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Parallel version of the universal Vassiliev–Kontsevich invariant

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Abstract

Let $\hat{Z}_f$ be the universal Vassiliev–Kontsevich invariant for framed links in [13], which is a generalization of Kontsevich's invariant in [10, 1]. Let K be a framed knot and $K^{(r)}$ be its r-parallel. Then we show $\hat{Z}_f(K^{(r)}) = \Delta_{(r)}(\hat{Z}_f(K))$, where $\Delta_{(r)}$ is an operation of chord diagrams which replace the Wilson loop by r copies. We calculate the values of $\hat{Z}_f$ of the Hopf links and the change of $\hat{Z}_f$ under the Kirby moves. An explicit formula of an important normalization factor, which is the value of the trivial knot, in the universal enveloping algebra $U(\mathfrak{g})$ of any Lie algebra is given. © 1997 Elsevier Science B.V.

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1. Introduction

Kontsevich found an invariant $\hat{Z}$ of knots with values in the chord diagram algebra $\mathscr{A}_0$ which is as powerful as the set of all Vassiliev invariants (invariants of finite types). This invariant is now called the universal Vassiliev–Kontsevich invariant, although it is not of finite type. We can also recover many knot invariants coming from quantum groups from $\hat{Z}$. So it may be very natural to think about extending $\hat{Z}$ to 3-manifold invariants. Towards this goal, we extended $\hat{Z}$ to an invariant of framed links in our previous paper [13]. In the Reshetikhin–Turaev construction for Witten's 3-manifold invariant in [18], the tensor product structure of a certain category plays an important role. In Lickorish's construction in [15], invariants of parallel links take the same role. Here we would like to prove the formula for the universal Vassiliev–Kontsevich invariant of parallel links, which was announced in [14, Theorem 4].

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This formula would be useful to study 3-manifold invariants related to the universal Vassiliev–Kontsevich invariant. We discuss the group-like property of $\hat{Z}_f$, which is closely related to the parallel formula (for knots this property had been known in 3). Then we calculate the values of the universal invariants of the Hopf links and the tangles that express the Kirby moves. These values are expressed in terms of an element v which is the value of the universal invariant of the trivial knot. We also give a proof of a Theorem announced in [14, Theorem 11]. Finally, we give a formula which describes the element v in the universal enveloping algebra of sl_2 through Bernoulli polynomials on the Casimir central element. Generalization for other simple Lie algebras is given.

The definition of universal invariants involves a special element, called an *associator*, an analogue of associators that appeared in Drinfeld's theory of quasi-Hopf algebras, although we do not have any quasi-Hopf algebra. In order to establish the formula of parallel version of tangles in the most general form, we have to use a special "associator" (see Section 3) which possesses more symmetry than the well-known associator derived from the Knizhnik–Zamolodchikov equation. The existence of such an associator was proved by Drinfeld. We would like to emphasize that the parallel formula is not valid for tangles in general, if one uses the associator derived from the Knizhnik–Zamolodchikov equation. For *links* only, one can use any associator, since in [14] we proved that the value of the universal Vassiliev–Kontsevich invariant of links does not depend on the associators. In practice, very often one has to deal with tangles with non-empty boundary.

2. Preliminary definitions and facts

We recall some definitions from theory of the universal Kontsevich–Vassiliev invariants for *framed* oriented q-tangles. For details we refer the reader to [13, 14], see also [2].

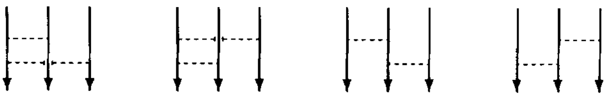
Suppose X is a one-dimensional compact oriented smooth manifold whose components are numbered. *A chord diagram with support* X is a set consisting of a finite number of unordered pairs of distinct non-boundary points on X, regarded up to orientation and component preserving homeomorphisms. We view each pair of points as a chord on X and represent it as a dashed line connecting the two points.

Let $\mathscr{A}(X)$ be the vector space over the field $\mathbb{C}$ of complex numbers spanned by all chord diagrams with support X, subject to the 4-term relation

$$D_1 - D_2 + D_3 - D_4 = 0,$$

where D_j are four-chord diagrams identical outside a ball in which they differ as indicated in Fig. 1 (see [4]).

The vector space $\mathscr{A}(X)$ is graded by the number of chords, and, abusing notation, we use the same $\mathscr{A}(X)$ to denote the completion of this vector space with respect to this grading.



D_1 D_2 D_3 D_4

Fig. 1. Chord diagrams in the four-term relation.

When $X = S^1$, we denote $\mathscr{A}(X)$ simply by $\mathscr{A}$. When X is n numbered lines, $\mathscr{A}(X)$ is denoted by $\mathscr{P}_n$ (it is denoted by $\mathscr{M}^{(n)}$ in [13]). All the $\mathscr{P}_n$ are algebras: the product of two-chord diagrams D_1 and D_2 is obtained by placing D_1 on top of D_2. The algebra $\mathscr{P}_1$ is commutative (see [1, 10]).

Suppose C is a component of X. Reversing the orientation of C, we get X'. Let $S_{(C)} : \mathscr{A}(X) \to \mathscr{A}(X')$ be the linear mapping which transfers every chord diagram D in $\mathscr{A}(X)$ to $S_{(C)}(D)$ obtained from D by reversing the orientation of C and multiplying by $(-1)^m$, where m is the number of end points of chords on the component C.

Replacing C by r copies $C^1, \dots, C^r$ of C, from X we get $X^{(r,C)}$, with a projection $p : X^{(r,C)} \to X$. If q is a point on C then $p^{-1}(q)$ consists of r points, while if q is a point of other components, then $p^{-1}(q)$ consists of one point. Let D be a chord diagram with support X with n chords; then D has $2n$ vertices (end points of chords); and suppose that m of which are on C. Consider all possible collections of $2n$ points on $X^{(r,C)}$ whose projection onto X are exactly the $2n$ vertices of D. There are r^m such collections; each defines a chord diagram with support $X^{(r,C)}$ where the pairing of points is corresponding to the pairing of points in D. The sum of these r^m chord diagrams is denoted by $\Delta_{(r,C)}(D)$. Define $\Delta_{(r,C)}$ for every element of $\mathscr{A}(X)$ using linearity. For a point q on C other than vertices of D, the inverse image $p^{-1}(q)$ consists of r points; and we remove r small arcs on C^i's that contain these points but do not contain any points of the inverse image of vertices of D. We mount a chord diagram D' in $\mathscr{P}_r$ (whose support is r lines) to the places of the r removed arcs. By this way, from $\Delta_{(r,C)}(D)$, we get an element of $\mathscr{A}(X^{(r,c)})$, denoted by $\Delta_{(r,C)}(D) \underset{q}{\times} D'$.

Lemma 2.1. *In the above setting,* $\Delta_{(r,C)}(D) \underset{q}{\times} D'$ *does not depend on the choice of the point* q.

For $r = 1$, this lemma comes from the proof of Lemma 3.1 in [1]. Proof for $r \geq 2$ case is given in the last section. As a corollary of this lemma, we obtain Theorem 1 of [14] which states that the image of $\Delta_{(n)} : \mathscr{P}_1 \to \mathscr{P}_n$ is in the center of $\mathscr{P}_n$.

