}_m \subset \mathfrak{u}_{m-1} + p(\mathscr{P}_m) + p(\mathscr{Q}_m). \tag{5.4}$$

Hence to prove Lemma 5.8, one only needs to prove (5.4).

Claim 1: One has

$$\mathfrak{u}_m \;\subset\; \left[\mathfrak{u}_{m-1} + p(\mathscr{P}_m) + (\mathbb{C}(M)\text{-span of } \{y^{m-2}z^2, y^{m-1}z, y^{m-2}\})\right],$$

i.e. modulo $(\mathfrak{u}_{m-1} + p(\mathscr{P}_m))$, $\mathfrak{u}_m$ is $\mathbb{C}(M)$-spanned by $y^{m-2}z^2, y^{m-1}z, y^{m-2}$.

Proof. (of Claim 1) Modulo $\mathfrak{u}_{m-1}$, $\mathfrak{u}_m$ is $\mathbb{C}(M)$-spanned by

$$\mathfrak{M}_m = \{z^m, yz^{m-1}, \dots, y^{m-2}z^2, y^{m-1}z\} \cup \{y^{m-2}\}.$$

Note that $P \in p(\mathscr{P}_3)$, with leading term z^3. Modulo $p(\mathscr{P}_m)$, any monomial in $\mathfrak{M}_m$ with z-degree ≥ 3 can be reduced to a sum of terms with z-degree less than 3. Since the elements of $\mathfrak{M}_m$ with z-degree less than 3 are $y^{m-2}z^2, y^{m-1}z$ and y^{m-2}, Claim 1 follows. □

Claim 2: One has

$$y^{m-2}z^2 \;\in\; \left[\mathfrak{u}_{m-1} + p(\mathscr{Q}_m) + (\mathbb{C}(M)\text{-span of } \{y^{m-1}z, y^{m-2}\})\right].$$

Proof. (of Claim 2) From Lemma 5.7 one has

$$Q\, y^{m-n} = p(\tilde{Q} \star y^{m-n}) \in p(\mathscr{Q}_m).$$

Note that $Q\, y^{m-n}$ has leading monomial $y^{m-2}z^2$. Hence modulo $p(\mathscr{Q}_m)$, $y^{m-2}z^2$ can be reduced to a sum of terms in $\mathfrak{u}_m$ such that each of them is $< y^{m-2}z^2$ (by the lexicographic order of the pair $(\deg_{yz}, \deg_z)$). Since a term in $\mathfrak{u}_m$ that is $< y^{m-2}z^2$ is either $y^{m-1}z, y^{m-2}$, or a term in $\mathfrak{u}_{m-1}$, Claim 2 follows. □

Claim 3: One has

$$y^{m-1}z \;\in\; \left[\mathfrak{u}_{m-1} + p(\mathscr{P}_m) + p(\mathscr{Q}_m) + (\mathbb{C}(M)\text{-span of } \{y^{m-2}\})\right].$$

Proof. (of Claim 3) From Lemma 5.7 one has

$$(d_P P + d_Q Q) y^{m-(2n-1)} = \varepsilon(\tilde{P} \star d_P y^{m-(2n-1)}) + \varepsilon(\tilde{Q} \star d_Q y^{m-(2n-1)}) \in p(\mathscr{P}_m) + p(\mathscr{Q}_m).$$

By Lemma 5.4(b), one has

$$(d_P P + d_Q Q) y^{m-(2n-1)} \stackrel{\text{lt}}{=} y^{m-1}z.$$

Hence modulo $p(\mathscr{P}_m) + p(\mathscr{Q}_m)$, $y^{m-1}z$ can be reduced to a sum of terms in $\mathfrak{u}_m$ such that each of them is $< y^{m-1}z$. Since a term in $\mathfrak{u}_m$ that is $< y^{m-1}z$ is either y^{m-2} or a term in $\mathfrak{u}_{m-1}$, Claim 3 follows. □

Claim 4: One has

$$y^{m-2} \in [\mathfrak{u}_{m-1} + p(\mathscr{P}_m) + p(\mathscr{Q}_m)].$$

Proof. (of Claim 4) From Lemma 5.7 one has

$$(c_P P + c_Q Q) y^{m-3n} = \varepsilon(\tilde{P} \star c_P y^{m-3n}) + \varepsilon(\tilde{Q} \star c_Q y^{m-3n}) \in p(\mathscr{P}_m) + p(\mathscr{Q}_m).$$

