A QUANTUM sl_2-INVARIANT OF 3-MANIFOLDS WHICH CONTAINS ALL THE WITTEN-RESHETIKHIN-TURAEV INVARIANTS

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Abstract. We define a 3-manifold invariant with values in a functional space. The quantum sl_2 invariants of Witten and Reshetikhin-Turaev can be regarded as special values of this invariant.

1. Introduction

Using mathematically non-rigorous path integrals, Witten [Wi] defined quantum invariants of 3-manifolds, depending on a Lie algebra and a quantum parameter q. Reshetikhin and Turaev [RT1] defined 3-manifold invariant τ_r which can be considered as Witten invariant corresponding to the case when the Lie algebra is sl_2 and $q = \exp(2\pi i/r)$ is a root of unity. There are an open question about the behavior of τ_r when $r \to \infty$, and a related question how to define quantum sl_2 invariants not at roots of unity.

For various treatments on quantum sl_2 invariants at roots of unity, the reader is referred to [RT1, KM, Li, Mo, Tu].

Here we define a quantum sl_2 invariant $\tau(M)$, in some sense, not at roots of unity. The values of the invariant are in a functional space which can be considered as some completion of the space of Laurent power series in h, where $q = \exp(2\pi i h)$. When $h = 1/r$, where r is an integer, the "function" (which is not a function in the usual sense) gives rise to a complex value, and $\tau(M)(1/r) = \tau_r(M)$.

To define $\tau(M)$ we follows the line in [KM]. The first observation is that the colored Jones polynomial can be defined when the colors are not integers, although when the colors are not integers, we get a formal power series in h which is not necessarily convergent. This has been proved in [MM]; and can be proved in general case for arbitrary Lie algebra using relation between the Kontsevich integral and quantum invariants of links [LM, Ka]. Then we define formally the invariant as in [KM], showing that all the steps in [KM] can be done. The most difficult is to prove that some relations between some quantities are polynomial. By a trivial reason we don't need any analog of the symmetry principle of [KM].

2. Some functional spaces

2.1. **Power series with polynomial coefficients.** Denote by $\mathbb{N}$ the set of all positive integers. The following is trivial.

Lemma 2.1. *Suppose $f, g \in \mathbb{C}[x_1, \dots, x_n]$ are two polynomials such that $f(k_1, \dots, k_n) = g(k_1, \dots k_n)$ for every $(k_1, k_2, \dots, k_n) \in \mathbb{N}^n$ with $k_1 \geq k_2$, then $f = g$.*

Lemma 2.2. *Suppose $f(z)$ is a polynomial in z, then there is a polynomial $g(z_1, z_2)$ such that $g(k_1, k_2) = \sum_{j=k_1}^{k_2} f(j)$ for every $(k_1, k_2) \in \mathbb{N}^2$.*

Proof. This follows from the identity:

$$\sum_{j=k_1}^{k_2} j^m = [B_{m+1}(k_2+1) - B_{m+1}(k_1)]/(m+1),$$

where $B_m(z)$ is the m-th Bernoulli polynomial, (see, for example [Ra]). □

For each positive integer n let R_n be the subset of $\mathbb{C}[x_1,\dots,x_n][[h]]$ consisting of $f(x_1,\dots,x_n;h)$ such that for every tuple $\mathbf{k}=(k_1,\dots,k_n)\in\mathbb{N}^n$ the formal power series $g(h)=f(k_1,\dots,k_m;h)$ in h is an entire function of h.

Lemma 2.3. *Two elements f,g in R_n are equal if and only if $f(\mathbf{k})=g(\mathbf{k})$ as two functions in h, for every $\mathbf{k}\in\mathbb{N}^n$.*

Proof. This follows from the fact that the coefficients of h in f,g are polynomials in $x_1,\dots,x_n$. □

Lemma 2.4. *Let $f(x_1,\dots,x_n;h)\in R_n$. There exists an element $\sigma(f)\in R_{n+1}$ such that for every $\mathbf{k}=(k_1,\dots,k_n,k_{n+1})\in\mathbb{N}^{n+1}$ we have*

$$\sigma(f)(k_1,\dots,k_n,k_{n+1};h) = \sum_{j=k_1-k_2+1}^{k_1+k_2-1} f(j,k_3,\dots,k_{n+1};h),$$

where the sum is over all $j=k_1-k_2+1, k_1-k_2+3, k_1-k_2+5,\dots,k_1+k_2-1$.