For $r = 1$ (then D' is in $\mathscr{P}_1$ and $\Delta_{(1,C)}(D) = D$) , we call $D \underset{q}{\times} D'$ the result of action of D' on D at component C.

If $f : X \to X'$ is an embedding, then there is an obvious associated mapping $f_* : \mathscr{A}(X) \to \mathscr{A}(X')$.

A *non-associative word* on some symbols is an element of the free non-associative magma (see [20]) generated by these symbols. If w is a non-associative word and v_1

is obtained from w by replacing a symbol in w by another associative word v_2, then we say that v_2 is a *sub-word* of v_1.

We fix an oriented three-dimensional Euclidean space $\mathbb{R}^3$ with coordinates (x, y, t). *A tangle* is a smooth one-dimensional compact oriented manifold $L \subset \mathbb{R}^3$ lying between two horizontal planes $\{t = a\}, \{t = b\}, a < b$ such that all the boundary points are lying on two lines $\{t = a, y = 0\}, \{t = b, y = 0\}$, and at every boundary point L is orthogonal to these two planes. These lines are called the top and the bottom lines of the tangle.

A normal vector field on a tangle L is a smooth vector field on L which is nowhere tangent to L (and, in particular, is nowhere zero) and at every boundary point is given by the unit vector in the plane $\mathbb{R}^2(x, t)$ which together with the tangent vector at the boundary point forms the positive orientation of the plane $\mathbb{R}^2(x, t)$. *A framed tangle* is a tangle equipped with a normal vector field. Two framed tangles are isotopic if they can be deformed by a 1-parameter family of diffeomorphisms into one another within the class of framed tangles.

We will consider a *tangle diagram* as the projection onto $\mathbb{R}^2(x, t)$ of a tangle in generic position. Every double point is provided with a sign + or − indicating an over or under crossing.

If U is a tangle diagram, then U defines a unique class of isotopic *framed* tangles $T(U)$: let $T(U)$ be a tangle which projects into T and is coincident with U except for a small neighborhood of the double points where one resolves the singularity according to the sign there, the normal vector at every point of $L(T)$ is the unit vector which is perpendicular to the tangent vector and together with the tangent vector forms the positive orientation of $\mathbb{R}^2(x, t)$.

One can assign a symbol + or − to all the boundary points of a tangle diagram according to whether the tangent vector at this point directs downwards or upwards. Then on the top boundary line of a tangle diagram we have a sequence of symbols consisting of + and −. Similarly, on the bottom boundary line there is also a sequence of symbols + and −.

A *q-tangle diagram* U is a tangle diagram enhanced with two non-associative words $w_b(U)$ and $w_t(U)$ such that when forgetting about the non-associative structure from $w_t(U)$ (resp. $w_b(U)$) we get the sequence of symbols on the top (resp. bottom) boundary line. Similarly, *a framed q-tangle* T is a framed tangle enhanced with two non-associative words $w_b(T)$ and $w_t(T)$ such that when forgetting about the non-associative structure from $w_t(L)$ (resp. $w_b(L)$) we get the sequence of symbols on the top (resp. bottom) boundary line.

Suppose U is a q-tangle diagram. *A trivial extension* of U is another q-tangle diagram U' obtained from U by adding a number of vertical straight lines on the left-hand and right-hand sides of U. The non-associative word $w_t(U')$ can be any word which contains $w_t(U)$ as a subword, and $w_b(U')$ is obtained from $w_t(U')$ by replacing $w_t(U)$ by $w_b(U)$.

If U_1, U_2 are q-tangle diagrams such that $w_b(U_1) = w_t(U_2)$, we can define the product $U = U_1 \times U_2$ as the q-tangle obtained by placing U_1 on top of U_2. The product is a

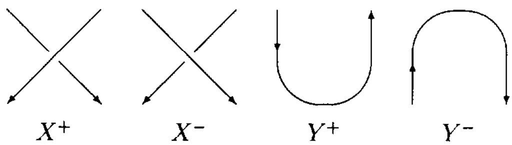


Fig. 2.

q-tangle diagram with $w_t(U) = w_t(U_1), w_b(U) = w_b(U_2)$. Moreover, if D_1, D_2 are two chord diagrams in, respectively, $\mathscr{A}(U_1), \mathscr{A}(U_2)$, then one can define the product D_1D_2 which is a chord diagram with support $U = U_1U_2$ by the same rule.

Let X^+, X^-, Y^+, Y^- be q-tangles as in Fig. 2. For any non-empty non-associative words w_1, w_2, w_3 let $U_{w_1w_2w_3}$ be the q-tangle diagram consisting of vertical straight lines with $w_t = w_1(w_2w_3)$ and $w_b = (w_1w_2)w_3$. Finally, let $U^-_{w_1w_2w_3}$ be the same as $U_{w_1w_2w_3}$ with w_t and w_b exchanged.

Every q-tangle diagram can be decomposed (non-uniquely) as the product of *elementary* q-tangle diagrams which are trivial extensions of the X^+, X^-, Y^+, Y^-, $U_{w_1w_2w_3}$, $U^-_{w_1w_2w_3}$ or any of them with inverse orientation on some components.

To every q-tangle diagram U we will assign an element $Z_f(U)$ of $\mathscr{A}(U)$ satisfying two requirements. First, if U is the product of U_1 and U_2, then $Z_f(U) = Z_f(U_1)Z_f(U_2)$. If U' is a trivial extension of U, then there is an obvious mapping f from U to U'. The second requirement is that $f_*(Z_f(U)) = Z_f(U')$.

Define $Z_f(X^+)$, $Z_f(X_-)$, $Z_f(Y^+)$, $Z_f(Y_-)$, $Z_f(U_{w_1w_2w_3})$, and $Z_f(U^-_{w_1w_2w_3})$ as follows:

$$Z_f(X^+) = \exp(\Omega/2) = 1 + \Omega/2 + \cdots + \frac{\Omega^n}{2^n n!} + \cdots$$

Here Ω^n stands for the chord diagram in $\mathscr{A}(X^+)$ with n horizontal chords:

$$Z_f(X^-) = \exp(-\Omega/2) = 1 - \Omega/2 + \cdots + \frac{\Omega^n}{(-2)^n n!} + \cdots,$$

$$Z_f(Y^+) = 1, Z_f(Y^-) = 1,$$

where 1 is the chord diagram with appropriate support without any chord:

$$Z_f(U_{w_1w_2w_3}) = \Delta_{(l_1,C_1)}\Delta_{(l_2,C_2)}\Delta_{(l_3,C_3)}\Phi,$$

where l_i is the number of symbols in the non-associative words w_i, C_i's are the components of support of chord diagrams in $\mathscr{P}_3$, $i = 1,2,3$, and Φ is a special element of $\mathscr{P}_3$ called an *associator*, which is an analogue of Drinfeld's associator in quasi-Hopf algebras [5, 6]. For a definition of associator in our sense, see [14]. There are many associators, and we use a special one which is discussed in the next section:

$$Z_f(U^-_{w_1w_2w_3}) = \Delta_{(l_1,C_1)}\Delta_{(l_2,C_2)}\Delta_{(l_3,C_3)}\Phi^{-1}.$$

Finally, if U' is obtained from U by reversing the orientation of a component C, let $Z_f(U') = S_{(C)}(Z_f(U))$.