By Lemma 5.4(a), one has

$$(c_P P + c_Q Q) y^{m-3n} \overset{\text{lt}}{=} y^{m-2}.$$

Modulo $p(\mathscr{P}_m) + p(\mathscr{Q}_m)$, y^{m-2} can be reduced to a sum of terms in $\mathfrak{u}_m$ such that each of them is $< y^{m-2}$. Since a term in $\mathfrak{u}_m$ that is $< y^{m-2}$ is a term in $\mathfrak{u}_{m-1}$, Claim 4 follows. □

From Claims 1, 2, 3 and 4, one has (5.4). Lemma 5.8 then follows. □

5.8. **Proof of Proposition 5.1.** By Lemma 5.5 and Lemma 5.8, the map j_m is an isomorphism for $m \geq 3n$. It follows that

$$\bar{\mathcal{S}}_{3n} = \bar{\mathcal{S}}_{3n+1} = \bar{\mathcal{S}}_{3n+2} = \dots.$$

Hence $\bar{\mathcal{S}} = \bar{\mathcal{S}}_{3n}$, which is finitely generated over $\bar{D}$.

Appendix A. Character varieties of two-bridge knots

We first review the description of character varieties of two-bridge knots from [Le3]. Suppose $K = \mathfrak{b}(p,m)$ is a two-bridge knot. Let $X = S^3 \setminus K$. Then

$$\pi_1(X) = \langle a, b \mid wa = bw \rangle,$$

where both a and b are meridians. The word w has the form $a^{\varepsilon_1} b^{\varepsilon_2} \cdots a^{\varepsilon_{p-2}} b^{\varepsilon_{p-1}}$, where $\varepsilon_j := (-1)^{\lfloor jm/p \rfloor}$. In particular, if we read w from right to left and interchange a and b then we get w again. For example, $\mathfrak{b}(p,1)$ is the torus knot $T(2,p)$, and in this case $w = (ab)^d$, where $d := (p-1)/2$.

We adopt the convention that if $r : \pi_1(X) \to SL_2(\mathbb{C})$ is a representation and u is a word then we write u also for $r(u)$ and $|u|$ for $\operatorname{tr} r(u)$. If u is a word then u' denotes the word obtained from u by deleting the two letters at the two ends.

Let $x := |a| = |b|$ and $z := |ab|$. It was shown in [Le3] that the non-abelian character variety, i.e. the set of characters of non-abelian representations, of $\pi_1(X)$ is the zero set of the polynomial

$$\Phi_{(p,m)}(x,z) = |w| - |w'| + \cdots + (-1)^{d-1}|w^{(d-1)}| + (-1)^d.$$

Moreover $\Phi_{(p,m)}(x,z)$ is a polynomial in $\mathbb{Z}[x^2, z]$ with z-leading term z^d.

A.1. **Irreducibility over $\mathbb{Q}$.** Let $\Phi_d(x,z) = \Phi_{(p,1)}(x,z)$, where $d = (p-1)/2$. It was shown in [Le3, Proposition 4.3.1] (also see below) that $\Phi_d(x,z)$ does not depend on x.

Proposition A.1. *The polynomial $\Phi_d(z)$ is irreducible over $\mathbb{Q}$ if and only if $p = 2d+1$ is prime.*

Proof. It is immediate from [Le3, Proposition 4.3.1] that $\Phi_d(z) = S_d(z) - S_{d-1}(z)$, where $S_n(z)$'s are the Chebyshev polynomials defined by $S_0(z) = 1$, $S_1(z) = z$ and $S_{n+1}(z) = z\,S_n(z) - S_{n-1}(z)$. By similar arguments as in the proof of Lemma 4.13 one can show that $\Phi_d(z)$ is an integer polynomial of degree d with exactly d roots given by $z = 2\cos\left(\frac{2j+1}{2d+1}\pi\right)$, $0 \le j \le d-1$. It follows that the splitting field of $\Phi_d(z)$ is $\mathbb{Q}(\cos\eta)$, where $\eta := \pi/p$. Hence $\Phi_d(z)$ is irreducible over $\mathbb{Q}$ if and only if the extension field degree $[\mathbb{Q}(\cos\eta):\mathbb{Q}]$ is exactly the degree of Φ_d, which is d.

