ON POSITIVITY OF KAUFFMAN BRACKET SKEIN ALGEBRAS OF SURFACES

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ABSTRACT. We show that the Chebyshev polynomials form a basic block of any positive basis of the Kauffman bracket skein algebras of surfaces.

1. Introduction

1.1. **Positivity of Kauffman bracket skein algebras.** Let $\mathbb{Z}[q^{\pm 1}]$ be the ring of Laurent polynomial in an indeterminate q with integer coefficients. Suppose Σ is an oriented surface. The Kauffman bracket skein algebra $\mathscr{S}_{\mathbb{Z}[q^{\pm 1}]}(\Sigma)$ is a $\mathbb{Z}[q^{\pm 1}]$-algebra generated by unoriented framed links in $\Sigma \times [0,1]$ subject the Kauffman skein relations [Kau] which are recalled in Section 2. The skein algebras were introduced by Przyticki [Pr] and independently Turaev [Tu] in an attempt to generalize the Jones polynomial to links in general 3-manifold, and have connections and applications to many interesting objects like character varieties, topological quantum field theories, quantum invariants, quantum Teichmüller spaces, and many others, see e.g. [BW, Bu, PS, LT, Mu, Pr, Th].

The quotient $\mathscr{S}_{\mathbb{Z}}(\Sigma) = \mathscr{S}(\Sigma)/(q-1)$, which is a $\mathbb{Z}$-algebra, is called the skein algebra at $q=1$.

For $R=\mathbb{Z}$ let $R_+ = \mathbb{Z}_+$, the set of non-negative integers, and for $R = \mathbb{Z}[q^{\pm 1}]$ let $R_+ = \mathbb{Z}_+[q^{\pm 1}]$, the set of Laurent polynomials with non-negative coefficients, i.e. the set of $\mathbb{Z}_+$-linear combinations of integer powers of q. An R-algebra, where $R = \mathbb{Z}[q^{\pm 1}]$ or $R = \mathbb{Z}$, is said to be *positive* if it is free as an R-module and has basis $\mathcal{B}$ in which the structure constants are in R_+, i.e. for any $b, b', b'' \in \mathcal{B}$, the b-coefficient of the product $b'b''$ is in R_+. Such a basis $\mathcal{B}$ is called a *positive* basis of the algebra.

The important positivity conjecture of Fock and Goncharov [FoG] states that $\mathcal{S}_R(\Sigma)$ is positive for the case $R = \mathbb{Z}[q^{\pm 1}]$ and $R = \mathbb{Z}$. The first case obviously implies the second. The second case, on the positivity over $\mathbb{Z}$, was proved by D. Thurston [Th]. Moreover, D. Thurston make precise the positivity conjecture by specifying the positive basis as follows.

A *normalized sequence of polynomials over* R is a sequence $\mathbf{P} = (P_n(z)_{n\ge 0})$, such that for each n, $P_n(z) \in R[z]$ is a monic polynomial of degree n. Every normalized sequence $\mathbf{P}$ over R gives rise in a canonical way a basis $\mathcal{B}_{\mathbf{P}}$ of the R-module $\mathscr{S}_R(\Sigma)$, see Section 2.

Definition 1. *A sequence* $\mathbf{P} = (P_n(z)_{n\ge 0})$ *of polynomials in* $R[z]$ *is* positive over R *if it is normalized and the basis* $\mathcal{B}_{\mathbf{P}}$ *is a positive basis of* $\mathscr{S}_R(\Sigma)$*, for any oriented surface* Σ.

Then D. Thurston proved the following.

Theorem 1.1 (D. Thurston [Th])**.** *The sequence* (T_n) *of Chebyshev's polynomials is positive over* $R = \mathbb{Z}$.

Here in this paper, Chebyshev's polynomials $T_n(z)$ are defined recursively by

$$T_0(z) = 1,\ T_1(z) = z, T_2(z) = z^2 - 2, \quad T_n(z) = zT_{n-1}(z) - T_{n-2}(z) \quad \text{for} \quad n \ge 3.$$

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If T_0 is the constant polynomial 2, then the above polynomials are Chebyshev's polynomials of type 1. Here we have to set $T_0 = 1$ since by default, all normalized sequence begins with 1.

D. Thurston suggested the following conjecture, making precise the positivity conjecture.

Conjecture 1. *The sequence* (T_n) *of Chebyshev's polynomials is positive over* $\mathbb{Z}[q^{\pm 1}]$.

