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Local Rules for Pentagonal Quasi-Crystals

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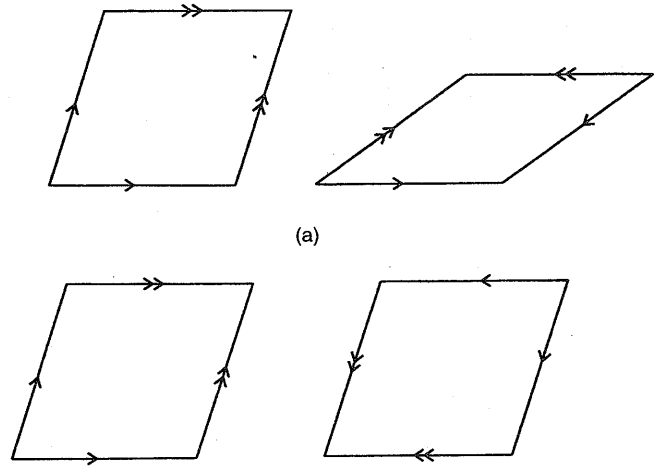
Abstract. The existence of different kinds of local rules is established for many sets of pentagonal quasi-crystal tilings. For each $t \in \mathbb{R}$ there is a set $\bar{\mathcal{T}}_t$ of pentagonal tilings of the same local isomorphism class; the case $t = 0$ corresponds to the Penrose tilings. It is proved that the set $\bar{\mathcal{T}}_t$ admits a local rule which does not involve any colorings (or markings, decorations) if and only if $t = m + n\tau$. In other words, this set of tilings is totally characterized by patches of some finite radius, or r-maps. When $t = (m + n\sqrt{5})/q$ the set $\bar{\mathcal{T}}_t$ admits a local rule which involves *colorings*. For the set of Penrose tilings the construction here leads exactly to the Penrose matching rules. Local rules for the case $t = \frac{1}{2}$ are presented.

Introduction

The aim of this paper is to find *local rules* (or *matching rules*) which force tilings to belong to a specific set of tilings. The tilings considered in this paper are the sets of pentagonal quasi-crystal tilings, also sometimes called quasi-periodic tilings with fivefold symmetry, or generalized Penrose two-dimensional tilings. These tilings have received a great deal of attention among mathematicians and physicists (see [dB1], [KP], and [IS]). The method used in this paper can be applied to other classes of quasi-periodic tilings of two or higher dimensions. The infinitely many sets of pentagonal quasi-crystals are parametrized by $t \in \mathbb{R}$; we denote them by $\bar{\mathcal{T}}_t$. The exact definition of these tilings are given in Section 1. They have many nice properties, and they are quasi-periodic by perhaps all the known definitions of quasi-periodicity. For each t the set $\bar{\mathcal{T}}_t$ consists of tilings of the same local isomorphism class: every finite part of a tiling in $\bar{\mathcal{T}}_t$ can be found in any other tiling in $\bar{\mathcal{T}}_t$. All these tilings are obtained by the strip projection method [dB1], [KP].



(a)

(b)

Fig. 1. The Penrose local rule.

The set of tilings $\bar{\mathcal{T}}_0$ was discovered by Penrose; it is perhaps the best known among nonperiodic tilings. As an example we give here the local rule which forces tilings to belong to $\bar{\mathcal{T}}_0$ (see [P] and [dB1]). Let us consider two rhombs whose acute angles are $\pi/5$ and $2\pi/5$, and whose edges are of the same length. The edges are equipped with arrows as in Fig. 1(a). A tiling of a two-dimensional plane by these "arrowed" rhombs is said to satisfy the *Penrose local rule* if every edge of this tiling has a definite arrow, that is, the arrows of an edge coming from two rhombs incident to this edge are coincident.

Suppose T is a tiling of the plane by copies of these rhombs, without any arrows. We say that T satisfies the Penrose local rule if there is a way to put arrows on edges of rhombs of T such that the arrowed tiling satisfies the Penrose local rule. A fundamental result of De Bruijn states that the set $\bar{\mathcal{T}}_0$ is the set of all tilings satisfying the Penrose local rule.

The main question of this paper is whether there are similar *local rules* for other sets $\bar{\mathcal{T}}_t$ of pentagonal quasi-crystals. This question is also studied in [KP] and [IS].

Note that a "local rule," in some sense, contains information in a local finite radius. It is far from trivial to decide when a local rule forces tilings, say, to be nonperiodic, or quasi-periodic, or to belong to a specific set consisting of tilings of the same isomorphism class. The question whether a set of (usually quasi-periodic in some sense) tilings admits a local rule also has importance for physics. It seems that only such sets of tilings can serve as model for *real* quasi-crystals such as those discovered in 1984 (see discussions in [K], [Lev], and [LPS2]). The similar question of finding a set of prototiles such that every tiling by these prototiles must be aperiodic, or, more difficult, quasi-periodic in some sense but not periodic, seems interesting and has been investigated by many authors.

There are two types of local rules discussed in this paper; their definition are given in Section 1. The main difference between them is the first one does not involve any coloring (marking, decorations) while the second one does.