Note that $\cos\eta = (e^{i\eta} + e^{-i\eta})/2$. We need to study the extension field $\mathbb{Q}(e^{i\eta})/\mathbb{Q}$. It is well-known that the minimal polynomial over $\mathbb{Q}$ of $e^{i\eta}$ is the cyclotomic polynomial

$$C_{2p}(t) = \prod_{1\le j\le 2p,\ \gcd(j,2p)=1} (t - e^{j\pi i/p}),$$

see e.g. [La]. This is an integer polynomial whose degree is $\varphi(2p) = \varphi(p)$, where φ is the Euler totient function. Thus the degree of the extension field is $[\mathbb{Q}(e^{i\eta}):\mathbb{Q}] = \varphi(p)$. From the identity $(t-e^{i\eta})(t-e^{-i\eta}) = t^2 - 2(\cos\eta)t + 1$, we see that $[\mathbb{Q}(e^{i\eta}):\mathbb{Q}(\cos\eta)] = 2$, thus $[\mathbb{Q}(\cos\eta):\mathbb{Q}] = \varphi(p)/2$. Therefore $\Phi_d(z)$ is irreducible over $\mathbb{Q}$ if and only if $\varphi(p) = p-1$, which occurs if and only if p is prime. ∎

Proposition A.2. *One has $\Phi_{(p,m)}(0,z) = \Phi_{(p,1)}(z)$. Hence if $\Phi_{(p,1)}(z)$ is irreducible in $\mathbb{Q}[z]$ then $\Phi_{(p,m)}(x,z)$ is irreducible in $\mathbb{Q}[x,z]$.*

Proof. If $x = |a| = |b| = 0$ then $a^{-1} = -a$ and $b^{-1} = -b$. (This follows from the Cayley-Hamilton theorem applying for matrices in $SL_2(\mathbb{C})$: $a + a^{-1} = |a|\, I_{2\times 2}$, where $I_{2\times 2}$ is the 2×2 identity matrix.)

Recall that $\Phi_{(p,m)}(x,z) = |w| - |w'| + \cdots + (-1)^{d-1}|w^{(d-1)}| + (-1)^d$. From the definition of the word w, it is easy to see that a^{-1} and b^{-1} appear in pairs in w. This is also true for a^{-1} and b^{-1} in each word $w^{(j)}$, $0 \le j \le d-1$, hence $w^{(j)}$ does not change if one simultaneously replaces a^{-1} by a and b^{-1} by b. Thus $w^{(j)} = (ab)^{d-j}$. Note that for the torus knot $\mathfrak{b}(2d+1,1)$ we have $w = (ab)^d$, hence the proposition follows. ∎

A.2. **Irreducibility over $\mathbb{C}$.** For a word u, let $\overleftarrow{u}$ be the word obtained from u by writing the letters in u in reversed order. Then, by [Le3, Lemma 3.2.2], $|\overleftarrow{u}| = |u|$ for any word u in 2 letters a and b.

Suppose $\nu_1, \nu_2, \dots, \nu_d \in \{-1, 1\}$. Let $\nu_{d+j} := \nu_{d+1-j}$ for $j = 1, \dots, d$. Let

$$w_j = a^{\nu_j} b^{\nu_{j+1}} \dots a^{\nu_{2d-j}} b^{\nu_{2d+1-j}}.$$

Then $w_1 = a^{\nu_1} b^{\nu_2} \dots a^{\nu_{2d-1}} b^{\nu_{2d}}$ and $w_{j+1} = (w_j)'$.

Let $\mu_j := \nu_j \nu_{j+1}$ for $j = 1, \dots, d$. Note that $\mu_d = 1$. Let c_j be the number of $\mu_k = -1$ among $\mu_j, \dots, \mu_d$.

Recall that $x = |a| = |b|$ and $z = |ab|$. Let $X := x^2$.

Proposition A.3. *$|w_j|$ is a polynomial in X, z of total degree $d+1-j$ and*

$$|w_j| = z^{d+1-j-c_j}(z - X)^{c_j} + l.o.t.$$

Here l.o.t. is the term of total degree $< d+1-j$.