Together with the above two requirements, these define the mapping Z_f uniquely. The well-definedness of Z_f is more difficult to establish.

Z_f itself is not an invariant of framed q-tangles. Let $v \in \mathscr{P}_1$ be an element such that

$$v^{-1} = Z_f(U_1),$$

where U_1 is the first q-tangle diagram in Fig. 5. Note that $Z_f(U_1)$ is of the form 1+ elements of grading ≥ 1 in $\mathscr{P}_1$, hence it has a unique inverse in $\mathscr{P}_1$. Since $\mathscr{P}_1$ is commutative, there are only 2 elements in $\mathscr{P}_1$ whose square is v, and we denote the one which has the form 1+ higher degree by $v^{1/2}$.

Let T be a framed q-tangle which is defined by the q-tangle diagram U (i.e., $T = T(U)$). Suppose U has k components, and the ith component has m_i critical points, i.e. points where the tangent to U is parallel to the plane (x, y). Define

$$\hat{Z}_f(T) = (v^{m_1/2} \otimes \cdots \otimes v^{m_k/2}) \times Z_f(U),$$

where the right-hand side denote the element obtained by successively acting $v^{m_i/2}$ on $Z_f(U)$ at the ith component, $i = 1, \ldots, k$.

Theorem 2.1. *The mapping $\hat{Z}_f$ is well-defined and is an isotopy invariant of framed q-tangles.*

For the proof, see [13, 2]. For a knot in S^3, there is a natural way to identify the set of framings with $\mathbb{Z}$, and we will use this identification. If K is a trivial knot with framing 0, then $\hat{Z}_f(K) = v$. If L' is obtained from L by increasing the framing of component C by 1, then $\hat{Z}_f(L')$ is the result of action of $\exp(\omega/2)$ at the component C of $\hat{Z}_f(L)$. Here ω is the only chord diagram in $\mathscr{P}_1$ with one chord.

Remark. (1) Each associator gives rise to an invariant $\hat{Z}_f$ of framed q-tangles. There is a slight difference between the definition of $\hat{Z}_f$ here and that of [13, 14]. First, we use another associator, second, we use another normalization for q-tangles with non-empty boundary (this normalization was used in [2]). But for framed *links* or q-tangles with exactly one boundary point on top and on bottom line, there is no difference between the definition of $\hat{Z}_f$ here and that of [13, 14] (see Theorem 8 of [14]). For general framed q-tangles, this does not hold true. It follows that v is independent of the associator. The difference between the values of $\hat{Z}_f$, when the associator is changed, is described just by a simple twist which depends only on the non-associative structure of the q-tangle (see Theorem 7 of [14]).

(2) It is proved [14] that the coefficients of $\hat{Z}_f$ of a framed *links* are rational, that is, if one fixes any base of $\mathscr{A}(L)$ consisting of chord diagrams and expresses $\hat{Z}_f(L)$ as an infinite series on this base, then all the coefficients are rational. This, in general, does not hold true for q-tangles and general associators.

3. On an associator of Drinfeld

By an associator we mean an element Φ in $\mathscr{P}_3$ satisfying a system of equations (see [14]). We record here only two of them:

$$\Delta_{(2,C_1)}(R) = \Phi^{312}R^{13}(\Phi^{132})^{-1}R^{23}\Phi, \tag{3.1}$$

$$\Delta_{(2,C_2)}(R) = (\Phi^{231})^{-1}R^{13}(\Phi^{213})R^{12}\Phi^{-1}, \tag{3.2}$$

where C_i are the components of the support of chord diagrams in $\mathscr{P}_2$, $R = \exp(\Omega/2) \in \mathscr{P}_2$ with Ω being the chord diagram with one chord connecting the two components, Φ^{ijk} is obtained from Φ by permuting the components: C_1 to C_i, C_2 to C_j, C_3 to C_k, and $R^{ij} = \exp(\Omega_{ij}/2)$ with Ω_{ij} being the chord diagram in $\mathscr{P}_3$ with one chord connecting C_i and C_j.

Let $\mathfrak{fr}(A,B)$ be the algebra of Lie formal series over the field $\mathbb{Q}$ of rational numbers, and $\mathrm{Fr}(A,B) = \exp \mathfrak{fr}(A,B)$ which is a subalgebra of the algebra $\mathbb{Q}\langle\langle A,B\rangle\rangle$ of formal power series on two non-commuting variables A,B. There is a natural grading on $\mathbb{Q}\langle\langle A,B\rangle\rangle$. A deep result of Drinfeld [6, Proposition 5.4], shows that there exists $\varphi \in \mathrm{Fr}(A,B)$ such that

$$\varphi(-A,-B) = \varphi(A,B), \tag{3.3}$$

$$\varphi(B,A) = \varphi^{-1}(A,B), \tag{3.4}$$

and $\varphi(\Omega_{12},\Omega_{23})$ is an associator. Note that the well-known φ_{KZ} (see [6, 12, 14]) derived from the Knizhnik–Zamolodchikov equation does not satisfy (3.3), although it satisfies (3.4). Let σ be the *anti-isomorphism* of $\mathbb{Q}\langle\langle A,B\rangle\rangle$ with $\sigma(A) = A, \sigma(B) = B$.

Lemma 3.1. $\varphi^{-1} = \sigma(\varphi)$.

Proof. We have $\varphi(A,B) = \exp(P(A,B))$, where $P(A,B)$ is in $\mathfrak{fr}(A,B)$. From (3.3) it follows that $P(A,B)$ is a sum of elements of even grading. It is not hard to see that $\sigma([X,Y]) = -[\sigma(X),\sigma(Y)]$ for any X,Y in $\mathbb{Q}\langle\langle A,B\rangle\rangle$. Every element of even grading in $\mathfrak{fr}(A,B)$ is a non-associative word on A,B with an odd number of Lie brackets. Hence, for an element X of $\mathfrak{fr}(A,B)$ of even grading we have $\sigma(X) = -X$. Then $\sigma(P(A,B)) = -P(A,B)$, and $\sigma(\varphi) = \exp(-P) = \varphi^{-1}$. □

From now on we fix φ and use the associator $\Phi = \varphi(\Omega_{12},\Omega_{23})$ in the definition of Z_f. Then the associator Φ is a sum of chord diagrams of even grading. As a corollary we see that v has even grading. Note that v does not depend on the particularly chosen associator.

Consider the following symmetry group G of the plane $\mathbb{R}^2(x,t)$ generated by mirror reflections with respect to horizontal and vertical lines. If U is a q-tangle diagram, $f \in G$, then $f(U)$ is also a q-tangle diagram. The non-associative structure on the boundary points is induced from that of U. Any mapping $f \in G$ also defines an obvious isomorphism f_* between $\mathscr{A}(U)$ and $\mathscr{A}(f(U))$.