We will show

Theorem 1.2. *Let* $R = \mathbb{Z}$ *or* $R = \mathbb{Z}[q^{\pm 1}]$ *and* $\mathbf{P} = (P_n(z)_{n\geq 0})$ *be a sequence of polynomials in* $R[z]$. *If* $\mathbf{P}$ *is positive over* R, *then* $P_n(z)$ *is an* R_+*-linear combination of* $T_0(z), T_1(z), \dots, T_n(z)$. *Besides,* $P_1(z) = T_1(z) = z$.

For the case $R = \mathbb{Z}$, this result complements well Theorem 1.1 above, as they together claim that the sequence of Chebyshev polynomials is the minimal one in the set of positive sequence over $\mathbb{Z}$. For $R = \mathbb{Z}[q^{\pm 1}]$, our result says that the sequence of Chebyshev polynomial should be the minimal positive sequence.

Remark 1.1. The positivity conjecture considered here is different from the one discussed in [MSW], which claims that every element of a certain basis is a $\mathbb{Z}_+$-linear combination of monomials of cluster variables.

1.2. **Marked surfaces.** *A marked surface* is a pair $(\Sigma, \mathcal{P})$, where Σ is a compact oriented surface with (possibly empty) boundary $\partial\Sigma$ and $\mathcal{P}$ is a finite set in $\partial\Sigma$. G. Muller [Mu] defined the skein algebra $\mathscr{S}_R(\Sigma, \mathcal{P})$, extending the definition from surfaces to marked surfaces. When $R = \mathbb{Z}$, this algebra had been known earlier, and actually, the above mentioned result of D. Thurston (Theorem 1.1) was proved also for the case of marked surfaces. However, there are two types of basis generators of the skein algebras, namely loops and arcs, and one needs two normalized sequences of polynomials $\mathbf{P}$ and $\mathbf{Q}$ to define an R-basis of the skein algebra $\mathscr{S}_R(\Sigma, \mathcal{P})$. Here $\mathbf{P}$ is applicable to loops, and $\mathbf{Q}$ is applicable to arcs, see Section 4. A pair of polynomials $(\mathbf{P}, \mathbf{Q})$ are *positive over* R if they are normalized and the basis they generate is positive for any marked surface. D. Thurston result says that with $\mathbf{P} = (T_n)$, the sequence of Chebyshev polynomials, and $\mathbf{Q} = (Q_n)$ defined by $Q_n(z) = z^n$, the pair $(\mathbf{P}, \mathbf{Q})$ are positive over $\mathbb{Z}$. We obtained also an extension of Theorem 1.2 to the case of marked surface as follows.

Theorem 1.3. *Let* $R = \mathbb{Z}$ *or* $R = \mathbb{Z}[q^{\pm 1}]$. *Suppose a pair* $(\mathbf{P}, \mathbf{Q})$ *of sequences of polynomials in* $R[z]$, $\mathbf{P} = (P_n(z)_{n\geq 0})$ *and* $\mathbf{Q} = (Q_n(z)_{n\geq 0})$, *are positive over* R. *Then* $P_n(z)$ *is an* R_+*-linear combination of* $T_0(z), T_1(z), \dots, T_n(z)$ *and* $Q_n(z)$ *is an* R_+*-linear combination of* $1, z, \dots, z^n$. *Moreover,* $P_1(z) = Q_1(z) = z$.

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1.4. **Plan of the paper.** In Section 2 we recall basic facts about the Kauffman bracket skein algebras of surfaces. We present the proofs of Theorem 1.2 and Theorem 1.3 in respectively Section 3 and Section 4.

2. Skein algebras

We recall here basic notions concerning the Kauffman bracket skein algebra of a surface.

2.1. **Ground ring.** Let R be any commutative domain over $\mathbb{Z}$ containing an invertible element q and a subset R_+ such that

- R_+ is closed under addition and multiplication
- $q, q^{-1} \in R_+$, and
- $R_+ \cap (-R_+) = \{0\}$.

For example, we can take $R = \mathbb{Z}$ with $q = 1$ and $R_+ = \mathbb{Z}_+$, or $R = \mathbb{Z}[q^{\pm 1}]$ with $R_+ = \mathbb{Z}_+[q^{\pm 1}]$. The reader might think of R as one of these two rings.