Proof. Let $u_j := w_{j+1}a^{\nu_j} = a^{\nu_{j+1}}\dots b^{\nu_{j+1}}a^{\nu_j}$ and $v_j := b^{\nu_j}w_{j+1} = b^{\nu_j}a^{\nu_{j+1}}\dots b^{\nu_{j+1}}$ for $j=1,\dots,d$, where $w_{d+1}:=1$. We will show that

1) $x|u_j|$ and $x|v_j|$ are polynomials in X,z of total degree $\le d+1-j$,
2) $|w_j|$ is a polynomial in X,z of total degree d_j and

$$|w_j| = z^{d+1-j-c_j}(z-X)^{c_j} + l.o.t.,$$

by induction on $1\le j\le d$, beginning with $j=d$ which is obvious.

Suppose $j\le d-1$. Consider the following 2 cases: $\nu_j\nu_{j+1}=1$ and $\nu_j\nu_{j+1}=-1$.

Case 1: $\nu_j\nu_{j+1}=1$, i.e. $\nu_j=\nu_{j+1}$. Then $c_j=c_{j+1}$ and

$$\begin{aligned}
x|u_j| &= x|(a^{\nu_{j+1}}b^{\nu_{j+2}}\dots a^{\nu_{j+2}}b^{\nu_{j+1}})a^{\nu_{j+1}}| \\
&= x|(a^{\nu_{j+1}}b^{\nu_{j+2}}\dots a^{\nu_{j+2}}b^{\nu_{j+1}})(xI_{2\times2}-a^{-\nu_{j+1}})| \\
&= x^2|a^{\nu_{j+1}}b^{\nu_{j+2}}\dots a^{\nu_{j+2}}b^{\nu_{j+1}}| - x|b^{\nu_{j+2}}\dots a^{\nu_{j+2}}b^{\nu_{j+1}}| \\
&= x^2|w_{j+1}| - x|\overleftarrow{v_{j+1}}| = x^2|w_{j+1}|-x|v_{j+1}|, \\
x|v_j| &= x|b^{\nu_{j+1}}(a^{\nu_{j+1}}b^{\nu_{j+2}}\dots a^{\nu_{j+2}}b^{\nu_{j+1}})| \\
&= x|(xI_{2\times2}-b^{-\nu_{j+1}})(a^{\nu_{j+1}}b^{\nu_{j+2}}\dots a^{\nu_{j+2}}b^{\nu_{j+1}})| \\
&= x^2|a^{\nu_{j+1}}b^{\nu_{j+2}}\dots a^{\nu_{j+2}}b^{\nu_{j+1}}| - x|a^{\nu_{j+1}}b^{\nu_{j+2}}\dots a^{\nu_{j+2}}| \\
&= x^2|w_{j+1}| - x|\overleftarrow{u_{j+1}}| = x^2|w_{j+1}|-x|u_{j+1}|, \\
|w_j| &= |(a^{\nu_{j+1}}b^{\nu_{j+1}})a^{\nu_{j+2}}\dots b^{\nu_{j+2}}(a^{\nu_{j+1}}b^{\nu_{j+1}})| \\
&= |(a^{\nu_{j+1}}b^{\nu_{j+1}})a^{\nu_{j+2}}\dots b^{\nu_{j+2}}(zI_{2\times2}-(a^{\nu_{j+1}}b^{\nu_{j+1}})^{-1})| \\
&= z|a^{\nu_{j+1}}b^{\nu_{j+1}}a^{\nu_{j+2}}\dots b^{\nu_{j+2}}| - |a^{\nu_{j+2}}\dots b^{\nu_{j+2}}| \\
&= z|b^{\nu_{j+1}}a^{\nu_{j+2}}\dots b^{\nu_{j+2}}a^{\nu_{j+1}}| - |w_{j+2}| \\
&= z|\overleftarrow{w_{j+1}}| - |w_{j+2}| = z|w_{j+1}|-|w_{j+2}|.
\end{aligned}$$

It follows that 1) and 2) hold true for j, by induction hypothesis.