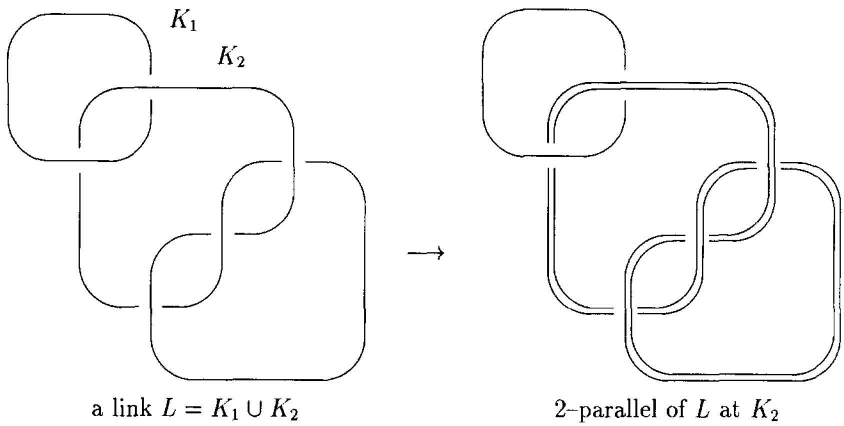


Fig. 3. Parallel of a link.

Proposition 3.1. *For every f in G we have*

$$Z_f(f(U)) = f_*(Z_f(U)).$$

Proof. It is enough to consider the case when U is an elementary q-tangle. When U is one of X^+, X^-, Y^+, Y^- the Proposition is trivial. Let U be the q-tangle consisting of 3 straight lines directed downwards, with the top non-associative word $w_t = (+(++))$ and the bottom $w_b = ((++)+)$. In other words, $U = U_{+++}$. It is enough to prove the proposition for U, since all the other $U_{w_1w_2w_3}$ are obtained from U by various actions of Δ. When f is the reflection with respect to a vertical line, $Z_f(U) = \Phi$, $Z_f(f(U)) = \Phi^{-1}$ while the mapping f_* is the linear mapping from $\mathscr{P}_3$ to itself which exchanges the first and the third strings. Now equality (3.4), with a careful consideration, shows that $Z_f(f(U)) = f_*(Z_f(U))$. The case when f is a reflection with respect to a horizontal line is similar (though more difficult), using Lemma 3.1 instead of equality (3.4). □

Remark. This proposition does not hold true if we use Φ_{KZ} instead of Φ.

4. Parallel of framed links and chord diagrams

Let L be an ℓ-component framed link and $K_1, K_2, \ldots, K_\ell$ be its connected components. By adding $r - 1$ strings parallel to a component K_i which are the push-offs of K_i using the frame, as in Fig. 3, we get an $(\ell + r - 1)$-component link, called the r-parallel of L at K_i and denoted by $L^{(r,K_i)}$.

We will prove the following.

Theorem 4.1. *Let K be a link with components $K_1, \ldots, K_m$. Then*

$$\hat{Z}_f(K^{(r,K_i)}) = \Delta_{(r,K_i)}(\hat{Z}_f(K)).$$

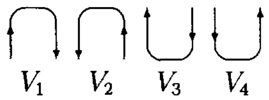


Fig. 4. Critical points.

Remark. The normalization in which the trivial knot takes value v is important for this theorem. Change of this normalization will invalidate this theorem.

We give a proof of Theorem 4.1 for $r = 2$ case, since $\Delta_{(r)}$ is a composition of $(r-1)$ $\Delta_{(2)}$'s.

For a q-tangle T and its component C, we make the 2-parallel $T^{(2)}$, (or $T^{(2,C)}$) of T at component C by adding a string C' parallel to C with respect to the frame of T. The non-associative words $w_t(T^{(2)}), w_b(T^{(2)})$ are obtained from $w_t(T)$ and $w_b(T)$, respectively, by replacing the symbol u corresponding to C by the word (uu). Here u is a sign $+$ or $-$.

Theorem 4.2. *In the above setting, one has*

$$\hat{Z}_f(T^{(2,C)}) = \Delta_{(2,C)}(\hat{Z}_f(T)).$$

Certainly Theorem 4.1 is a corollary of this theorem. It is sufficient to prove the theorem for elementary tangles. The difficulty came only from the case when T contains a maximum or minimum point. We will first deal with this case.

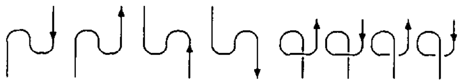
Remark. The above theorem becomes false if we use Φ_{KZ} instead of Φ. See, however, the remark after the proof of the theorem.

Let v_1, v_2, v_3, v_4 be respectively the value of the non-normalized Z_f of 2-parallels of q-tangles V_1, V_2, V_3, V_4 in Fig. 4. We consider v_1, v_2, v_3, v_4 as elements of $\mathscr{P}_2$, where the inner string is numbered first. It follows from Proposition 3.1 that $v_1 = v_2 = v_3 = v_4 = v$.

Lemma 4.1. *Let U be a one-component q-tangle diagram with k critical points. One has*

$$Z_f(U^{(2)}) = v^k \Delta_{(2)}(Z_f(U)).$$

Proof. Decompose U into elementary q-tangle diagrams. Note that here we use the non-normalized Z_f, not $\hat{Z}_f$. The value of Z_f of U comes from the parts which do not contain critical points. Now take 2-parallel of the tangle and decompose $U^{(2)}$ by the same manner. Using Lemma 2.1 (which says that the image of Δ is commutative with everything) we can regroup all the contribution coming from critical points, and we get the result. □



U_1 U_2 U_3 U_4 U_5 U_6 U_7 U_8

Fig. 5.

$$Z_f(U_1^{(2)}) = v^2\Delta_{(2)}(\nu^{-1}) \in \mathcal{P}_2.$$

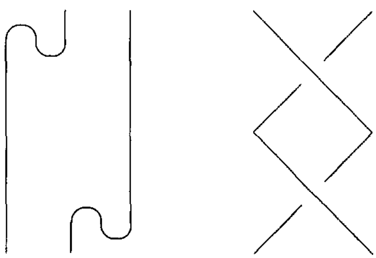


Fig. 6. Deformations of $U_1^{(2)}$ and $U_5^{(2)}$.

Lemma 4.2. *The values of* v *is defined by the following*:

$$v = (\nu^{-1/2} \otimes \nu^{-1/2})\Delta_{(2)}(\nu^{1/2}).$$

Proof. Let $U_1, \ldots, U_8$ be the q-tangle diagrams depicted in Fig. 5.

Since $\hat{Z}_f(U_1) = \hat{Z}_f(U_2) = \hat{Z}_f(U_3) = \hat{Z}_f(U_4) = 1$, it follows that

$$Z_f(U_1) = Z_f(U_2) = Z_f(U_3) = Z_f(U_4) = \nu^{-1}.$$

Applying Lemma 4.1, we have $Z_f(U_1^{(2)}) = v^2\Delta_{(2)}(\nu^{-1}) \in \mathscr{P}_2$.