This can be proved easily using Lemma 2.2.

Example . 1) $u(x_1;h):=1-x_1h$ is an element of R_1. It is easy to see that u is non-decomposable and is not a unit of R_1.

2) Denote by $[x_1]_h$ the formal power series in $\mathbb{C}[x_1][[h]]$

$$[x_1]_h = \frac{\exp(\pi i x_1 h) - \exp(-\pi i x_1 h)}{\exp(\pi i h) - \exp(-\pi i h)}. \tag{2.1}$$

It is easy to see that $[x_1]_h$ is an element of R_1.

Let $\mathcal{S}=R_1/(u)$ be the quotient algebra. Actually, $\mathcal{S}$ is an integral domain. For an element $f\in R_1$ denote by $\hat{f}$ its image in the quotient $\mathcal{S}$.

Formally, if $f(x_1;h)\in R_1$, then $\hat{f}$ should be $f(1/h;h)$. But we cannot replace x_1 by $1/h$ in ALL terms of the power series $f(x_1;h)$; we can do that only for any finite number of terms. If we replace x_1 by $1/h$ in all terms of the power series $f(x_1;h)$, the coefficient of a power of h is an infinite sum which may be divergent.

The following fact shows that $\mathcal{S}$ is not a trivial algebra.

For each positive integer r let μ_r be the homomorphism from $\mathcal{S}$ to $\mathbb{C}$ defined by:

$$\mu_r(\hat{f}(x_1;h)) = f(r;1/r).$$

It is well-defined: if $g(x_1;h)-f(x_1;h)$ is divisible by $u=1-x_1h$ in R_1, then it is easy to see that $\mu_r(f)=\mu_r(g)$. The image of each μ_r is the whole $\mathbb{C}$, so $\mathcal{S}$ is not trivial.

2.2. **Some computations.** Let $t = \exp(\pi i h/2)$; it is an entire function of h and can be regarded as element of R_1. Then $t^{2x_1} := \exp(\pi i\, x_1 h)$ is an element of R_1, and $t^{2(x_1 \pm x_2)} := \exp(\pi i\,(x_1 \pm x_2)h)$ are elements of R_2.

Lemma 2.5. *$t^2 - t^{-2}$ is not equal to 0 in $\mathcal{S}$. Other words, $\widehat{(t^2 - t^{-2})} \neq 0$.*

Lemma 2.6. *One has*

$$t^{2x_1} + 1 = 0 \pmod{1 - x_1 h}.$$

Proof. One has to prove that $\exp(\pi i x_1 h) + 1$ is divisible by $u = 1 - x_1 h$ in R_1. Observe that $\exp(\pi i(x_1 h)) + 1$ is an entire function of $x_1 h$ and vanishes when $x_1 h = 1$, hence $\exp(\pi i x_1 h) + 1 = u(x_1; h) g(x_1 h)$, where $g(z)$ is an entire function. It follows that $g(x_1 h)$ is in R_1. This completes the proof. □

Corollary 2.1. *In R_2*

$$[x_2]_h = [x_1 - x_2]_h = -[x_1 + x_2]_h \pmod{(1 - x_1 h)}.$$

Proof. Note that $t^{2(x_1 - x_2)} = t^{2x_1} t^{-2x_2}$. By the previous Lemma, the latter is equal to $-t^{-2x_2}$, $\mod 1 - x_1 h$. The Corollary follows easily from this observation. □

Proposition 2.1. *There are $c_+, c_- \in R_1$ and $s \in R_2$ such that for every positive integers j, r one has:*

$$c_+(r; h) = \sum_{k=1}^{r} t^{k^2-1} [k]_h^2,$$

$$c_-(r; h) = \sum_{k=1}^{r} t^{-(k^2-1)} [k]_h^2,$$

$$s(r, j; h) = \sum_{k=1}^{4r} t^{(k+j)^2}.$$

Proof. Expand t, $[k]_h$ as formal power series in h, then apply Lemma 2.2. □

For example, here is an explicit formula of $s(x_1, x_2; h)$:

$$s(x_1, x_2; h) = \sum_{m=0}^{\infty} \frac{1}{m!} (\frac{\pi i h}{2})^m \sum_{q=0}^{2m} \begin{pmatrix} 2m \\ q \end{pmatrix} x_2^{2m-q} [B_{q+1}(4x_1 + 1) - B_{q+1}(1)].$$