Case 2: $\nu_j\nu_{j+1}=-1$, i.e. $\nu_j=-\nu_{j+1}$. Then $c_j=c_{j+1}+1$ and

$$\begin{aligned}
x|u_j| &= x|(a^{\nu_{j+1}}b^{\nu_{j+2}}\dots a^{\nu_{j+2}}b^{\nu_{j+1}})a^{-\nu_{j+1}}| = x|b^{\nu_{j+2}}\dots a^{\nu_{j+2}}b^{\nu_{j+1}}| \\
&= x|\overleftarrow{v_{j+1}}| = x|v_{j+1}|, \\
x|v_j| &= x|b^{-\nu_{j+1}}(a^{\nu_{j+1}}b^{\nu_{j+2}}\dots a^{\nu_{j+2}}b^{\nu_{j+1}})| = x|a^{\nu_{j+1}}b^{\nu_{j+2}}\dots a^{\nu_{j+2}}| \\
&= x|\overleftarrow{u_{j+1}}| = x|u_{j+1}|, \\
|w_j| &= |a^{-\nu_{j+1}}b^{\nu_{j+1}}a^{\nu_{j+2}}\dots b^{\nu_{j+2}}a^{\nu_{j+1}}b^{-\nu_{j+1}}| \\
&= |(xI_{2\times2}-a^{\nu_{j+1}})b^{\nu_{j+1}}a^{\nu_{j+2}}\dots b^{\nu_{j+2}}a^{\nu_{j+1}}(xI_{2\times2}-b^{\nu_{j+1}})| \\
&= x^2|b^{\nu_{j+1}}a^{\nu_{j+2}}\dots b^{\nu_{j+2}}a^{\nu_{j+1}}| + |a^{\nu_{j+1}}b^{\nu_{j+1}}a^{\nu_{j+2}}\dots b^{\nu_{j+2}}a^{\nu_{j+1}}b^{\nu_{j+1}}| \\
&\quad - x|a^{\nu_{j+1}}b^{\nu_{j+1}}a^{\nu_{j+2}}\dots b^{\nu_{j+2}}a^{\nu_{j+1}}| - x|b^{\nu_{j+1}}a^{\nu_{j+2}}\dots b^{\nu_{j+2}}a^{\nu_{j+1}}b^{\nu_{j+1}}|. \qquad \text{(A.1)}
\end{aligned}$$

We have

$$|b^{\nu_{j+1}}a^{\nu_{j+2}}\dots b^{\nu_{j+2}}a^{\nu_{j+1}}| = |\overleftarrow{w_{j+1}}| = |w_{j+1}|. \qquad \text{(A.2)}$$

By Case 1,

$$|a^{\nu_{j+1}}b^{\nu_{j+1}}a^{\nu_{j+2}}\dots b^{\nu_{j+2}}a^{\nu_{j+1}}b^{\nu_{j+1}}| = z|w_{j+1}|-|w_{j+2}|. \qquad \text{(A.3)}$$

We have

$$
\begin{aligned}
x|a^{\nu_{j+1}}b^{\nu_{j+1}}a^{\nu_{j+2}}\dots b^{\nu_{j+2}}a^{\nu_{j+1}}| &= x|(xI_{2\times 2}-a^{-\nu_{j+1}})b^{\nu_{j+1}}a^{\nu_{j+2}}\dots b^{\nu_{j+2}}a^{\nu_{j+1}}| \\
&= x^2|b^{\nu_{j+1}}a^{\nu_{j+2}}\dots b^{\nu_{j+2}}a^{\nu_{j+1}}| - x|b^{\nu_{j+1}}a^{\nu_{j+2}}\dots b^{\nu_{j+2}}| \\
&= x^2|\overleftarrow{w_{j+1}}| - x|v_{j+1}| = x^2|w_{j+1}| - x|v_{j+1}|. \qquad \text{(A.4)}
\end{aligned}
$$

Similarly,

$$x|b^{\nu_{j+1}}a^{\nu_{j+2}}\dots b^{\nu_{j+2}}a^{\nu_{j+1}}b^{\nu_{j+1}}| = x^2|w_{j+1}| - x|u_{j+1}|. \qquad \text{(A.5)}$$

From (A.1), (A.2), (A.3), (A.4) and (A.5), we get

$$|w_j| = (z-x^2)|w_{j+1}| - |w_{j+2}| + x|u_{j+1}| + x|v_{j+1}|. \qquad \text{(A.6)}$$

Hence, by induction hypothesis, $|w_j|$ is a polynomial in X, z of total degree $d+1-j$ and

$$|w_j| = (z-X)z^{d-j-c_{j+1}}(z-X)^{c_{j+1}} + l.o.t. = z^{d+1-j-c_j}(z-X)^{c_j} + l.o.t.$$

where l.o.t. is the term of total degree $< d+1-j$, since $c_j = c_{j+1}+1$. □

Applying Proposition A.3 with $\nu_j = \varepsilon_j = (-1)^{\lfloor jm/p \rfloor}$,

$$\Phi_{(p,m)}(x,z) = |w| - |w'| + \dots + (-1)^{d-1}|w^{(d-1)}| + (-1)^d$$

is a polynomial in X, z of total degree $d = \frac{p-1}{2}$.