Our chief results can be summarized as follows. First we prove that if $t = (m + n\sqrt{5})/q$, then the set $\bar{\mathcal{T}}_t$ admits a local rule of type 2. Furthermore, this local rule consists of patches containing only two neighboring tiles. The case $t = 0$ corresponds to the Penrose tilings, and the construction of local rules here leads exactly to the Penrose local rule. Hence De Bruijn's result can be regarded as a special case of our results. We prove that if $t = m + n\tau$ where $\tau = (1 + \sqrt{5})/2$, then this local rule can be realized by a local rule of type 1; hence $\bar{\mathcal{T}}_t$ admits a local rule of type 1. It follows from a result of Ingersent and Steinhardt [IS] that $t = m + n\tau$ is also a necessary condition for the existence of local rules of type 1. Hence we have a criterion for the existence of local rules of type 1. As an illustration, we describe the local rule (of type 2) for the case $t = \frac{1}{2}$. Actually, we have to refine the method so that we could get a simple local rule. This local rule is the Kleman–Pavlovitch local rule [KP] enhanced with some condition on the vertex which is very similar to the Ammann and Socolar local rules for eight-fold and twelve-fold tilings.

The existence of local rules of type 2 has been established for sets of quasi-crystals based on quadratic irrationality of any dimensions [LPS1], [LPS2]. The eight-fold two-dimensional and icosahedral three-dimensional tilings are special cases of these results.

We would like to emphasize the fact that many sets of tilings that do not admit local rules of the first type do admit local rules of the second type. An example is the set of tilings with eight-fold symmetry (see [B], [dB2], [Le1], and [So1]), or any set $\bar{\mathcal{T}}_t$ with $t = (m + n\sqrt{5})/q$ where t is not of the form $m + n\tau$. For those sets of tilings, the problem of *coloring* a tiling to get a *colored tiling* satisfying the local rule of type 2 is *not* a local problem: one cannot decide how to color a tile by inspecting a bounded region around this tile. As observed by Senechal [Se] there is a tiling in $\bar{\mathcal{T}}_t$ which can be colored in two different ways. This is very different from the set of penrose tilings. For this class, we can decide how to arrow the edges of a tile by inspection around this tile within the radius 2; and every tiling can be arrowed in a unique way.

Here is a very rough sketch of the proof of the existence of a type 2 local rule. By the cut method, there is a periodic tiling $\mathcal{O}_t$ of $\mathbb{R}^4$ whose tiles are prisms with the base parallel to a two-dimensional plane $\mathbf{E}$; and every tiling in $\bar{\mathcal{T}}_t$ is obtained as a slice of $\mathcal{O}_t$ by a two-dimensional plane parallel to $\mathbf{E}$. In other words, a tiling in $\bar{\mathcal{T}}_t$ is the projection of the tiles of $\mathcal{O}_t$ which meet a fixed 2-plane parallel to $\mathbf{E}$. Suppose T is a tiling such that every pair of neighboring tiles of T is a translate of a pair of neighboring tiles of some tiling in $\bar{\mathcal{T}}_t$ (i.e., T satisfies some special local rule). Then we can lift T onto $\mathcal{O}_t$: we can choose a collection of tiles of $\mathcal{O}_t$ which project onto tiles of T. The lift has an important property: for every pair of neighboring members in this collection, there is a plane parallel to $\mathbf{E}$ which meets the interior of both of them. In some cases this is enough to prove that:

($\bigstar$) There is a plane parallel to $\mathbf{E}$ meeting all the members of this collection.

This means T is defined by a slice of $\mathcal{O}_t$, i.e., T belongs to $\bar{\mathcal{T}}_t$.

Technically the proof of (★) is rather complicated. First we have to prove that our local rule is a weak local rule in the sense of Levitov [Lev]. Actually this was established in a general setting in [LPS2]; we give a proof for our case in the Appendix. Then we have to study the boundary of the tiles in $\mathcal{O}_t$ and in general we have to replace $\mathcal{O}_t$ by its "refinement" so that the boundary matches some conditions. This step corresponds to the coloring of the local rules. The method used here can be easily generalized for other sets of quasi-periodic tilings obtained by the projection method. There is no specific property of the case investigated here that is used for the proof, except that in this case, due to some symmetry, we can reduce the number of verifications. The general result of the existence of type 2 local rules (local rules involving coloring) is formulated in Section 7, Theorem 7.1.

To establish the existence of a type 1 local rule we do need some specific property of the case under investigation. Actually we prove that if $t = m + n\tau$, there is some finite radius such that the coloring of a tile is uniquely defined by the configuration of the tiling inside the disk of this radius around the tile. It also follows from the proof that the radius is linearly dependent on n.

The paper is organized as follows. In Section 1 we introduce definitions and preliminary facts. In Section 2 some facts about the cut method are recalled. We follow the paper [ODK]. In Section 3 we give the proof of the existence of a local rule of type 2 for the case $t = 0$; the proof is readily generalized in Section 4 for the case $t = (m + n\sqrt{5})/q$. We also prove that the local rule obtained for $t = 0$ is equivalent to the Penrose local rule. Section 5 is devoted to the case $t = m + n\tau$; the existence of a local rule of type 1 is proved. In Section 6 we prove a technical result used in Sections 3 and 4. Section 7 contains the example $t = \frac{1}{2}$, some generalizations, and concluding remarks. In the Appendix we prove a generalization of a result of Levitov concerning the weak local rule.

1. Definitions and Preliminary Facts

For technical convenience some of our definitions (tilings, prototiles, etc.) are more special than is generally the case. In particular, we use translational congruence instead of the usual congruence which involves rotations and reflections (our results can be easily reformulated in terms of the usual congruence).

1.1. *On Tilings and Local Rules*

Two subsets of $\mathbb{R}^k$ are called *congruent* if the second is a translate of the first. We always distinguish between two congruent polyhedra.