Lemma 2.7. *Modulo $1 - x_1 h$, the element $s(x_1, x_2; h)$ does not depend on x_2. This means*

$$s(x_1, x_2; h) = s(x_1, 0; h) \pmod{1 - x_1 h}.$$

Proof. By Lemma 2.6, $t^{4x_1} = 1 \pmod{1 - x_1 h}$. Hence there is $g \in R_2$ such that for every positive integers r, m one has

$$t^{(m+4r)^2} - t^{m^2} = (1 - rh) g(r, m; h). \tag{2.2}$$

For every positive integers r, j note that

$$s(r, j; h) = \sum_{k=1}^{4r} t^{(k+j)^2} = \sum_{k=1}^{4r} t^{k^2} + \sum_{m=1}^{j} (t^{(m+4r)^2} - t^{m^2}). \tag{2.3}$$

There is $f \in R_2$ such that for every positive integers j, r we have $f(r, j; h) = \sum_{m=1}^{j} g(r, m; h)$. Equalities (2.2) and (2.3) show that $s(x_1, x_2; h) = s(x_1, 0; h) + (1 - x_1 h) f(x_1, x_2; h)$. □

Let $s_0(x_1;h) = s(x_1,0;h)$. We consider s_0 as an element of R_1.

Lemma 2.8. *One has*

$$(t-t^5)c_+ = s_0/2 = (t^{-1}-t^{-5})c_- \pmod{(1-x_1h)}.$$

Proof. Let $d_+(x_1,h)$ be the element in R_1 such that for every $r\in\mathbb{N}$, $d_+(r;h)=\frac{1}{4}\sum_{k=1}^{4r} t^{k^2-1}[k]_h^2$.

First observe that $c_+ = d_+ \pmod{1-x_1h}$.

In fact, by Corollary 2.1 and Lemma 2.6 there is $f\in R_2$ such that for every positive integers k,n:

$$t^{k^2-1}[k]_h^2 = t^{(2r-k)^2-1}[2r-k]_h^2 + (1-rh)f(r,k;h).$$

(The first term of the right hand side is obtained from the left hand side by replacing k by $(2r-k)$). When k runs the set $\{1,2,\dots,r\}$, $(2r-k)$ runs the set $\{r,r+1,\dots,2r-1\}$. Summing from $k=1$ to $k=r$ one gets

$$c_+(r;h) = \sum_{k=r}^{2r-1} t^{k^2-1}[k]_h^2 + (1-rh)g(r;h),$$

where $g\in R_1$. Note that, by Lemma 2.6, $[x_1]_h=[2x_1]_h=0 \pmod{1-x_1h}$. Hence from the above identity one has

$$c_+(r;h) = \sum_{k=r+1}^{2r} t^{k^2-1}[k]_h^2 + (1-rh)g_1(r;h),$$

for some $g_1\in R_1$. Similarly, we can see that

$$c_+(r,h) = \sum_{k=2r+1}^{3r} t^{k^2-1}[k]_h^2 + (1-rh)g_2(r;h),$$

$$c_+(r,h) = \sum_{k=3r+1}^{4r} t^{k^2-1}[k]_h^2 + (1-rh)g_3(r;h),$$

for some $g_2,g_3\in R_1$. Summing up the last three equalities, we get $c_+=d_+ \pmod{1-x_1h}$.