Deforming $U_1^{(2)}$ into the tangle in the left-hand side of Fig. 6, we see that

$$Z_f(U_1^{(2)}) = \nu^{-1} \otimes \nu^{-1}.$$

Here by $a \otimes b$, where a, b are chord diagrams in $\mathscr{P}_1$, we mean the chord diagram in $\mathscr{P}_2$ with a on the first component and b on the second component. Hence, we obtain:

$$v^2 = (\nu^{-1} \otimes \nu^{-1})\,\Delta_{(2)}(\nu).$$

Now note that $w = (\nu^{-1/2} \otimes \nu^{-1/2})\Delta_{(2)}(\nu^{1/2})$ is in the center of $\mathscr{P}_2$ (see Lemma 2.1 and the remark after it). One has that $v^2 = w^2$, or $(v - w)(v + w) = 0$. Hence, either $v = w$ or $v = -w$. Considering the 0th grading of both v and w leads to $v = w$. $\square$

Now we can proceed to prove Theorem 4.2. It is sufficient to consider the case when T is one of $X^+, X^-, Y^+, Y^-, U_{w_1,w_2w_3}, U^-_{w_1w_2w_3}$ described in Section 2. The last two cases follow straightforwardly from the definition of $\hat{Z}_f$. When T is one of $V_1, \dots, V_4$, say, $T = V_1$:

$$\hat{Z}_f(V_1^{(2)}) = (v^{1/2} \otimes v^{1/2}) Z_f(V_1^{(2)}) = (v^{1/2} \otimes v^{1/2}) v = \Delta_{(2)}(v^{1/2}) = \Delta_{(2)}(\hat{Z}_f(V_1)).$$

That is, the theorem holds true for $T = Y^+$ and $T = Y^-$.

When $T = X^+$ or $T = X^-$, let us decompose $T^{(2)}$ into elementary q-tangles and write out the expression of $\hat{Z}_f(T)$, using the definition. Identities (3.1) and (3.2) now show that the theorem is true for this case. Theorem 4.2 is proved. □

Important remarks. Theorem 4.2 does not hold true if we use the KZ associator instead of the above one. In fact, one can easily see that, for the KZ associator, $\hat{Z}_f(\Delta(V_1))$ contains non-trivial chord diagrams of degree 3, while $\Delta(\hat{Z}_f(V_1)) = \Delta(v)$ contains only chord diagrams of even degrees.

We have used a special associator for the proof of the theorem. For any other associator (for which the values of v_1, v_2, v_3, v_4 maybe different), one can prove a weaker form: Theorem 4.2 remains true for q-tangles T each component of which has the same numbers of maximal and minimal points, and hence for all links. In fact, a similar argument of the previous lemma, applied to $U_2, \dots, U_8$ instead of U_1 (with the deformation of U_5 in the right-hand side of Fig. 6) would lead to $v_1 = v_2 = a$, $v_3 = v_4 = b$ and $ab = ba = (v^{-1} \otimes v^{-1})\Delta_{(2)}(v)$. So the value of a stands for a maximal critical point, and b for a minimal point. And for those q-tangles of the above type we can always pair a maximal point with a minimal point, and get a proof of the theorem. In most applications in this paper (Theorems 4.1, 6.1, 6.2, 6.3) the q-tangles involved are of the above type, so these theorems are valid for every associator.

5. All values of the universal Kontsevich–Vassiliev invariant are group like elements

We now define a co-multiplication $\hat{\Delta}$ in $\mathcal{A}(X)$. A *chord subdiagram* of a chord diagram $D \in \mathcal{A}(X)$ is any chord diagram obtained from D by removing some chords. The *complement chord subdiagram* of a chord subdiagram D' is the chord subdiagram obtained by removing chords in D'. Let

$$\hat{\Delta}(D) = \sum D' \otimes D''.$$

Here the sum is over all chord subdiagrams D' of D, and D'' is the complement of D'. This co-multiplication is co-commutative. The corresponding co-unit $\varepsilon : \mathcal{A}(X) \to \mathbb{Q}$ is $\varepsilon(D) = 1$ if the chord diagram D does not contain any dashed chord, otherwise, $\varepsilon(D) = 0$.

So far, an algebra structure was introduced in $\mathcal{A}(X)$ only for the case when X is n lines (in this case $\mathcal{A}(X) = \mathcal{P}_n$). While the co-algebra structure is defined for every X.

It is not hard to see that the co-multiplication is compatible with the algebra structure in $\mathcal{P}_n$. Hence, all the $\mathcal{P}_n$ are co-commutative Hopf algebra; the antipode is obtained

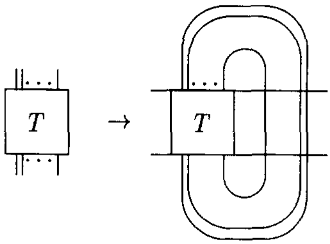


Fig. 7. Closing a tangle.

by successively applying the mapping $S_{(C)}$, where C runs the set of all the support lines. When $n = 1$ the algebra $\mathscr{P}_n$ is commutative, and is isomorphic to $\mathscr{A}(S^1)$; it was first considered by Kontsevich.

Applying Theorem 4.2 successively to every component of a q-tangle T, then deleting all the chords which connect the two copies of T, we get the following.

Theorem 5.1. *For every q-tangle T, the universal Vassiliev–Kontsevich invariant $\hat{Z}_f$ (T) is a group-like element in the co-algebra $\mathscr{A}(T)$, i.e.*

$$\hat{\Delta}[\hat{Z}_f(T)] = \hat{Z}_f(T) \otimes \hat{Z}_f(T).$$

This result is not quite new. After having finished this paper, we learnt that this result, for the case when T is a knot, had been formulated and used in [3]. There the result is derived from the integral formula of the universal Vassiliev–Kontsevich invariant. Note that since one is dealing with *framed links and tangles*, one must use the regularized form of the integral formula, as described in [13].

When K is a framed knot, $\hat{Z}_f(K)$ is group-like in the commutative co-commutative connected Hopf algebra $\mathscr{A}(S^1)$. Hence, $\hat{Z}_f(K) = \exp(y)$, where y is a *primitive element*, i.e. elements x such that $\hat{\Delta}(x) = x \otimes 1 + 1 \otimes x$. It follows that $\hat{Z}_f(K)$ always contains non-prime chord diagrams (which are products of non-trivial chord diagrams), even in the case when K is a prime knot. Also, $y = \log(\hat{Z}_f(K))$ is primitive; so it is an infinite sum of *connected Chinese character diagrams* (see [1]); and maybe it is easier to study $\log(\hat{Z}_f(K))$ than $\hat{Z}_f(K)$.

6. Closing a tangle, the Hopf link and the Kirby moves

Let T be a framed q-tangle with n top and n bottom boundary points. We suppose that the non-associative words on the top and boundary points are the same. Now we close T as in Fig. 7 to get a link $\langle T \rangle$. The question now is what is the relation between the values of $\hat{Z}_f$ of T and its closure link. The method using straight definition of $\hat{Z}_f$ would lead to very complicated chord diagrams.

Recall that $v = \hat{Z}_f(U_1)$ is an element of $\mathscr{P}_1$. Let $v_n = \Delta_{(n)}(v) \in \mathscr{P}_n$.

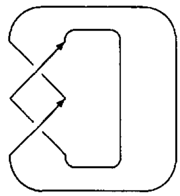
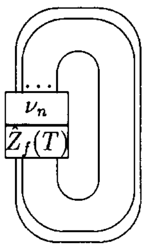


Fig. 8. The Hopf link.