A *tiling* of $\mathbb{R}^k$ is a family of k-dimensional polyhedra, called the *tiles* of this tiling, which covers $\mathbb{R}^k$ without overlaps (that is, the interiors of two different tiles have empty intersection). In this paper, except in the case of the oblique periodic tilings

which appear later, all tilings are assumed to be face-to-face type, i.e., the intersection of every two tiles is a common facet of lower dimension, if not empty. A *vertex*, *edge*, *facet*, etc., of a tiling is, respectively, any vertex, edge, facet, etc., of one of its tiles. For a given tiling the translation classes of tiles are called the *prototiles* of this tiling. All the tilings encountered in this paper are tilings of some Euclidean space with a fixed origin **0**. We use the following definition of convergence of tilings (compare [Ra1] and [Ro]).

Definition. A sequence of tilings $T_1, T_2, \ldots,$ of $\mathbb{R}^k$ converges to a tiling T if, for every $r > 0$, there is a natural number N such that, for $i > N$, the tiling T_i coincides with T inside the disk U_r with center at **0** and radius r.

We define the closure $\overline{\mathcal{U}}$ of a set $\mathcal{U}$ of tilings as the set of all the limits of sequences of tilings belonging to $\mathcal{U}$. A set of tilings is *closed* if it is coincident with its closure.

An r-map of a tiling T at a vertex v is the collection of all the tiles lying inside the ball of radius r centered at v. More generally, *an r-map* is any r-map of any tiling at any vertex. Two r-maps are *congruent* if the second is a translate of the first.

Definition. A local rule of type 1 of radius r is a *finite set of r-maps*. A tiling T satisfies a local rule $\mathcal{A}$ of type 1 of radius r if every r-map of T is congruent to one from $\mathcal{A}$.

By "a local rule of type 1" we mean a local rule of type 1 of some radius.

1.2. *Coloring*

A colored polyhedron is a pair (P, j) where P is a polyhedron and j is an arbitrary element, called the color of this polyhedron. Two colored polyhedra are *congruent* if their colors are the same and the second is a translate of the first. A *colored tiling* (resp. *a colored r-map*) is a tiling (resp. r-map) whose tiles are colored polyhedra. Two colored r-maps are congruent if the second is a translate of the first and colors of the corresponding tiles are coincident. As in the previous section, we can define colored r-maps of colored tilings, limits of sequences of colored tilings, and closures of sets of colored tilings.

The following definition of a local rule of type 2 is introduced only for the two-dimensional case.

An edge-configuration is a collection of two colored polygons having a full common edge. Two edge-configurations are congruent if the second is a translate of the first and the corresponding colors are the same. For a colored tiling of $\mathbb{R}^2$ the edge-configuration of an edge is the pair of colored tiles incident to this edge.

Definition. A local rule of type 2 is a finite set of edge-configurations. A colored tiling satisfies a local rule of type $2\mathcal{B}$ if the edge-configuration of every its edge is congruent to one from $\mathscr{B}$. A noncolored tiling satisfies this local rule if it can be colored to become a colored tiling satisfying this local rule.

Definition. A set of noncolored tilings admits a local rule of any type if it is the set of all tilings satisfying this local rule.

It is easy to see that if a set of tilings admits a local rule of any type, then this set is closed.

Remarks.

1. The coloring makes the number of prototiles become larger. For example the two Penrose tiles in Fig. 1(b) are not congruent.
2. The Penrose local rule is a local rule of type 2. All the proofs for the absence of local rules (maybe under some restrictions) in [B], [L], [Le1], and [IS] are only for local rules of type 1.
3. In [Le1], [Le2], and [LPS2] local rules of type 1 are called simply "local rule"; local rules of type 2 are special cases of "local rules with coloring" there. Danzer [D2] calls local rules of type 2 "strictly local matching rules." Our definition of a local rule of type 1 corresponds to the r-rule of Levitov [Lev].

1.3. *The Superspace* $\mathbb{R}^5$

In the Euclidean space $\mathbb{R}^5$ with origin $\mathbf{0}$ we fix a standard base $\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$. Let $\mathbb{Z}^5$ be the lattice of integer points, and let $\mathbb{Q}\sqrt{5}$ be the set of all numbers of the form $a + b\sqrt{5}$, where a and b are rational numbers. If x is a point in $(\mathbb{Q}\sqrt{5})^5$, then $x = y + z\sqrt{5}$ where y and z are rational points in $\mathbb{R}^5$, i.e., points with rational coordinates. The point $y - z\sqrt{5}$ is called the conjugate of x. If X and Y are subsets of $\mathbb{R}^5$ put $X + Y = \{x + y | x \in X, y \in Y\}$, and $-X = \{-x | x \in X\}$.

A subspace (as a vector space) of $\mathbb{R}^5$ is called a *homogeneous plane* and its translates are called planes; an n-plane is a plane of dimension n. A plane F is called *rational* if F is homogeneous and spanned by vectors with rational coordinates.

Consider the action of cyclic group $\mathbb{Z}_5 = \langle g | g^5 = 1 \rangle$ on $\mathbb{R}^5$ by cyclic permutation of the base: $g(\varepsilon_i) = (\varepsilon_{i+1}) \pmod 5$. The space $\mathbb{R}^5$ decomposes into three invariant subspaces $\mathbf{E}$, $\overline{\mathbf{E}}$, and Δ. Here Δ is the one-dimensional subspace spanned by $\delta = (\varepsilon_0 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)/5$, $\mathbf{E}$ is the 2-plane spanned by two vectors with coordinates $(4, \sqrt{5} - 1, -\sqrt{5} - 1, -\sqrt{5} - 1, \sqrt{5} - 1)$ and $(\sqrt{5} - 1, 4, \sqrt{5} - 1, -\sqrt{5} - 1, -\sqrt{5} - 1)$, $\overline{\mathbf{E}}$ is the 2-plane spanned by the conjugates of these two vectors. The element g acts on $\mathbf{E}$ as rotation by $2\pi/5$, on $\overline{\mathbf{E}}$ as rotation by $4\pi/5$, and on Δ as the identity.