Now

$$\begin{aligned} t^3(t^2-t^{-2})^2 d_+(r;h) &= \frac{1}{4}\sum_{k=1}^{4r} t^{k^2+2}(t^{2k}-t^{-2k})^2 \\ &= \frac{1}{4}\sum_{k=1}^{4r}[t^{k^2+4k+2}+t^{k^2-4k+2}-2t^{k^2+2}] \\ &= \frac{1}{4}\sum_{k=1}^{4r}[t^{-2}(t^{(k+2)^2}+t^{(k-2)^2})-2t^2t^{k^2}] \\ &= \frac{t^{-2}}{4}[s(r,2;h)+s(r,-2;h)] - \frac{t^2}{2}s_0(r;h). \end{aligned}$$

Using Lemma 2.7 one gets

$$t^3(t^2-t^{-2})^2 d_+(x_1;h) = \frac{1}{2}(t^{-2}-t^2)s_0(x_1;h) \pmod{(1-x_1h)}.$$

Since t^2-t^{-2} is not 0 in $\mathcal{S}$ one can divide by t^2-t^{-2} and get the desired result for c_+. The computation for c_- is similar. □

Lemma 2.9. *There exists $v_+(x_1, x_2; h) \in R_2$ such that for every $j, r \in \mathbb{N}$ one has*

$$\sum_{k=1}^{r} t^{j^2+k^2-2} \frac{[jk]_h[k]_h}{[j]_h} = c_+(r; h) + u(r; h)v_+(j, r; h).$$

Both sides are considered as entire functions in h. Similarly, there exists $v_-(x_1, x_2; h) \in R_2$ such that for every $j, r \in \mathbb{N}$ one has

$$\sum_{k=1}^{r} t^{-(j^2+k^2-2)} \frac{[jk]_h[k]_h}{[j]_h} = c_-(r; h) + u(r; h)v_-(r, j; h).$$

Proof. As in the proof of the previous Lemma, we note that

$$\sum_{k=1}^{r} t^{(j^2+k^2-2)} \frac{[jk]_h[k]_h}{[j]_h} = \frac{1}{4} \sum_{k=1}^{4r} t^{(j^2+k^2-2)} \frac{[jk]_h[k]_h}{[j]_h}.$$

Hence it suffices to prove that there is $w_+(x_1, x_2; h) \in R_2$ such that for every $(j, r) \in \mathbb{N}^2$ one has

$$\sum_{k=1}^{4r} t^{j^2+k^2-2}[jk]_h[k]_h = c_+(r; h)[j]_h + u(r; h)w(r, j; h).$$

We have

$$t^3(t^2 - t^{-2})^2 \sum_{k=1}^{4r} t^{j^2+k^2-2}[jk]_h[k]_h = \sum_{k=1}^{4r} t^{j^2+k^2+1}(t^{2jk} - t^{-2jk})(t^{2k} - t^{-2k})$$

$$= \sum_{k=1}^{4r} [t^{-2j}(t^{(j+k+1)^2} + t^{(j-k+1)^2}) - t^{2j}(t^{(j+k-1)^2} + t^{(j-k-1)^2})].$$

$$= t^{-2j}[s(r, j + 1; h) + s(r, -j - 1; h)] - t^{2j}[s(r, j - 1; h) + s(r, -j + 1; h)].$$

Using Lemma 2.7, we get

$$t^3(t^2 - t^{-2})^2 \sum_{k=1}^{4r} t^{j^2+k^2-2}[jk]_h[k]_h = (t^2 - t^{-2})[j]_h 2s + (1 - rh)f(r, j; h),$$

for some $f \in R_2$.

Now applying Lemma 2.8, we see that

$$\sum_{k=1}^{4r} t^{j^2+k^2-2}[jk]_h[k]_h = c_+(r; h)[j]_h + (1 - rh)w_+(r, j; h),$$

for some $w_+ \in R_2$.

The proof for c_- is similar. □

3. The colored Jones polynomial

Let L be a framed non-oriented link in S^3 whose components are numbered: $L_1, \ldots, L_n$. For every $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$ denote by $J_{L,\mathbf{k}}(q)$ the quantum sl_2 invariant of the framed link L with colors $\mathbf{k}$, (see [KR, MM, KM, RT2]). Here q is a formal parameter. $J_{L,\mathbf{k}}$ is also known as the colored Jones polynomial.

We introduce another parameter h with $q = \exp(2\pi i h)$.