Theorem 6.1. *The value of $\hat{Z}_f$ of the closure link can be calculated by the following formula*:

$\hat{Z}_f(< T >) =$ 

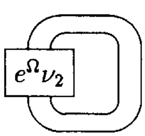
Proof. We decompose $\langle T \rangle$ into 3 q-tangles as in Fig. 7. Note that the first and the third part are n-parallel of q-tangles of Fig. 4. Applying the $\hat{Z}_f$ to each part and using Theorem 4.2 we get the result. □

Note that Theorem 11 in [14] is a corollary of this theorem. The fact that ν_n is in the center of $\mathscr{P}_n$ explains why $\hat{Z}_f$ is invariant under the second Markov move of braids.

The Hopf link and the Kirby moves play an important role in the search for 3-manifold invariants. Here we calculate the value of $\hat{Z}_f$ of the Hopf link and the change of $\hat{Z}_f$ under the Kirby moves. Again, the method using straight definition of $\hat{Z}_f$ would give a very complicated chord diagrams.

Consider the Hopf link in Fig. 8.

Theorem 6.2. *The value of $\hat{Z}_f$ of the Hopf link is*



where Ω is the only chord diagram in $\mathscr{P}_2$ with one chord connecting the two components of the support.

Proof. This follows from the previous theorem. □

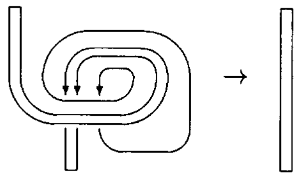
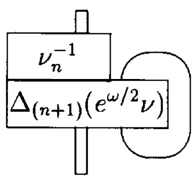


Fig. 9. The Kirby move.

Let T be framed q-tangle on the left-hand side of Fig. 9 which depicts the Kirby move. Here the band stands for n parallel lines of the same orientation. We assume that the non-associative words of the top and the bottom of T are the same.

Theorem 6.3. *The value $\hat{Z}_f$ of the q-tangle T on the left-hand side of Fig. 9 is*

$\hat{Z}_f(T) =$ 

Here ω is the chord diagram in $\mathscr{A}$ with one chord.

Proof. It suffices to prove the theorem for the case when $n = 1$, since the $n > 1$ case follows from the $n = 1$ case by applying Theorem 4.2 to the open component of the tangle T of the case $n = 1$.

Let $X = S^1 \sqcup S^1$ and Y be the disjoint union of a line and a loop. Closing the open component of Y, we get a mapping f from Y to X which induces a mapping f_* from $\mathscr{A}(Y)$ to $\mathscr{A}(X)$. An important observation is that f_* is an isomorphism between vector spaces. The proof is exactly the same as the proof of $\mathscr{A}(S^1) \cong \mathscr{P}_1$ in [1].

Denote the element on the right-hand side of the formula in the theorem, with $n = 1$, by a. One has to prove $\hat{Z}_f(T) = a$. Both are elements of $\mathscr{A}(Y)$. It is enough to prove that

$$(v \otimes 1)\hat{Z}_f(T) = (v \otimes 1)a,$$

where multiplication by $(v \otimes 1)$ means that we take the result of action of v at the open component of Y. Since f_* is an isomorphism between $\mathscr{A}(Y)$ and $\mathscr{A}(X)$, the above equality is equivalent to

$$(v \otimes 1)f_*(\hat{Z}_f(T)) = (v \otimes 1)f_*(a). \tag{6.1}$$

Note that the right-hand side of (6.1) is just $\Delta_{(2)}(e^{\omega/2}v)$, where $\Delta_{(2)} : \mathscr{A}(S^1) \to \mathscr{A}(S^1 \sqcup S^1)$ is defined in Section 2. The left-hand side is $\hat{Z}_f(\langle T \rangle)$, by Theorem 6.1. The closure link of T is the parallel of the trivial knot K with framing 1; and it is known that $\hat{Z}_f(K) = e^{\omega/2}v$. Hence, formula (6.1) follows from Theorem 4.1, applied to K. □

Remark. When the orientation of some of strings in the band is reversed, one can use Theorem 4 of [14], combined with this theorem, to calculate the Kirby move. For the Kirby move with a negative framing, one needs only to replace ω by $-\omega$ in the formula of the theorem.

7. Element v in the universal enveloping algebras of Lie algebras

We have seen that the element v is very important in the whole theory of Vassiliev invariants. In [12] we calculated the coefficients of Drinfeld's Knizhnik–Zamolodchikov associator via multiple zeta values, and a formula for v can be easily derived. Using this formula, one can, say, compute some low degree parts of v, or get some estimate of coefficients of v. However, that formula may not be practical, since all the coefficients of v are rational, while the multiple zeta values are, in general, transcendental; and rational relations between multiple zeta values are not known. Here we give a compact formula for the image of v in the universal enveloping algebra of a simple Lie algebra $\mathfrak{g}$ under the "weight system" mapping.

Suppose $\mathfrak{g}$ is a simple Lie algebra associated to a Cartan matrix. Let us fix a Cartan subalgebra $\mathfrak{h}$, a base of the root system. There are scalar products on $\mathfrak{h}$ and on $\mathfrak{h}^*$, associated to the Cartan matrix. The scalar product on $\mathfrak{h}$ defines an invariant scalar product on $\mathfrak{g}$.

Denote by Δ_+ the set of all positive roots. Let $\delta = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$.

Using the invariant bilinear form, one can define a linear mapping $\mu : \mathscr{P}_1 \cong \mathscr{A} \to \mathfrak{z}(\mathfrak{g})[[h]]$, where $\mathfrak{z}(\mathfrak{g})$ is the center of the universal enveloping algebra of $\mathfrak{g}$; h is a formal parameter, as follows. Let $g_1,\dots,g_k$ be an orthonormal base of $\mathfrak{g}$. Suppose $D \in \mathscr{P}_1$ is a chord diagram with n chords; and hence there are $2n$ end points of chords which, if we follow the line from top to bottom, are $a_1, a_2, \dots, a_{2n}$. *A state* is any mapping $f : \{a_1, a_2, \dots, a_{2n}\} \to \{1, 2, \dots, k\}$ which takes the same value on the two vertices of a chord. Let

$$\mu(D) = h^n \sum_{\text{states } f} g_{f(a_1)} g_{f(a_2)} \cdots g_{f(a_{2n})}.$$

It is known that μ is well-defined, and $\mu(D)$ is in $h^n\mathfrak{z}(\mathfrak{g})$ (see [10, 1]).

We want to calculate $\mu(v)$. First we consider the case when $\mathfrak{g} = sl_2$.

The Lie algebra sl_2 is generated by X, Y, H with relations: $[H,X] = 2X, [H,Y] = -2Y, [X,Y] = H$. The standard invariant bilinear form, which is one fourth of the standard Killing form, has the corresponding symmetric tensor $t = \frac{1}{2}H \otimes H + X \otimes Y + Y \otimes X$.