Putting $\mathbf{E}^\perp = \overline{\mathbf{E}} + \Delta$, $\mathbb{R}^5 = \mathbf{E} \oplus \overline{\mathbf{E}} \oplus \Delta = \mathbf{E} \oplus \mathbf{E}^\perp$. Let $\mathbf{p}, \overline{\mathbf{p}}, \mathbf{p}^\perp, \mathbf{p}_\Delta$ be, respectively, the projectors of $\mathbb{R}^5$ on $\mathbf{E}, \overline{\mathbf{E}}, \mathbf{E}^\perp, \Delta$. We define $e_i = \mathbf{p}(\varepsilon_i)$, $e_i^\perp = \mathbf{p}^\perp(\varepsilon_i)$, and $\bar{e}_i = \overline{\mathbf{p}}(\varepsilon_i)$. The projection of every of ε_i on Δ is δ. Hence $\mathbf{p}_\Delta(\xi)$ is a multiple of δ for every $\xi \in \mathbb{Z}^5$.

For a real number t let $\overline{\mathbf{E}}_t = \overline{\mathbf{E}} + t\delta$. Denote the 4-plane $\mathbf{E} + \overline{\mathbf{E}}_t$ by $\mathbb{R}^4_t$. Then $\mathbb{R}^5 = \bigcup_{t \in \mathbb{R}} \mathbb{R}^4_t$. For example, $\mathbb{R}^4_0$ is the rational 4-plane $\mathbf{E} \oplus \overline{\mathbf{E}}$; it is the set of all points whose coordinates sum to 0. Let $\Lambda = \mathbb{R}^4_0 \cap \mathbb{Z}^5$ be the set of all integer points lying in $\mathbb{R}^4_0$; it is the lattice generated by $\varepsilon_0 - \varepsilon_1, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4$.

The following is easy to check.

Proposition 1.1. $\mathbf{p}(\xi) = \mathbf{p}(\eta)$, *where* $\xi, \eta \in \mathbb{Z}^5$, *if and only if* $\xi - \eta$ *is a multiple of* 5δ.

Definition. Let $x = \mathbf{p}(\xi)$ where $\xi = \sum_{i=0}^4 n_i \varepsilon_i, n_i \in \mathbb{Z}$, is an integer point. The index(x) is the remainder of the sum $\sum_{i=0}^4 n_i$ modulo 5 (see also [dB1] and [KP]).

It follows from Proposition 1.1 that the index is well defined; it is defined only for points in $\mathbf{p}(\mathbb{Z}^5)$. For $x, y \in \mathbf{p}(\mathbb{Z}^5)$ we have $\text{index}(x + y) \equiv \text{index}(x) + \text{index}(y) \pmod 5$.

The projection $\mathbf{p}(\mathbb{Z}^5)$ on $\mathbf{E}$ is a dense $\mathbb{Z}$-module generated by five vectors $e_1, \ldots, e_5$ which point to the vertices of a regular pentagon (see Fig. 2). The set $\mathbf{p}^\perp(\mathbb{Z}^5)$ is not dense in $\mathbf{E}^\perp$ but is contained and dense in the union of parallel and equidistant 2-planes $\overline{\mathbf{E}} + k\delta$, where $k \in \mathbb{Z}$.

The following is also easy to check.

Proposition 1.2.

(a) *If* $\mathbf{p}_\Delta(\xi) = m\delta$, *then* $m \in \mathbb{Z}$ *and* $m \equiv \textit{index}(\mathbf{p}(\xi)) \pmod 5$.
(b) *If* $\mathbf{p}_\Delta(\xi) = \mathbf{0}$, *where* $\xi \in \mathbb{Z}^5$, *then* ξ *belongs to* Λ (*recall that* $\Lambda = \mathbb{R}^4_0 \cap \mathbb{Z}^5$).
(c) *If two rational points* ξ, ξ' *have the same projection on* $\mathbf{E}^\perp$, *then* $\xi = \xi'$.
(d) *If a rational k-plane F contains* $\mathbf{E}$, *then it contains* $\mathbb{R}^4_0 = \mathbf{E} \oplus \overline{\mathbf{E}}$.

1.4. *The Strip Projection Method*

If $v_1, v_2, \ldots, v_k$ are vectors of $\mathbb{R}^5$ let $\text{Pol}(v_1, \ldots, v_k)$ be the polyhedron:

$$\text{Pol}(v_1, \ldots, v_k) = \left\{ \sum_{i=1}^{i=k} \lambda_i v_i, \lambda_i \in [0, 1] \right\}.$$

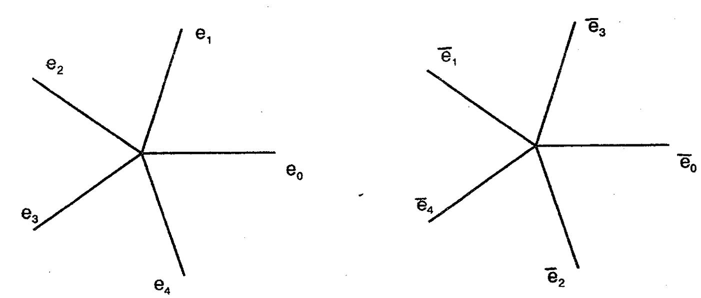


Fig. 2. The projections of the base.