Theorem 3.1. *For every framed link L of n components, there is $F_L \in R_n$ such that for every $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{N}^n$ one has*

$$F_L(\mathbf{k}; h) = J_{L,\mathbf{k}}(\exp(2\pi i h)). \tag{3.1}$$

This has been proved in [MM]. Another proof can be deduced from theory of Kontsevich integral, see [LM]. Using this Theorem, one can define $F_L(x_1, \dots, x_n; h)$ when x_i's are not integer; the result is a formal power series in h, not necessarily an entire function.

Theorem 3.2. *Let $L^{(2)}$ be the link obtained from L by replacing component L_1 by 2 parallel push-offs of L_1, using the frame. We have*

$$F_{L'} = \sigma(F_L).$$

Proof. Recall that σ is defined in Lemma 2.4. We have to prove that

$$F_{L^{(2)}}(k_1, k_2, \dots, k_{n+1}; h) = \sigma(F_L)(k_1, k_2, \dots, k_{n+1}; h), \tag{3.2}$$

for every $(k_1, k_2, \dots, k_{n+1}) \in \mathbb{N}^{n+1}$. This is proved in [KM, MS, Mu] for $k_1 \geq k_2$. Lemma 2.1 now shows that (3.2) is true for every $k_1, k_2, \dots, k_{n+1} \in \mathbb{N}$. □

Suppose $L, L_{(1)}, L_{(2)}, L_{(3)}$ are 4 framed links which agree everywhere except for a small ball in which they are as in Figure 1, with blackboard framing.

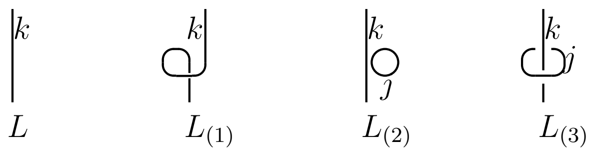

FIGURE 1.

We suppose that L has n components, then L_2 and L_3 have $n+1$ components, and we enumerate them so that the extra components in Figure 1 are the $(n+1)$-th. The colors of components are there shown.

Theorem 3.3. *One has:*

$$F_{L_{(1)}} = \exp[(\pi i h/2)(k^2 - 1)] F_L = t^{k^2-1} F_L,$$

$$F_{L_{(2)}} = [j]_h F_L,$$

$$F_{L_{(3)}} = \frac{[jk]_h [k]_h}{[j]_h} F_L.$$

See [KM] for a proof.

Let us denote $\mathbf{2}_n = (2, \dots, 2) \in \mathbb{N}^n$. Let $E_L(h) = F_L(\mathbf{2}; h)$. This is a form of the Jones polynomial for non-oriented links.

For a framed link L of n components $L_1, \dots, L_n$ and a tuple $\mathbf{j} = (j_1, \dots, j_n) \in \mathbb{N}^n$ let $L^{(\mathbf{j})}$ be the link obtained by replacing each component L_i of L by j_i parallels of L_i.

We use multi-index notation: $\begin{pmatrix} \mathbf{n} \\ \mathbf{k} \end{pmatrix} = \prod \begin{pmatrix} n_i \\ k_i \end{pmatrix}$, $(-1)^{\mathbf{j}} = \prod (-1)^{j_i}$, and $\sum_{\mathbf{j}=\mathbf{1}}^{\mathbf{n}}$ is the sum over all $\mathbf{j}$ with $1 \leq j_i \leq n_i$.

Theorem 3.4. *For every* $\mathbf{k} \in \mathbb{N}^n$

$$F_L(\mathbf{k}; h) = \sum_{\mathbf{j}=\mathbf{1}}^{(\mathbf{k}-\mathbf{1})/2} (-1)^{\mathbf{j}} \begin{pmatrix} \mathbf{k}-\mathbf{1}-\mathbf{j} \\ \mathbf{j} \end{pmatrix} E_{L(\mathbf{k}-\mathbf{1}-\mathbf{2j})}(h).$$

This is proved in [KM].

A well-known property of E_L is that it satisfies some skein relation. We formulate this property in the following way.