2.2. **Kauffman bracket skein algebra.** Suppose Σ is an oriented surface. The *Kauffman bracket skein module* $\mathscr{S}_R(\Sigma)$ is the R-module freely spanned by isotopy classes of non-oriented framed links in $\Sigma \times [0,1]$ modulo the *skein relation* and the *trivial loop relation* described in Figure 1. In all

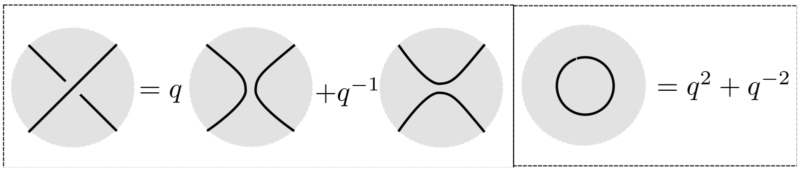

Figure 1. Skein relation (left) and trivial loop relation

Figures, framed links are drawn with blackboard framing. More precisely, the trivial loop relation says if L is a loop bounding a disk in $\Sigma \times [0,1]$ with framing perpendicular to the disk, then $L = -q^2 - q^{-2}$. And the skein relation says

$$L = qL_+ + q^{-1}L_-$$

if L, L_+, L_- are identical except in a ball in which they look like in Figure 2.

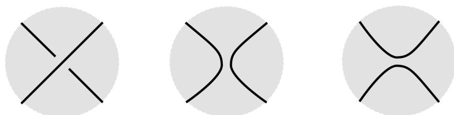

Figure 2. From left to right: L, L_+, L_-.

For future reference, we say that the diagram L_+ (resp. L_-) in Figure 2 is obtained from L by the positive (resp. negative) resolution of the crossing.

The R-module $\mathscr{S}_R(\Sigma)$ has an algebra structure where the product of two links α_1 and α_2 is the result of stacking α_1 atop α_2 using the cylinder structure of $\Sigma \times [0,1]$. Over $R = \mathbb{Z}$, the skein algebra $\mathscr{S}_R(\Sigma)$ is commutative and is closely related to the $SL_2(\mathbb{C})$-character variety of Σ, see [Bu, PS, Tu, BFK]. Over $R = \mathbb{Z}[q^{\pm 1}]$, $\mathscr{S}_R(\Sigma)$ is not commutative in general and is closely related to quantum Teichmüller spaces [CF].

2.3. **Bases.** We will consider Σ as a subset of $\Sigma \times [0,1]$ by identifying Σ with $\Sigma \times \{1/2\}$.

By [PS, Theorem 5.2], $\mathscr{S}_R(\Sigma)$ is free over R with basis the set of all essential links in Σ with vertical framing. Here *an essential link α in Σ* is a closed 1-dimensional non-oriented submanifold of Σ containing no trivial component, i.e. a component bounding a disk in Σ. The framing is *vertical* if at every point x, the framing is parallel to $x \times [0,1]$, with the direction equal to the positive direction of $[0,1]$. As usual, links in Σ are considered up to ambient isotopies of Σ. By convention, the empty set is considered an essential link.

This basis can be parameterized as follows. An *integer lamination* of Σ is an unordered collection $\mu = (n_i, C_i)_{i=1}^m$, where

- each n_i is a positive integer
- each C_i is a non-trivial knot in Σ
- no two C_i intersect
- no two C_i are ambient isotopic.

For each integer lamination μ, define an element $b_\mu \in \mathscr{S}(\Sigma)$ by

$$b_\mu = \prod_i (C_i)^{n_i}.$$

Then the set of all b_μ, where μ runs the set of all integer laminations including the empty one, is the above mentioned basis of $\mathscr{S}(\Sigma)$.

Suppose $\mathbf{P} = (P_n(z)_{n\geq 0})$ is a normalized sequence of polynomials in $R[z]$. Then we can twist the basis element b_μ by $\mathbf{P}$ as follows:

$$b_{\mu,\mathbf{P}} := \prod_i P_{n_i}(C_i).$$

As $\{P_n(z)\} \mid n \in \mathbb{Z}_+\}$, just like $\{z^n\}$, is a basis of $R[z]$, the set $\mathcal{B}_{\mathbf{P}}$ of all $b_{\mu,\mathbf{P}}$, when μ runs the set of all integer laminations, is a free R-basis of $\mathscr{S}(\Sigma)$.

3. Proof of Theorem 1.2

We present here the proof of Theorem 1.2, using a result from [Le1] which we recall first.

3.1. **Skein algebra of the annulus.** Let $\mathbb{A} \subset \mathbb{R}^2$ be the annulus $\mathbb{A} = \{x \in \mathbb{R}^2, 1 \leq |x| \leq 2\}$. Let z be the core of the annulus, $z = \{x, |x| = 3/2\}$. It is easy to show that, as an algebra, $\mathscr{S}_R(\mathbb{A}) = R[z]$.

Let $p_1 = (0,1) \in \mathbb{R}^2$ and $p_2 = (0,2) \in \mathbb{R}^2$, which are points in $\partial\mathbb{A}$. Then $\mathbb{A}_{io} = (\mathbb{A}, \{p_1, p_2\})$ is an example of a marked surface. See Figure 3, which also depicts the arcs θ_0, θ_{-1}, θ_2. Here θ_0 is the straight segment p_1p_2, and θ_n, for $n \in \mathbb{Z}$, is an arc properly embedded in $\mathbb{A}$ beginning at p_1 and winding clockwise (resp. counterclockwise) $|n|$ times if $n \geq 0$ (resp. $n < 0$) before getting to p_2.