The center $\mathfrak{z}(sl_2)$ is the free polynomial algebra on the Casimir element c. Combining μ with $\hat{Z}_f$, we get an invariant of knots which are formal power series of h whose

nth coefficient is a polynomial of c:

$$\mu(\hat{Z}_f(K)) = \sum_{n=0}^{\infty} p_{K,n}(c)h^n$$

From the definition it follows that the polynomial degree of $p_{K,n}(c)$ is at most n. This polynomial $p_{K,n}(c)$ is a Vassiliev invariant of order n. Note that in our setting, the value of $\hat{Z}_f$ of the trivial knot is v. The question about the polynomial $p_{K,n}(c)$, when K is the trivial knot, is the question about the image of v in $U(sl_2)$.

Let $B_n(x)$ be the Bernoulli polynomial (see, for example, [17]). One has

$$\sum_{k=M}^{N} k^n = [B_{n+1}(N+1) - B_{n+1}(M)]/(n+1). \tag{7.1}$$

The polynomial B_n has order n, and

$$B_{2n+1}(1-x) = -B_{2n+1}(x).$$

This means $B_{2n+1}(\frac{1}{2}+\frac{1}{2}x) = -B_{2n+1}(\frac{1}{2}-\frac{1}{2}x)$ is an odd polynomial. Hence, there is a polynomial $q_n(x)$ of degree n such that

$$q_n\left(\frac{(x^2-1)}{2}\right) = \frac{2}{(2n+1)!}[B_{2n+1}(\tfrac{1}{2}+\tfrac{1}{2}x)]/x.$$

Theorem 7.1. *For the trivial knot* K,

$$p_{K,2n}(c) = q_n(c), \qquad p_{K,2n+1} = 0,$$

that is,

$$\mu(v) = \sum_{n=0}^{\infty} q_n(c)h^{2n}.$$

Proof. Let V_m be the unique irreducible of sl_2 of dimension m. The Casimir c in this representation is $(m^2-1)/2$ times the identity. Hence, $\mathrm{tr}_{V_m} c^k = m((m^2-1)/2)^k$, and $\mathrm{tr}_{V_m}\, p_{K,n}(c) = m\, p_{K,n}((m^2-1)/2)$.

By Theorem 10 and formula (12) of [14] (see also [9, Theorem XX.8.3], and [11, 12]), $\mathrm{tr}_{V_m}(\mu(\hat{Z}_f(K)))$ is the invariant of knot obtained using the quantum group approach as in [19]. In [19], the value of the trivial knot has been calculated, and is equal to $\mathrm{tr}_{V_m}(\exp(hH/2))$, where δ is half sum of the positive co-roots. In this case $\delta = H/2$. We have

$$\mathrm{tr}_{V_m}(\exp(hH/2)) = \sum_{n=0}^{\infty} \mathrm{tr}_{V_m}\, p_{K,n}(c)\, h^n = \sum_{n=0}^{\infty} m\, p_{K,n}((m^2-1)/2)\, h^n.$$

Note that H, in the representation V_m, is a diagonal matrix with the following numbers on the diagonal: $(m-1), (m-3), \ldots, (1-m)$. Comparing the coefficients of h^n in the above equality, one gets

$$\frac{1}{n!2^n}[(m-1)^n + (m-3)^n + \cdots + (1-m)^n] = m\, p_{K,n}((m^2-1)/2).$$

If n is odd, the left-hand side is 0. If n is even, using (7.1), one can show that the left-hand side is $(1/(n+1)!)[B_{n+1}(\frac{1}{2}+m/2)-B_{n+1}(\frac{1}{2}-\frac{1}{2}m)]$. Hence, we get

$$p_{K,n}((m^2-1)/2)=\frac{2}{m(n+1)!}[B_{n+1}(\tfrac{1}{2}+\tfrac{1}{2}m)].$$

This holds for every positive integer m, hence for every real numbers, and we get the theorem. □

From the formula for generating functions of the Bernoulli polynomials one can easily see that

$$2\sum_{n=0}^{\infty}\frac{B_{2n+1}(x)}{(2n+1)!}h^{2n}=\frac{\mathrm{e}^{h(x-1/2)}-\mathrm{e}^{h(1/2-x)}}{\mathrm{e}^{h/2}-\mathrm{e}^{-h/2}}.$$

Combining this with Theorem 7.1 we see that the image of v in $U(sl_2)$ is

$$\mu(v)=\frac{1}{\sqrt{2c+1}}\frac{\mathrm{e}^{(h/2)(\sqrt{2c+1})}-\mathrm{e}^{-(h/2)(\sqrt{2c+1})}}{\mathrm{e}^{h/2}-\mathrm{e}^{-h/2}}. \tag{7.2}$$

If, as in quantum groups theory, $[x]$ stands for

$$[x]=\frac{\exp(hx/2)-\exp(-hx/2)}{\exp(h/2)-\exp(-h/2)}, \tag{7.3}$$

then the previous equality can be rewritten in the following compact form:

$$\mu(v)=\frac{[\sqrt{2c+1}]}{\sqrt{2c+1}}. \tag{7.4}$$

Now we consider the general case, when $\mathfrak{g}$ is an arbitrary simple Lie algebra. Let ψ be the Harish–Chandra isomorphism between the center of the universal enveloping algebra of $\mathfrak{g}$ and the algebra $S(\mathfrak{h})^W$ of polynomials on $\mathfrak{h}$ invariant under the Weyl group W. See, for example, [7].

Let δ be the half sum of the positive roots. A generalization of (7.4) is the following.

Theorem 7.2. *The image of* v *in the center of* $U(\mathfrak{g})$ *is*

$$\mu(v)=\psi^{-1}\left(\prod_{\alpha\in\Delta_+}\frac{[(\lambda,\alpha)]}{(\lambda,\alpha)}\frac{(\delta,\alpha)}{[(\delta,\alpha)]}\right).$$

Here we regard

$$u=u(\lambda)=\prod_{\alpha\in\Delta_+}\frac{[(\lambda,\alpha)]}{(\lambda,\alpha)}\frac{(\delta,\alpha)}{[(\delta,\alpha)]}$$

as a function of the weight $\lambda\in\mathfrak{h}^*$. The element u can be expressed as a formal power series of h with coefficients being polynomial on $\mathfrak{h}$. It is easy to see that u is invariant under actions of the Weyl group. Hence, all the coefficients of powers of h are in

$S(\mathfrak{h})^W$, and one can apply the inverse of the Harish–Chandra isomorphism to these coefficients.

Proof. One needs to prove that

$$\psi(\mu(v)) = \prod_{\alpha\in\Delta_+} \frac{[(\lambda,\alpha)]}{(\lambda,\alpha)} \frac{(\delta,\alpha)}{[(\delta,\alpha)]}. \tag{7.5}$$

Let $\mu(v) = \sum_{n=0}^{\infty} c(n)h^n$, where $c(n)$ are elements of $\mathfrak{z}(\mathfrak{g})$. For a dominant weight λ let V_λ be the irreducible finite dimensional $\mathfrak{g}$-module whose highest weight is λ. Then

$$\mathrm{tr}_{V_\lambda}(\mu(v)) = \deg(\lambda)\sum_{n=0}^{\infty} c(n)_\lambda h^n, \tag{7.6}$$

where tr_{V_λ} is the trace in this representation, and $\deg(\lambda)$ is the dimension of V_λ.