The set $\gamma = \mathrm{Pol}(\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ is the unit hypercube of the lattice $\mathbb{Z}^5$. The set $\mathrm{Pol}(\varepsilon_i, \varepsilon_j)$ (for $0 \le i < j \le 4$) is called a 2-facet of γ and its translates $\xi + \mathrm{Pol}(\varepsilon_i, \varepsilon_j)$, where $\xi \in \mathbb{Z}^5$, are called the 2-facets of the lattice $\mathbb{Z}^5$.

The set of all pairs (i, j) with $0 \le i < j \le 4$ is denoted by M. For convenience a rhomb in this paper always refers to a translate of one of ten $P_I = \mathrm{Pol}(e_i, e_j)$ with $I = (i, j) \in M$. Up to rotation there are only two rhombs from the ten P_i, they are shown in Fig. 1(a).

For every $\alpha \in \mathbf{E}^{\perp}$ let $\mathbf{S}_\alpha$ be the strip defined by shifting the unit hypercube γ along the 2-plane $\mathbf{E} + \alpha$: $\mathbf{S}_\alpha = \mathbf{E} + \alpha + \gamma$, $\alpha \in \mathbf{E}^{\perp}$.

Definition. $\alpha \in \mathbf{E}^{\perp}$ is *regular* if the boundary of the strip $\mathbf{S}_\alpha$ does not contain any integer point. Otherwise α is *singular.*

The following facts are fundamental in the strip projection method. For regular α the union of all the 2-facets of the lattice $\mathbb{Z}^5$ lying in the strip $\mathbf{S}_\alpha$ is a two-dimensional continuous surface. This surface contains all the vertices of the lattice $\mathbb{Z}^5$ falling inside $\mathbf{S}_\alpha$ and has an obvious polyhedral structure. By projecting this polyhedral structure along $\mathbf{E}^{\perp}$ onto $\mathbf{E}$ we get a tiling T_α of $\mathbf{E}$. Note that there are no overlaps: the restriction of $\mathbf{p}$ to this surface is one-to-one. The prototiles are the ten rhombs P_I, $I \in M$. This method of obtaining the tilings T_α is the strip projection method, applied to our setting (see [dB1], [GR], and [ODK]).

Denote by $\mathscr{T}$ (resp. $\mathscr{T}_t$) the set of all tilings T_α for regular $\alpha \in \mathbf{E}^{\perp}$ (resp. $\alpha \in \overline{\mathbf{E}}_t$) and their translates.

Definition. A tiling of the closure $\overline{\mathscr{T}}$ of $\mathscr{T}$ is called a pentagonal quasi-crystal.

Of course if $\alpha \in \overline{\mathbf{E}}_t$ is regular, then any translate of T_α is in $\overline{\mathscr{T}}_t$, but other tilings in $\overline{\mathscr{T}}_t$ exist as well. We describe such tilings in Section 2.5.

Two important properties of pentagonal quasi-crystals are [dB1], [KP], [ODK]:

Proposition 1.3.

(a) *Local isomorphism: for every two tilings in $\overline{\mathscr{T}}_t$, for every $r > 0$, every r-map of the first tiling is congruent to an r-map of the second.*

(b) *Symmetry: if T is a tiling of $\overline{\mathscr{T}}_t$, then the rotation of T by $2\pi/5$ also belongs to $\overline{\mathscr{T}}_t$.*

Remark. Some authors consider only regular cases, that is, the sets $\mathscr{T}_t$ and $\mathscr{T}$ but not their closures. However, as argued above, the question of finding a local rule for $\mathscr{T}_t, \mathscr{T}$ is not relevant, because these sets are not closed. In each set of $\overline{\mathscr{T}}_t, \overline{\mathscr{T}}$ there are two different tilings coincident in a half-plane.

2. The Cut Method

2.1. *The Cut Method in the Superspace*

A set X in $\mathbb{R}^5$ is called a *prism* if $X = \mathbf{p}(X) + \mathbf{p}^{\perp}(X)$. If X and Y are prisms, then their intersection is also a prism, and we have the nice formula

$$X \cap Y = [\mathbf{p}(X) \cap \mathbf{p}(Y)] + [\mathbf{p}^{\perp}(X) \cap \mathbf{p}^{\perp}(Y)]. \tag{1}$$

This makes it easy to study the intersection of two prisms.

Definition. Suppose X is a prism such that $\mathbf{p}^{\perp}(X)$ is a polyhedron. We define the parallel boundary of X as

$$\mathbf{p}(X) + \partial(\mathbf{p}^{\perp}(\mathbf{X})),$$

where $\partial(\mathbf{p}^{\perp}(X))$ is the boundary of $\mathbf{p}^{\perp}(X)$.

The parallel boundary of a prism is a part of its boundary.

For $I = (i_0, i_1) \in M$ recall that P_I is the rhomb $\mathrm{Pol}(e_{i_0}, e_{i_1})$. Let

$$P_I^{\perp} = -\mathrm{Pol}\left(e_{j_0}^{\perp}, e_{j_1}^{\perp}, e_{j_2}^{\perp}\right),$$

where $(i_0, i_1, j_0, j_1, j_2)$ is a permutation of $(0, 1, 2, 3, 4)$. Put $C_I = P_I + P_I^{\perp}$; it is a prism.