Theorem 3.5. *(see* [KM]*)*

Let L, L', L'' *be 3 framed links which agree everywhere except for a small ball in which they are as in Figure 2. Suppose that they are colored by the same color outside the ball and that the components involved in the Figure have color 2. Then*

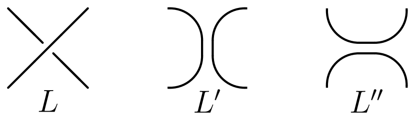

FIGURE 2.

a) $F_L = \exp(\pi i h/2) F_{L'} + \exp(-\pi i h/2) F_{L''}$ *if the two strands in the crossing come from different components of* L*, and*

b) $F_L = \varepsilon[\exp(\pi i h/2) F_{L'} - \exp(-\pi i h/2) E_{L''}]$ *if the two strands come from the same component of* L*, producing a crossing of sign* $\varepsilon = \pm 1$.

4. 3-MANIFOLD INVARIANTS

Suppose K is a framed link obtained from L by a Kirby move depicted in Figure 3, with the tangle in the box denoted by T. Here we use blackboard framing. The band outside the tangle T consists of many parallel lines. This is a negative second Kirby move. There is a similar positive second Kirby move. Suppose also that L is colored by $\mathbf{k}$.

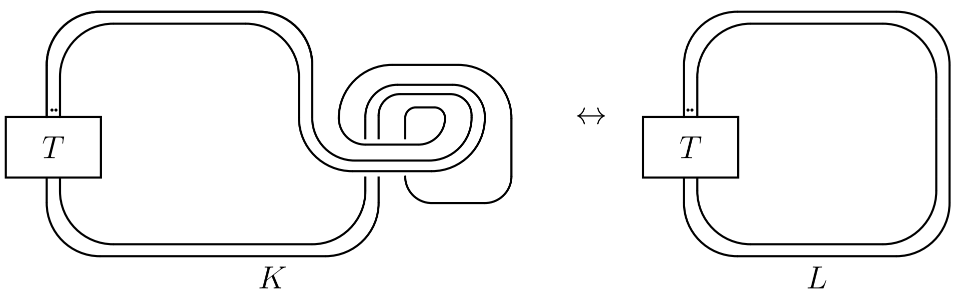

FIGURE 3.

Proposition 4.1. *There is an element* $a_T(x_1, \dots, x_{n+1}; h) \in R_{n+1}$ *such that*

$$\sum_{l=1}^{r} [l]_h F_K(\mathbf{k}, l; h) = c_- F_L(\mathbf{k}; h) + u(r; h) a_T(\mathbf{k}, r; h). \tag{4.1}$$

Here the extra component of K has color l, while other components have the same colors as those of components of L.

Proof. a) Suppose we have only one strand in the Kirby move. Using Theorem 3.3 one can easily express F_K in terms of F_L. Take $a_T = v_-(j,r;h)$ of Lemma 2.9. The statement now follows from this Lemma.

b) Suppose T_m is the trivial tangle with m strands. We show that the Proposition is valid for T_m using induction on m. The case $m=1$ has been covered by the previous case. Suppose a_{T_l} exists and formula (4.1) is valid for $l<m$. Define $a_{T_m} = \sigma(a_{T_{m-1}})$, where σ is defined in §2. Then Theorem 3.2 shows that the statement is true for T_m.

c) Suppose T has a diagram without double points. Then L is a trivial link, and one uses isotopy to remove all the maximal points in T. We are led to the previous case.

d) We show that for each tangle T there exists $b_T(x_1;h) \in R_1$, such that for every $r \in \mathbb{N}$

$$\sum_{k=1}^{r}[k]_h F_K(\mathbf{2}_n,k;h) = c_- F_L(\mathbf{2}_k;h) + u(r;h) b_T(r;h), \qquad (*)$$

using induction on the number of double points in a tangle diagram D of T. The case when there is no double point is covered by the previous case, with $b_T(x_1,h) = a_T(\mathbf{2}_m,x_1;h)$.

The induction step can be carried easily using the skein relation for F_L (see Theorem 3.5).

e) For every tuple $\mathbf{j} \in \mathbb{N}^n$ one can define the cablings $L^{(\mathbf{j})}$, $T^{(\mathbf{j})}$ and $K^{(\mathbf{j})}$. Here n is the number of components of L; and the cabling operation does not touch the extra component of K.