We will consider every polynomial on $h_1,\dots,h_l$ as a function on $\lambda \in \mathfrak{h}^*$, where $h_1,\dots,h_l$ form a base of $\mathfrak{h}$.

Lemma 7.1. *Let x be an element of $\mathfrak{z}(\mathfrak{g})$, then x acts as x_λ times the identity in V_λ, where*

$$x_\lambda = \psi(x)(\lambda+\delta).$$

Proof. This follows from the definition of the Harish–Chandra isomorphism, see [7, Section 23.3]. □

It suffices to prove (7.5) for a dominant weight λ. By Lemma 7.1, we have $\psi(\mu(v))(\lambda+\delta) = \psi(\sum_{n=0}^{\infty} c(n)h^n)(\lambda+\delta) = \sum_{n=0}^{\infty} c(n)_\lambda h^n$.

The right-hand side, by (7.6), is

$$\frac{\mathrm{tr}_{V_\lambda}(\mu(v))}{\deg(\lambda)}.$$

Using the formula for $\deg(\lambda)$ in [7] we get

$$\psi(\mu(v))(\lambda+\delta) = \mathrm{tr}_{V_\lambda}(\mu(v)) \prod_{\alpha\in\Delta_+} \frac{(\delta,\alpha)}{(\lambda+\delta,\alpha)}. \tag{7.7}$$

Note that $\mathrm{tr}_{V_\lambda}(\mu(v))$ is the value of the trivial knot in the representation V_λ, and is coincident with the value of the trivial knot of the invariant derived from the quantum group. This value has been calculated in [8] (there the authors computed the quantum dimension, which is exactly the value of the trivial knot):

$$\mathrm{tr}_{V_\lambda}(\mu(v)) = \prod_{\alpha\in\Delta_+} \frac{[(\lambda+\delta,\alpha)]}{[(\delta,\alpha)]},$$

where $[x]$ is defined as in (7.3). This formula can also be established using a formula for the twist element of the quantum group in [19] and the Weyl character formula.

Combined with (7.7) we have

$$\psi(\mu(\nu))(\lambda+\delta) = \prod_{\alpha\in\Delta_+} \frac{[(\lambda+\delta,\alpha)]}{[(\delta,\alpha)]}\frac{(\delta,\alpha)}{(\lambda+\delta,\alpha)}.$$

This holds true for every dominant weight λ, hence for every $\lambda \in \mathfrak{h}^*$, which implies (7.5). □

8. Proof of Lemma 2.1

Let q and q' be two points on the component C of the support of D other than the end points of the chords. Then we show that $D^{(r)} \underset{q}{\times} x$ and $D^{(r)} \underset{q'}{\times} x$ are equal. We may assume that there is only one end point between q and q'. The following lemma implies $\Delta_{(r,C)}(D) \underset{q}{\times} D' = \Delta_{(r,C)}(D) \underset{q'}{\times} D'$.

Lemma 8.1. *Let n be the number of chords in D'. Let $D_0^1, D_0^2, \ldots, D_0^r, D_{2n}^1, \ldots, D_{2n}^r$ be chord diagrams identical except within a ball where they are as in Fig. 10. Then*

$$\sum_{m=1}^{r} D_0^m = \sum_{m=1}^{r} D_{2n}^m \tag{8.1}$$

modulo the four term relation.

Proof. Let $\omega_1, \omega_2, \ldots, \omega_n$ be chords in D'. We place the components of D' vertically and $t_1 > t_2 > \cdots > t_{2n}$ be the levels of end points of the chords of D'. Let q_j^m be the point on C^m at level t_j for $j = 1, 2, \ldots, 2n$ and $m = 1, 2, \ldots, r$. Let $D_j^1, D_j^2, \ldots, D_j^r$ be the chord diagrams as in Fig. 10. Let $q_{\alpha(\ell)}^{m(\ell)}$ and $q_{\beta(\ell)}^{m'(\ell)}$ be the end points of ω_ℓ. Then, by the four-term relation,

$$\sum_{m=1}^{r} D_{\alpha(\ell)-1}^m - \sum_{m=1}^{r} D_{\alpha(\ell)}^m + \sum_{m=1}^{r} D_{\beta(\ell)-1}^m - \sum_{m=1}^{r} D_{\beta(\ell)}^m = 0. \tag{8.2}$$

By adding 8.2 for all $\ell = 1, 2, \ldots, n$, we get 8.1, since $\{\alpha(1), \ldots, \alpha(n), \beta(1), \ldots, \beta(n)\} = \{1, \ldots, 2n\}$. □

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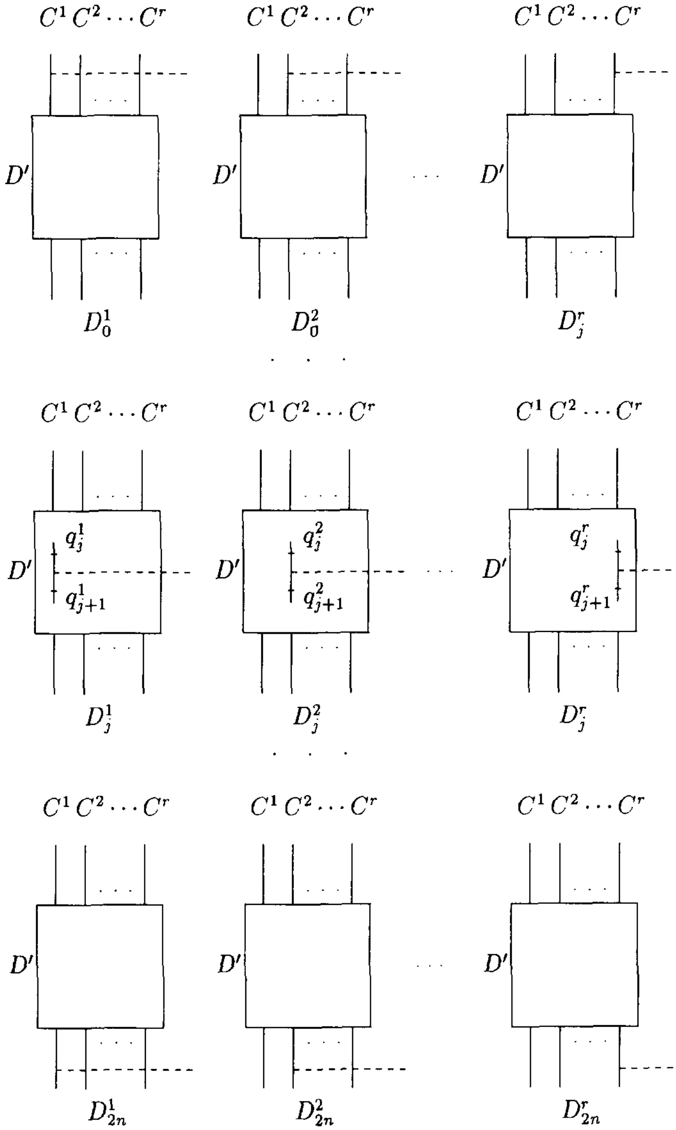

Fig. 10. Chord diagrams $D^1_0, D^2_0, \ldots, D^r_0, \ldots, D^1_{2n}, \ldots, D^r_{2n}$.

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