Consider the family $\mathscr{O}$ consisting of all prisms of the form $C_I + \xi$ for $I \in M$ and $\xi \in \mathbb{Z}^5$. It is proved in [ODK] that this family covers $\mathbb{R}^5$ without any overlaps, i.e., it is a tiling of $\mathbb{R}^5$. The tiling $\mathscr{O}$ is not of face-to-face type and is invariant under translations by vectors of $\mathbb{Z}^5$. The parallel boundary $\mathbf{B}$ of $\mathscr{O}$, by definition, is the union of the parallel boundaries of all the tiles of $\mathscr{O}$. It is a cellular complex of dimension 4. If $\alpha \in \mathbf{E}^{\perp}$ is such that $\mathbf{E} + \alpha$ does not meet $\mathbf{B}$, then all the intersections of $\mathbf{E} + \alpha$ with tiels of $\mathscr{O}$ form a cover of $\mathbf{E} + \alpha$ without overlaps and define a tiling of $\mathbf{E} + \alpha$. Projecting along $\mathbf{E}^{\perp}$ we get a tiling of $\mathbf{E}$, called the tiling defined by $\mathscr{O}$ and α. If $\mathbf{E} + \alpha$ meets $\mathbf{B}$, then all the intersections of $\mathbf{E} + \alpha$ with $\mathscr{O}$ covers $\mathbf{E} + \alpha$ with overlaps. The fundamental result of [ODK] can be stated as follows.

Theorem 2.1. *The 2-plane $\mathbf{E} + \alpha$ does not meet $\mathbf{B}$ if and only if α is regular and in this case the tiling defined by $\mathscr{O}$ and α is coincident with the tiling T_α obtained by the strip projection method.*

It follows that the set $\mathbf{Ir}$ of all singular points is $\mathbf{p}^{\perp}(\mathbf{B})$, $\mathbf{Ir} = \mathbf{p}^{\perp}(\mathbf{B})$. We have the following description of $\mathbf{Ir}$. For $I = (i, j) \in M$ let H_I be the 2-plane spanned by $e_i^{\perp}$ and $e_j^{\perp}$. Then $H_I + \mathbf{p}^{\perp}(\mathbb{Z}^5)$ is a dense family of parallel 2-planes in $\mathbf{E}^{\perp}$.

Proposition 2.2 (see [ODK]). *The set* **Ir** *of all singular points is the union of ten families of parallel planes* $H_I + \mathbf{p}^{\perp}(\mathbb{Z}^5)$, $I \in M$:

$$\mathbf{Ir} = \mathbf{p}^{\perp}(\mathbf{B}) = \bigcup_{I \in M} (H_I + \mathbf{p}^{\perp}(\mathbb{Z}^5)).$$

Of course **Ir** is a dense set in $\mathbf{E}^{\perp}$ but it has measure 0.

2.2. *Index of Rhombs*

We have $\overline{\mathscr{T}}_t = \overline{\mathscr{T}}_{t+1}$, hence we study $\overline{\mathscr{T}}_t$ for $-1 < t \le 0$.

Recall that a rhomb is always referred to as a translate of one of ten P_i, $I \in M$. Suppose a rhomb P has one vertex in $\mathbf{p}(\mathbb{Z}^5)$. Then all the four vertices are in $\mathbf{p}(\mathbb{Z}^5)$, hence they each have an index (see Section 1.3). Let v and v' be two vertices such that the segment $[v, v']$ is an edge of P. Then the vector $\overline{vv'}$ is either one of e_0, e_1, e_2, e_3, e_4 or one of $-e_0, -e_1, -e_2, -e_3, -e_4$. In the first case index(v') = index$(v) + 1$, in the second index(v') = index$(v) - 1$. Hence the four vertices can take only three values of the index $i, i+1, i+2 \pmod 5$ for some $i \in \{0,1,2,3,4\}$.

Definition. This number i is called the *index of this rhomb.*

In case $P = \mathbf{p}(\xi + C_I)$, $\xi \in \mathbb{Z}^5$, $I \in M$, it is easy to see that index(P) = index$(\mathbf{p}(\xi))$. The index is important due to the following proposition which is a consequence of Proposition 1.2(b).

Proposition 2.3. *Suppose* P, P' *are congruent rhombs,* $P = P' + v$. *Then* v *belongs to* $\mathbf{p}(\Lambda)$ *if and only if index*(P) = *index*(P').

Proposition 2.4. *For regular* $\alpha \in \overline{\mathbf{E}}_t$ *the tiles of* T_α *have indices* 1 *or* 2 *if* $t = 0$ *and indices* 0, 1, 2 *if* $-1 < t < 0$. (*The case* $t = 0$ *has been proved in* [*dB*1].)

Proof. Suppose P is a tile of T_α. Then $P = \mathbf{p}(\xi + C_I)$ for some prism $\xi + C_I$ meeting $\mathbf{E} + \alpha$, by the cut method. Since $\mathbf{E} + \alpha$ does not meet the parallel boundary of the prism $\xi + C_I$, it meets the interior of this prism. Projecting on Δ, we have $\mathbf{p}_\Delta(\mathbf{E} + \alpha) = t\delta$ and $\mathbf{p}_\Delta(C_I) = [\mathbf{0}, -3\delta]$. Let $\mathbf{p}_\Delta(\xi) = m\delta$, $m \in \mathbb{Z}$. Then $m\delta + [\mathbf{0}, -3\delta]$ contains $t\delta$ as an interior point. It follows that, when $t = 0$, m must be 1 or 2, and when $-1 < t < 0$, then m must be 0, 1, or 2. By Proposition 1.2(a) we have m = index$(\mathbf{p}(\xi))$ = index(P). □

2.3. *The Cut Method in Four-Dimensional Planes*

We have seen that every tiling T_α is the slice of a periodic structure in $\mathbb{R}^5$ by a 2-plane. For a fixed t every tiling T_α with $\alpha \in \overline{\mathbf{E}}_t$ can be obtained by a slice of a periodic structure in the 4-plane $\mathbb{R}^4_t$ as follows.