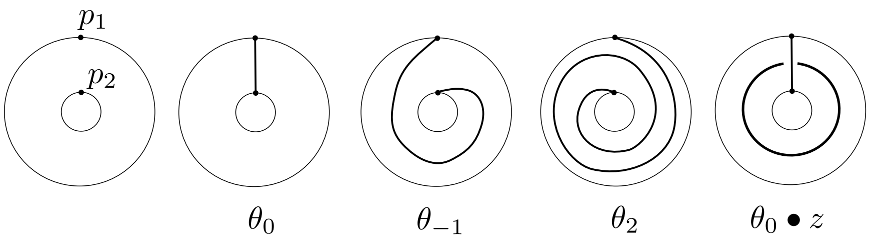

Figure 3. Marked annulus $\mathbb{A}_{io}$, arcs θ_0, θ_{-1}, θ_2, and element $\theta_0 \bullet z$

A $\mathbb{A}_{io}$-*arc* is a proper embedding of the interval $[0,1]$ in $\mathbb{A} \times [0,1]$ equipped with a framing such that one end point is in $p_1 \times [0,1]$ and the other is in $p_2 \times [0,1]$, and the framing is vertical at both end points. *A* $\mathbb{A}_{io}$-*tangle* is a disjoint union of a $\mathbb{A}_{io}$-arc and a (possibly empty) framed link in $\mathbb{A} \times [0,1]$. Isotopy of $\mathbb{A}_{io}$-tangles are considered in the class of $\mathbb{A}_{io}$-tangles.

Let $\mathscr{S}'_R(\mathbb{A}_{io})$ be the R-module spanned by isotopy classes of $\mathbb{A}_{io}$-tangles modulo the usual skein relation and the trivial knot relation. As usual, each θ_n is equipped with the vertical framing, and is considered as an element of $\mathscr{S}'_R(\mathbb{A}_{io})$.

For $\alpha \in \mathscr{S}(\mathbb{A})$ let $\theta_0 \bullet \alpha \in \mathscr{S}'(\mathbb{A}_{io})$ be the element obtained by placing θ_0 on top of α, see Figure 3 for an example. In [Le1] we proved the following.

Proposition 3.1. *For any integer $n \geq 1$, we have*

$$\theta_0 \bullet T_n(z) = q^n \theta_n + q^{-n} \theta_{-n}. \tag{1}$$

3.2. **Proof of Theorem 1.2.** We say that a sequence $\mathbf{P} = (P_n)$ of polynomials in $R[z]$ is *positive for* Σ over R if $\mathcal{B}_{\mathbf{P}}$ is a positive basis for $\mathscr{S}_R(\Sigma)$. Thus, $\mathbf{P} = (P_n)$ is positive if and only if it is positive for any oriented surface.

Let $\Sigma_{1,1}$ be surface obtained from a torus by removing the interior of a small disk. Topologically the interior of $\Sigma_{1,1}$ is a punctured torus. Theorem 1.2 follows from the following stronger statement.

Theorem 3.2. *Suppose $\mathbf{P} = (P_n(z))$ is positive for $\Sigma_{1,1}$. Then for every $n \geq 0$, $P_n(z)$ is an R_+-linear combination of $T_k(z)$ with $k \leq n$. Besides, $P_1(z) = z$.*

Proof. We present $\Sigma_{1,1}$ as the union $\Sigma_{1,1} = \mathbb{A} \cup \mathbb{A}'$, where $\mathbb{A}'$ is an annulus with core z', as in Figure 4, with $z' \cap \mathbb{A}$ equal to the arc $p_1 p_2$.

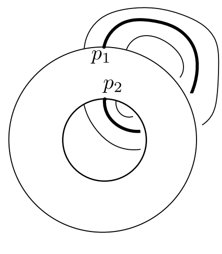

Figure 4. The punctured torus $\Sigma_{1,1} = \mathbb{A} \cup \mathbb{A}'$. The bold arc is $z' \setminus \mathbb{A}$.