Let $\Gamma_{(p,m)}(X,z) := \Phi_{(p,m)}(x,z) \in \mathbb{Z}[X,z]$. Then, also by Proposition A.3,

$$\Gamma_{(p,m)}(X,z) = z^{d-c}(z-X)^c + l.o.t.,$$

where l.o.t. is the term of total degree $< d$ and c is the number of $\mu_k = -1$ among $\mu_1, \dots, \mu_d$. Note that $c = \frac{m-1}{2}$, see e.g. [Bur].

Corollary A.4. *$(0, \frac{p-1}{2})$ and $(\frac{m-1}{2}, \frac{p-m}{2})$ are vertices of the Newton polygon of the polynomial $\Gamma_{(p,m)}(X,z) \in \mathbb{Z}[X,z]$.*

Theorem A.5. *(i) Suppose $\Phi_{(p,m)}(x,z)$ is irreducible in $\mathbb{Q}[x,z]$ and $\gcd(\frac{p-1}{2}, \frac{m-1}{2}) = 1$. Then $\Phi_{(p,m)}(x,z)$ is irreducible in $\mathbb{C}[x,z]$.*

(ii) Suppose p is prime and $\gcd(\frac{p-1}{2}, \frac{m-1}{2}) = 1$. Then $\Phi_{(p,m)}(x,z)$ is irreducible in $\mathbb{C}[x,z]$.

Proof. (i) Suppose $\Phi_{(p,m)}(x,z)$ is irreducible in $\mathbb{Q}[x,z]$. Then $\Gamma_{(p,m)}(X,z)$ is irreducible in $\mathbb{Q}[X,z]$. Note that $(0, \frac{p-1}{2})$ and $(\frac{m-1}{2}, \frac{p-m}{2})$ are vertices of the Newton polygon of the polynomial $\Gamma_{(p,m)}(X,z) \in \mathbb{Z}[X,z]$ by Corollary A.4, and $\gcd\big((0, \frac{p-1}{2}), (\frac{m-1}{2}, \frac{p-m}{2})\big) = 1$ since $\gcd(\frac{p-1}{2}, \frac{m-1}{2}) = 1$. Hence [BCG, Proposition 3] implies that $\Gamma_{(p,m)}(X,z)$ is irreducible in $\mathbb{C}[X,z]$.

Assume that $\Phi_{(p,m)}(x,z)$ is reducible in $\mathbb{C}[x,z]$ and $f(x,z)$ is an irreducible factor of $\Phi_{(p,m)}(x,z)$. Write $f(x,z) = g(X,z) + xh(X,z)$, where $g, h \in \mathbb{C}[X,z]$. If $h \equiv 0$ then $g(X,z) = f(x,z)$ is an irreducible factor of $\Gamma_{(p,m)}(X,z)$ in $\mathbb{C}[X,z]$ and the total degree of $g(X,z)$ is less than that of $\Gamma_{(p,m)}(X,z)$, a contradiction since $\Gamma_{(p,m)}(X,z)$ is irreducible in $\mathbb{C}[X,z]$. Hence $h \not\equiv 0$. Note that $f(-x,y) = g(X,z) - xh(X,z)$ is also an irreducible factor of $\Phi_{(p,m)}(x,z)$ and $f(-x,z) \neq f(x,z)$. It follows that $f(x,y)f(-x,y) \in \mathbb{C}[X,z]$ is a factor of $\Gamma_{(p,m)}(X,z)$. Since $\Gamma_{(p,m)}(X,z)$ is irreducible in $\mathbb{C}[X,z]$, we must have $f(x,z)f(-x,z) = \Gamma_{(p,m)}(X,z)$. In particular $f^2(0,z) = \Gamma_{(p,m)}(0,z) = \Phi_{(p,m)}(0,z)$. This is impossible since $\Phi_{(p,m)}(0,z)$ does not have repeated factors, according to the proof of Proposition A.1. Hence $\Phi_{(p,m)}(x,z)$ is irreducible in $\mathbb{C}[x,z]$.

(ii) Since p is prime, by Propositions A.1 and A.2, $\Phi_{(p,m)}(x,z)$ is irreducible in $\mathbb{Q}[x,z]$. The conclusion follows from Part (i). $\square$

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