The intersection of $\mathbb{R}^4_t$ with a tile $\xi + C_I$ of $\mathscr{O}$ is

$$[\mathbf{p}(\xi) + P_I] + \left[\overline{\mathbf{E}}_t \cap (\mathbf{p}^{\perp}(\xi) + P_I^{\perp})\right],$$

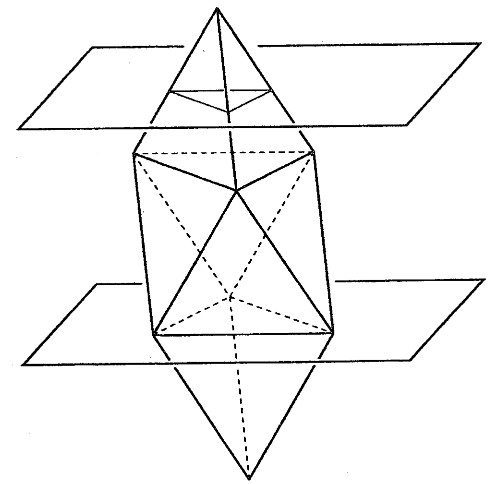


Fig. 3. Intersections of $\overline{\mathbf{E}}_t$ and $P_I^{\perp}$.

by formula (1). The first term is a translate of the rhomb P_I, the second term is either a point or a polygon, if it is not empty (see Fig. 3). Hence the intersection of $\mathbb{R}^4_t$ with a tile of $\mathscr{O}$ is either of dimension 2 or 4. Let $\mathscr{O}_t$ be the family of all the four-dimensional intersections of $\mathbb{R}^4_t$ with tiles of $\mathscr{O}$. The family $\mathscr{O}_t$ is a tiling of $\mathbb{R}^4_t$ and has two fundamental properties:

(i) It is invariant under translations from Λ (recall that $\Lambda = \mathbb{R}^4_0 \cap \mathbb{Z}^5$ is a lattice in $\mathbb{R}^4_0$).
(ii) All its members are prisms; there are a finite number of them up to congruence.

Definition. Any family of four-dimensional polyhedra in $\mathbb{R}^4_t$ covering $\mathbb{R}^4_t$ without overlaps and satisfying conditions (i) and (ii) above is called an *oblique periodic tiling of* $\mathbb{R}^4_t$.

It follows immediately from the construction that, for a regular $\alpha \in \overline{\mathbf{E}}_t$, the tiling T_α is obtained by intersecting $\mathbf{E} + \alpha$ with members of the family $\mathscr{O}_t$ and then projecting onto $\mathbf{E}$.

Definition. Suppose C is a prism in $\mathbb{R}^4_t$ such that $\mathbf{p}^{\perp}(C)$ is a polygon. The sum of $\mathbf{p}(C)$ and an edge of polygon $\mathbf{p}^{\perp}(C)$ is called a small wall of C. The union of all the small walls of C is its parallel boundary.

Suppose $\mathscr{U}$ is an oblique periodic tiling of $\mathbb{R}^4_t$ whose tiles are *colored* polyhedra. Let $\mathbf{B}(\mathscr{U})$ be the union of all the small walls of all the tiles of $\mathscr{U}$. If $\alpha \in \overline{\mathbf{E}}_t$ is such that $\mathbf{E} + \alpha$ does not meet $\mathbf{B}(\mathscr{U})$, then by intersecting $\mathbf{E} + \alpha$ with the colored tiles of $\mathscr{U}$ and then projecting onto $\mathbf{E}$ we get a colored tiling, called the colored tiling defined by $\mathscr{U}$ and α. Denote by $\mathscr{T}(\mathscr{U})$ the set of all such colored tilings and their translates, and by $\bar{\mathcal{T}}(\mathcal{U})$ the closure of $\mathcal{T}(\mathcal{U})$. When $\mathcal{U}$ is a noncolored tiling we use a similar definition, the set $\bar{\mathcal{T}}(\mathcal{U})$ is then a set of noncolored tilings.

Of course $\bar{\mathcal{T}}(\mathcal{O}_t) = \bar{\mathcal{T}}_t$. The projection on $\bar{\mathbf{E}}_t$ of the parallel boundary $\mathcal{B}(\mathcal{O}_t)$ is the set of singular points lying in $\bar{\mathbf{E}}_t$. Hence we get

Proposition 2.5. *The projection of the parallel boundary* $\mathbf{B}(\mathcal{O}_t)$ *onto* $\bar{\mathbf{E}}_t$ *is* $\mathbf{Ir} \cap \bar{\mathbf{E}}_t$:

$$\mathbf{p}^{\perp}(\mathbf{B}(\mathcal{O}_t)) = \mathbf{Ir} \cap \bar{\mathbf{E}}_t.$$

2.4. *Lifting a Tiling*

Suppose that $\mathcal{U}$ is an oblique periodic tiling of $\mathbb{R}_t^4$ whose tiles are colored polyhedra, and that T is a colored tiling of $\mathbf{E}$.