By definition $P_1(x) = x + a$, $a \in R$. Fix an integer $n \geq 1$. There are $c_k \in R$ such that

$$P_n(z) = \sum_{k=0}^{n} c_k T_k(z).$$

Let $z_{1,k}$ be the curve $(z' \setminus \mathbb{A}) \cup \theta_k$. We have

$$P_1(z')P_n(z) = aP_n(z) + z' \sum_{k=0}^{n} c_k T_k(z) = aP_n(z) + c_0 z' + \sum_{k=1}^{n} c_k z' T_k(z)$$

$$= aP_n(z_2) + c_0 z_1 + \sum_{k=1}^{n} c_k \left(q^k z_{1,k} + c_k q^{-k} z_{1,-k} \right), \tag{2}$$

where the last equality follows from Proposition 3.1. Using the homology, one sees that in the collection $\{z, z', z_{1,k}, z_{1,-k} \mid k \geq 1\}$ every curve is a non-trivial knot in $\Sigma_{1,1}$, and any two of them are non-isotopic. Rewriting $x = P_1(x) - a$, we have

$$P_1(z')P_n(z) = aP_n(z) + c_0(P_1(z') - a) + \sum_{k=1}^{n} c_k \left[q^k (P_1(z_{1,k}) - a) + q^{-k}(P_1(z_{1,-k}) - a) \right]$$

$$= aP_n(z) + c_0\, P_1(z') + \sum_{k=1}^{n} c_k \left[q^k (P_1(z_{1,k}) + q^{-k}(P_1(z_{1,-k}) \right] - ac_0 - \sum_{k=1}^{n} ac_k(q^k + q^{-k}).$$

By the positivity, the coefficients of $P_1(z')$, $P_n(z)$, $P_1(z_{1,k})$ (for every $k \geq 1$) and the constant coefficient are in R_+. This shows that $a, c_k \in R_+$ for every k, and

$$d := -ac_0 - \sum_{k=1}^{n} ac_k(q^k + q^{-k}) \in R_+.$$

Note that $-d = a(c_0 + \sum_{k=1}^n c_k(q^k + q^{-k})) \in R_+$, since $a, c_k \in R_+$. Thus, $d, -d \in R_+$, which implies $d = 0$. Hence a, as a factor of d, is 0. This completes the proof of the theorem. $\square$

4. Marked surfaces

4.1. **Skein algebra of marked surfaces.** Suppose $(\Sigma, \mathcal{P})$ is a marked surface, i.e. $\mathcal{P} \subset \partial\Sigma$ is a finite set. *A framed 3D $\mathcal{P}$-tangle α in* $\Sigma \times [0,1]$ is a framed proper embedding of a 1-dimensional non-oriented compact manifold into $\Sigma \times [0,1]$ such that $\partial\alpha \subset \mathcal{P} \times [0,1]$, and the framing at every boundary point of α is vertical. Two framed 3D $\mathcal{P}$-tangles are *isotopic* if they are isotopic through the class of framed 3D $\mathcal{P}$-tangles.

Just like the link case, a framed 3D $\mathcal{P}$-tangle α is depicted by its diagram on Σ, with vertical framing on the diagram. Here a diagram of α is a projection of α onto Σ in general position, with an order of strands at every crossing. Crossings come in two types. If the crossing is in $\Sigma \setminus \mathcal{P}$, then it is a usual double point (with usual over/under information). If the crossing is a point in $\mathcal{P}$, there may be two or more number of strands, which are ordered. The order indicates which strand is above which. When there are two strands, the lower one is depicted by a broken line, see e.g. Figure 5.

Let $\mathscr{S}_R(\Sigma, \mathcal{P})$ be the R-module spanned by the set of isotopy classes of framed 3D $\mathcal{P}$-tangles in $\Sigma \times [0,1]$ modulo the skein relation, the trivial knot relation (Figure 1), and the trivial arc relation of Figure 5.

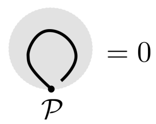

Figure 5. Trivial arc relation: If a framed 3D $\mathcal{P}$-tangle α has a trivial arc, then $\alpha = 0$.

The following relation holds, see [Le2].

Proposition 4.1. *In $\mathscr{S}(\Sigma, \mathcal{P})$, the reordering relation depicted in Figure 6 holds.*

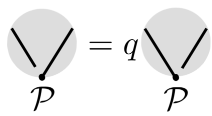

Figure 6. Reordering relation.

Again one defines the product $\alpha_1\alpha_2$ of of two skein elements α_1 and α_2 by stacking α_1 above α_2. This makes $\mathscr{S}(\Sigma, \mathcal{P})$ a R-algebra, which was first defined by Muller [Mu].