Note that $E_{L^{(\mathbf{j})}}(h)$ is a power series in h with coefficients being polynomials in $\mathbf{j}$. Similarly, $F_{K^{(\mathbf{j})}}(\mathbf{2},k;h)$ is a power series in h with coefficients being polynomials in k and $\mathbf{j}$. From (*) it follows that $b_{T^{(\mathbf{j})}}(r;h)$ is a power series in h with coefficients being polynomials in $\mathbf{j}$ and r.

f) In what follows, in essential, we reduce everything to the $\mathbf{k}=\mathbf{2}$ case.

Now for every tangle T, let:

$$a_T(k_1,\dots,k_n,r;h) = \sum_{\mathbf{j}=\mathbf{1}}^{(\mathbf{k}-\mathbf{1})/2} (-1)^{\mathbf{j}} \binom{\mathbf{k}-\mathbf{j}-\mathbf{1}}{\mathbf{j}} b_{T^{(\mathbf{k}-2\mathbf{j}-\mathbf{1})}}(r;h)$$

Lemma 2.2, together with step e) guarantees that a_T is in R_{n+1}.

Applying Theorem 3.4 (which expresses F_L through certain cablings) we get (4.1). □

For $\mathbf{k} = (k_1,\dots,k_n) \in \mathbb{N}^n$ let $[\mathbf{k}]_h = [k_1]_h \dots [k_n]_h$; it is an entire function in h. From Theorem 3.1 and Lemma 2.2 we get

Proposition 4.2. *For every framed link L of n components, there exists $G_L \in R_1$ such that for every $r \in \mathbb{N}$ one has*

$$G_L(r;h) = \sum_{i=1}^{n}\sum_{k_i=1}^{r}[\mathbf{k}]_h F_L(\mathbf{k};h).$$

Endow components of L with arbitrary orientation. Let $A = A_{ij}$ with $1 \le i,j \le n$ be the matrix defined by $A_{ij} = lk(L_i,L_j)$ if $i \neq j$ and $A_{ii} =$ the framing of L_i. Let σ_+, σ_- are respectively the numbers of $+$ and $-$ modes of matrix A.

Let

$$\tau(L) = \left(\frac{\widehat{G_L}}{c_+^{-\sigma_+} c_-^{-\sigma_-}} \right), \tag{4.2}$$

considered as an element of $s_0^{-1}\mathcal{S}$. The definition of $\mathcal{S}$ is given in §2.1.

Theorem 4.1. *$\tau(L)$ does not change under the two Kirby moves and hence defines an invariant $\tau(M_L) = \tau(L)$ of the 3-manifold obtained by Dehn surgery on L. Under the homomorphism μ_r, $\tau(M)$ is mapped to the Reshetikhin-Turaev invariant:*

$$\mu_r(\tau(M)) = \tau_r(M).$$

Proof. Using Theorem 3.3 one sees that $c_+ = G_{L+}$, where L_+ is the trivial knot with framing 1. Similarly $c_- = G_{L_-}$, where L_- is the trivial knot with framing -1.

Hence from (4.2) it is clear that $\tau(L)$ is invariant under the first Kirby move.

Let K be obtained from L by a second Kirby move as in Figure 3, with the tangle in the box denoted by T. Define

$$d_T(r;h) = \sum_{\mathbf{j=1}}^{r} [\mathbf{j}]_h a_T(\mathbf{j}, r; h),$$

where a_T is as in equation (4.1). By Lemma 2.2, d_T is in R_1.

Then from the definition of G_L and equality 4.1 we get

$$G_K(r;h) = c_- G_L(r;h) + u(r;h) d_T(r;h)$$

This means, in $s_0^{-1}\mathcal{S}$ one has $\tau(K) = \tau(L)$, or $\tau(L)$ is invariant under a negative second Kirby move.

It can be proved easily, using Lemma 2.9, that $\tau(L)$ is invariant under both positive and negative second Kirby moves when there is only one strand in the moves. (Or one can prove in a similar way that $\tau(L)$ is invariant under positive second Kirby moves.)

This is enough to conclude that $\tau(L)$ is invariant under all the Kirby moves (see [RT1, Tu]). Hence τ is an invariant of 3-manifolds.

The fact that $\mu_r(\tau(L)) = \tau_r(L)$, where τ_r is the Reshetikhin-Turaev invariant defined in [KM, RT1], normalized as in [KM], follows immediately from the definitions. ☐

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