Definition. A lift of a tile P of T into $\mathcal{U}$ is a tile C of $\mathcal{U}$ such that $\mathbf{p}(C) = P$ and the colors of P and C are the same. A lift of T into $\mathcal{U}$ is a map l: {tiles of T} $\to$ {prisms of $\mathcal{U}$} such that for every tile P of T the tile $l(P)$ is a lift of P into $\mathcal{U}$.

Of course the lift does not always exist, and when one does exist it may not be unique.

Definition. A lift l of a tiling T into $\mathcal{U}$ is *strongly connected* if, for every pair of tiles P_1, P_2 sharing a common edge, the polygons $\mathbf{p}^{\perp}(l(P_1)), \mathbf{p}^{\perp}(l(P_2))$ have a common *interior* point.

A simple but important example is the case when T is a colored tiling defined by α and $\mathcal{U}$, where $\mathbf{E} + \alpha$ does not meet the parallel boundary $\mathbf{B}(\mathcal{U})$. Then T has a lift into $\mathcal{U}$: $l(P)$, where P is a tile of T, is the tile of $\mathcal{U}$ which meets $\mathbf{E} + \alpha$ and projects into P. This lift is strongly connected because all the polygons $\mathbf{p}^{\perp}(l(P))$ contain α as an interior point.

Let P be a rhomb having vertices in $\mathbf{p}(\mathbb{Z}^5)$. There may be no lift of P into $\mathcal{O}_t$ but there are always lifts of P into $\mathcal{O}$; they are of the form $C + k5\delta$, $k \in \mathbb{Z}$, where C is congruent to one of ten prisms C_I. The projection $\mathbf{p}_{\Delta}(C_I)$ is the segment $[\mathbf{0}, -3\delta]$. Hence there is at most one prism from the set $C + k5\delta$ which meets $\mathbb{R}_t^4$. Thus we have:

Proposition 2.6. *Suppose P is a rhomb having vertices in* $\mathbf{p}(\mathbb{Z}^5)$. *There is at most one tile C of $\mathcal{O}_t$ such that* $\mathbf{p}(C) = P$.

This means the lift into $\mathcal{O}_t$ is always unique.

2.5. *Singular Cases*

For completeness we describe the "singular tilings," i.e., all the tilings in $\bar{\mathcal{T}} \setminus \mathcal{T}$, $\bar{\mathcal{T}}_t \setminus \mathcal{T}_t$. The results here are not used in what follows.

Suppose $\alpha \in \mathbf{Ir}$. Recall that $\mathbf{Ir}$ is a family of 2-planes in $\mathbf{E}^{\perp}$. There are several 2-planes from $\mathbf{Ir}$ going through α; they partition $\mathbf{E}^{\perp}$ into many pairs, called "corners."

Proposition 2.7. *Suppose α_i, $i = 1, 2, \ldots$ are regular, lie in one corner of* $\mathbf{E}^{\perp}$ *separated by 2-planes from* $\mathbf{Ir}$ *going through α, and converge to α when $i \to \infty$. Then the sequence of tilings T_{α_i} converges to a tiling, called the tiling defined by α and this corner. This tiling depends only on the corner containing α_i, but not on the choice of points α_i.*

Proof. We have to prove that for $r > 0$ a number N exists such that if $i > N$, then all the tilings T_{α_i} coincide inside the disk U_r with center at $\mathbf{0}$ and radius r. Consider the union Z of parallel boundaries of all prisms (from the family $\mathcal{O}$) which projects onto $\mathbf{E}$ into rhombs intersecting U_r. Let $Y = \mathbf{p}^{\perp}(Z)$. Of course $Y \subset \mathbf{Ir}$, but it is not a dense set in $\mathbf{E}^{\perp}$: for large r it divides a small neighborhood of α just as the 2-planes from $\mathbf{Ir}$ going through α do. Hence for large i all the points α_i lie in one part of $\mathbf{E}$ divided by Y. Obviously for these α_i the tilings T_{α_i} are the same inside U_r. □

It can be proved that no two tilings defined by an irregular α and two different corners are the same, and that the closure $\bar{\mathcal{T}}$ of $\mathcal{T}$ is the set of:

(a) All T_{α} with regular α, and their translates.
(b) All tilings defined by singular α and a corner of $\mathbf{E}^{\perp}$ divided by 2-planes from $\mathbf{Ir}$ going through α, and their translates.

The singular cases in $\bar{\mathcal{T}}_t$ are similar. Suppose $\alpha \in \bar{\mathbf{E}}_t$ is singular. The set $\mathbf{Ir} \cap \bar{\mathbf{E}}_t$ is a family of lines in $\bar{\mathbf{E}}_t$. There are several lines in $\mathbf{Ir} \cap \bar{\mathbf{E}}_t$ going through α; they partition $\bar{\mathbf{E}}_t$ into many parts. If α_i is a sequence of regular points lying in one part and converging to α, then the corresponding tilings converging to a tiling, called *the tiling defined by α and this part.* Of course this tiling belongs to $\bar{\mathcal{T}}_t$ and it can be proved that $\bar{\mathcal{T}}_t$, up to translations, is just the set of all such tilings plus "regular" ones. Using this picture the topology of $\bar{\mathcal{T}}, \bar{\mathcal{T}}_t$ can be easily described.

3. The Penrose Tilings

Theorem 3.1. *The set $\bar{\mathcal{T}}_0$ admits a local rule of type* 2.

This theorem was first proved by de Bruijn [dB1]. The proof here is different from that of de Bruijn and will be soon generalized for other cases.

We present a concrete local rule which is actually equivalent to the Penrose local rule. We make use of the Main Technical Theorem that is proven in Section 6.