4.2. **Basis for $\mathscr{S}(\Sigma, \mathcal{P})$.** *A $\mathcal{P}$-arc* is an immersion $\alpha : [0,1] \to \Sigma$ such that $\alpha(0), \alpha(1) \in \mathcal{P}$ and the restriction of α onto $(0,1)$ is an embedding into $\Sigma \setminus \mathcal{P}$. *A $\mathcal{P}$-knot* is an embedding of S^1 into $\Sigma \setminus \mathcal{P}$. A $\mathcal{P}$-knot or a $\mathcal{P}$-arc is *trivial* if it bounds a disk in Σ. Two $\mathcal{P}$-arcs (resp. $\mathcal{P}$-knots) are *$\mathcal{P}$-isotopic* if they are isotopic in the class of $\mathcal{P}$-arcs (resp. $\mathcal{P}$-knots).

A $\mathcal{P}$-arc is called *boundary* if it can be isotoped into $\partial\Sigma$, otherwise it is called *inner*. We consider every $\mathcal{P}$-arc and every $\mathcal{P}$-knot as an element of $\mathscr{S}_R(\Sigma, \mathcal{P})$ by equipping it with the vertical framing. In the case when the two ends of a $\mathcal{P}$-arc are the same point $p \in \mathcal{P}$), we order the left strand to be above the right one.

An *integer $\mathcal{P}$-lamination* of Σ is an unordered collection $\mu = (n_i, C_i)_{i=1}^m$, where

- each n_i is a positive integer
- each C_i, called a component of μ, is a non-trivial $\mathcal{P}$-knot or a non-trivial $\mathcal{P}$-arc
- no two C_i intersect in $\Sigma \setminus \mathcal{P}$
- no two C_i are $\mathcal{P}$-isotopic.

In an integer $\mathcal{P}$-lamination $\mu = (n_i, C_i)_{i=1}^m$, a knot component commutes with any other component (in the algebra $\mathscr{S}_R(\Sigma, \mathcal{P})$), while a $\mathcal{P}$-arc component C q-commutes with any other component C' in the sense that $CC' = q^k C'C$ for a certain $k \in \mathbb{Z}$. The q-commutativity follows from Proposition 4.1. Hence the element

$$b_\mu = \prod_i (C_i)^{n_i} \tag{3}$$

is defined up to a factor $q^k, k \in \mathbb{Z}$. To make b_μ really well-defined, we will fix once and for all a total order on the set of all $\mathcal{P}$-arcs in Σ such that any boundary $\mathcal{P}$-arc comes before any inner $\mathcal{P}$-arc. Now define b_μ by (3), where the product is taken in this order.

It follows from [Mu, Lemma 4.1] that the set of all b_μ, where μ runs the set of all integer $\mathcal{P}$-laminations including the empty one, is the a free R-basis of $\mathscr{S}_R(\Sigma, \mathcal{P})$.

Suppose $\mathbf{P} = (P_n(z))$ and $\mathbf{Q} = (Q_n)$ are normalized sequences of polynomials in $R[z]$. Define

$$b_{\mu,\mathbf{P},\mathbf{Q}} = \prod_i F_{n_i}(C_i), \tag{4}$$

where the product is taken in the above mentioned order, and

- $F_{n_i} = P_{n_i}$ if C_i is a knot,
- $F_{n_i} = Q_{n_i}$ if C_i is an inner $\mathcal{P}$-arc,
- $F_{n_i}(C_i) = (C_i)^{n_i}$ if C_i is a boundary $\mathcal{P}$-arc.

Then clearly $\mathcal{B}_{\mathbf{P},\mathbf{Q}}$ is a free R-basis of $\mathscr{S}_R(\Sigma, \mathcal{P})$.

4.3. **Positive basis in quotients.** The following follows right away from the definition.

Lemma 4.2. *Let $\mathcal{B}$ be a positive basis of an R-algebra A and $I \subset A$ be an ideal of A, with $\pi : A \to A/I$ the natural projection. Assume that I respects the base $\mathcal{B}$ in the sense that I is freely R-spanned by $I \cap \mathcal{B}$. Then $\pi(\mathcal{B} \setminus I)$ is a positive basis of A/I.*

4.4. **Ideal generated by boundary arcs.**

Lemma 4.3. *Suppose $\alpha_1, \dots, \alpha_l$ are boundary $\mathcal{P}$-arcs, and $I = \sum_{i=1}^l \alpha_i \mathscr{S}_R(\Sigma, \mathcal{P})$.*
(a) The set I is a 2-sided ideal of $\mathscr{S}_R(\Sigma, \mathcal{P})$.
(b) For any normalized sequences $\mathbf{P}, \mathbf{Q}$ of polynomials in $R[z]$, I respects the basis $\mathcal{B}_{\mathbf{P},\mathbf{Q}}$.

Proof. (a) Suppose α is a boundary $\mathcal{P}$-arc. Since α q-commutes with any basis element b_μ defined by (3), $\alpha \mathscr{S}_R(\Sigma, \mathcal{P})$ is a 2-sided ideal of $\mathscr{S}_R(\Sigma, \mathcal{P})$. Hence I is also a 2-sided ideal.

(b) In the chosen order, the boundary $\mathcal{P}$-arcs come before any inner arcs $\mathcal{P}$-arc. Hence I is freely R-spanned by all $b_{\mu,\mathbf{P},\mathbf{Q}}$ such that μ has one of the α_i as a component. Thus, I respects $\mathcal{B}_{\mathbf{P},\mathbf{Q}}$. □

4.5. **Proof of Theorem 1.3.**

Lemma 4.4. *One has $Q_1(z) = z$.*

Proof. Consider the marked surface $\mathbb{D}_1$, which is a disk with 4 marked points p_0, p_1, p_2 and q_1 as in Figure 7. Let x be the arc p_0p_2 and y the arc p_1q_1. Suppose $Q_1(z) = z + a$, where $a \in R$. Using the skein relation to resolve to only crossing of xy, we get

$$Q_1(x)Q_1(y) = aQ_1(x) + aQ_1(y) - a^2 \mod I_\partial. \tag{5}$$

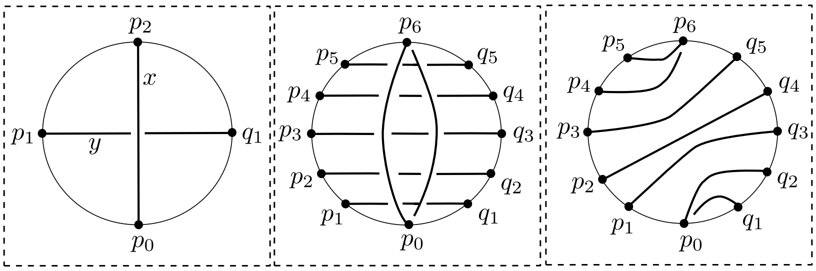

Figure 7. Left: $\mathbb{D}_1$, with product xy. Middle: $\mathbb{D}_5$, with product $x^2 y_5$. Right: Element $z_{2,5}$

Here I_∂ is the ideal of $\mathscr{S}(\mathbb{D}_1)$ generated by all the boundary $\mathcal{P}$-arcs, which respects $\mathcal{B}_{\mathbf{P},\mathbf{Q}}$ by Lemma 4.3. Lemma 4.2 and Equ. (5) show that $a \in R_+$ and $-a^2 \in R_+$. Hence both a^2 and $-a^2$ are R_+, which implies $a = 0$. $\square$

Now fix an integer number $n \geq 2$. Let $\mathbb{D}_n$ be the marked surface, which is a disk with $2n+2$ marked points which in clockwise orders are $p_0, p_1, \dots, p_{n+1}, q_n, \dots, q_1$. See Figure Figure 7 for an example of $\mathbb{D}_5$. Let x be the arc $p_0 p_{n+1}$, and y_n be the union of the n arcs $p_i q_i$, $i = 1, \dots, n$. For each $i = 0, \dots, n-1$ let γ_i be the arc $p_i p_{i+1}$, which is a boundary $\mathcal{P}$-arc. By Lemma 4.3, the set

$$I := \sum_{i=0}^{n-1} \gamma_i \mathscr{S}(\Sigma, \mathcal{P})$$

is a 2-sided ideal of $\mathscr{S}(\Sigma, \mathcal{P})$ respecting $\mathcal{B}_{\mathbf{P},\mathbf{Q}}$. It is important that the arc $p_n p_{n+1}$ is not in I.

Lemma 4.5. *For every $k \leq n$, one has*

$$x^k y_n = q^{-kn} z_{k,n} \mod I, \tag{6}$$

where $z_{k,n}$ is obtained from the diagram of $x^k y_n$ by negatively resolving all the crossings, see Figure 7.

Proof. We present x^k by k pairwise non-intersecting (except at the endpoints) arcs, each connecting p_0 and p_{n+1}, as in Figure 2. Label these arcs from left to right by $1, \dots, k$. There are kn crossings in the diagram of $x^k y_n$, and denote by $E_{l,m}$ the intersection of the l-th arc of x^k and the arc $p_m q_m$ of y_n. Order the set of all km crossings E_{lm} by the lexicographic pair $(l+m, l)$.

Suppose τ is one of the 2^{kn} ways of resolutions of all the kn crossings. Let D_τ be the result of the resolution τ, which is a diagram without crossing. Assume τ has at least one positive resolution. We will prove that $D_\tau \in I$. Let $E_{l,m}$ be the smallest crossing at which the resolution is positive. In a small neighborhood of $E_{l,m}$, D_τ has two arcs, with the lower left one denoted by δ, see Figure 8.

Figure 8. Left: Crossing $E_{l,m}$. Right: Its positive resolution, and the arc δ.

The resolution at any $E_{l',m'} < E_{l,m}$ is negative. These data are enough to determine the arc of D_τ containing δ: it is γ_{m-l} if $m \geq l$, and if $m < l$ then it is an arc whose two end points are p_0, which is 0. See Figure 9. Either case, $D_\tau \in I$.

Hence, modulo I, the only element obtained by resolving all the crossings of $x^k y_n$ is the all-negative resolution one, which is $z_{k,n}$. The corresponding factor coming from the skein relation is q^{-kn}. This proves Identity (6). $\square$

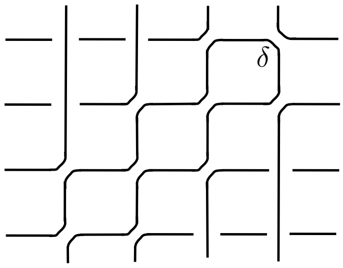

Figure 9. The top right corner crossing is $E_{l,m}$, with positive resolution. The arc of D_τ containing δ keeps going down the south-west direction.

Proof of Theorem 1.3. Theorem 1.2 implies that P_n is an R_+-linear combination of T_k with $k \le n$. Since $Q_1(z) = z$, each of $z_{k,n}$, y_n in Lemma 4.5 is an element of the basis $\mathcal{B}_{\mathbf{P},\mathbf{Q}}$.

Suppose $Q_n(z) = \sum_{k=0}^n c_k z^k$ with $c_k \in R$. From (6), we have

$$Q_n(x)y_n = \left(\sum_{k=0}^n c_k x^k\right) y_n = \sum_{k=0}^n c_k q^{-kn} z_{k,n} + I.$$

Note that $z_{k,n} \notin I$. Since I respects the basis $\mathcal{B}_{\mathbf{P},\mathbf{Q}}$, Lemma 4.2 shows that $c_k \in R_+$ for all k. This completes the proof of Theorem 1.3. □

References

[Bu] D. Bullock, *Rings of $Sl_2(\mathbb{C})$-characters and the Kauffman bracket skein module*, Comment. Math. Helv. **72** (1997), no. 4, 521–542.

[BFK] D. Bullock, C. Frohman, and J. Kania-Bartoszynska, *Understanding the Kauffman bracket skein module*, J. Knot Theory Ramifications **8** (1999), no. 3, 265–277.

[BW] F. Bonahon and H. Wong,*Quantum traces for representations of surface groups in SL_2(C)*, Geom. Topol. **15** (2011), no. 3, 1569–1615.

[CF] L. Chekhov and V. Fock, *Quantum Teichmüller spaces* (Russian) Teoret. Mat. Fiz. **120** (1999), no. 3, 511–528; translation in Theoret. and Math. Phys. **120** (1999), no. 3, 1245–1259.

[FoG] V. Fock and A. Goncharov, *Moduli spaces of local systems and higher Teichmüller theory*, Publ. Math. Inst. Hautes Études Sci. No. 103 (2006), 1–211.

[Kau] L. Kauffman, *States models and the Jones polynomial*, Topology, **26** (1987), 395–407.

[Le1] T. T. Q. Lê, *On Kauffman bracket skein modules at roots of unity*, Algebr. Geom. Topol. **15** (2015), 1093–1117.

[Le2] T. T. Q. Lê, *Quantum Teichmuller spaces and quantum trace map*, preprint arXiv:1511.06054, 2015.

[LT] T. T. Q. Lê and A. Tran, *On the AJ conjecture for knots*, Indiana Univ. Math. J. **64** (2015), 1103–1151.

[Mu] G. Muller, *Skein algebras and cluster algebras of marked surfaces*, Preprint arXiv:1204.0020, 2012.

[MSW] G. Musiker,R. Schiffler, and L. Williams, *Bases for cluster algebras from surfaces*, Compos. Math. **149** (2013), 217–263.

[Pr] J. Przytycki, *Fundamentals of Kauffman bracket skein modules*, Kobe J. Math. **16** (1999) 45–66.

[PS] J. Przytycki and A. Sikora, *On the skein algebras and $Sl_2(\mathbb{C})$-character varieties*, Topology **39** (2000), 115–148.

[Th] D. Thurston, *Positive basis for surface skein algebras*, Proc. Natl. Acad. Sci. USA **111** (2014), 9725–9732.

[Tu] V. Turaev, *Skein quantization of Poisson algebras of loops on surfaces*, Ann. Sci. Sc. Norm. Sup. (4) **24** (1991), no. 6, 635–704.

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