

# ON KAUFFMAN BRACKET SKEIN MODULES AT ROOT OF UNITY

THANG T. Q. LÊ

ABSTRACT. We reprove and expand results of Bonahon and Wong on central elements of the Kauffman bracket skein modules at root of 1 and on the existence of the Chebyshev homomorphism, using elementary skein methods.

## 0. INTRODUCTION

**0.1. Kauffman bracket skein modules.** Let us recall the definition of the Kauffman bracket skein module, which was introduced by J. Przytycki [Pr] and V. Turaev [Tu]. Let  $R = \mathbb{C}[t^{\pm 1}]$ . A *framed link* in an oriented 3-manifold  $M$  is a disjoint union of smoothly embedded circles, equipped with a non-zero normal vector field. The empty set is also considered a framed link. The Kauffman bracket skein module  $\mathcal{S}(M)$  is the  $R$ -module spanned by isotopy classes of framed links in  $M$  subject to the following relations

$$(1) \quad L = tL_+ + t^{-1}L_-$$

$$(2) \quad L \sqcup U = -(t^2 + t^{-2})L,$$

where in the first identity,  $L, L_+, L_-$  are identical except in a ball in which they look like in Figure 1, and in the second identity, the left hand side stands for the union of a link  $L$

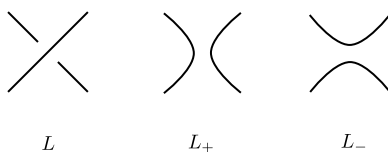


FIGURE 1. The links  $L, L_+$ , and  $L_-$

and the trivial framed knot  $U$  in a ball disjoint from  $L$ . When  $M = \mathbb{R}^3$ , then  $\mathcal{S}(\mathbb{R}^3) = R$ , and the value of a framed link  $L$  in  $\mathcal{S}(\mathbb{R}^3) = R = \mathbb{C}[t^{\pm 1}]$  is a version of Jones polynomial [Kau].

For a non-zero complex number  $\xi$ , let  $\mathcal{S}_\xi(M)$  be the quotient  $\mathcal{S}(M)/(t - \xi)$ , which is a  $\mathbb{C}$ -vector space.

For an oriented surface  $\Sigma$ , possibly with boundary, we define  $\mathcal{S}(\Sigma) := \mathcal{S}(M)$ , where  $M = \Sigma \times [-1, 1]$  is the cylinder over  $\Sigma$ . The skein module  $\mathcal{S}(\Sigma)$  has an algebra structure induced by the operation of gluing one cylinder on top of the other.

For a framed knot  $K$  in  $M$  and an integer  $m \geq 0$  let  $K^m$  be the link consists of  $m$  parallels of  $K$  (using the framing of  $K$ ) in a small neighborhood of  $K$ . A polynomial  $p(z)$  is a linear combination of powers  $z^m$ , hence one can define  $p(K)$  as an element in  $\mathcal{S}(M)$ . When  $L$  is a link, define  $p(L)$  by applying  $p$  to each component of  $L$ .

---

Supported in part by National Science Foundation.

2010 *Mathematics Classification*: Primary 57N10. Secondary 57M25.

*Key words and phrases*: Kauffman bracket skein module, Chebyshev homomorphism.

## 0.2. Bonahon and Wong results.

**Definition 1.** A polynomial  $p(z) \in \mathbb{C}[z]$  is called central at  $\xi \in \mathbb{C}^\times$  if for any oriented surface  $\Sigma$  and any framed link  $L$  in  $\Sigma \times [-1, 1]$ ,  $p(L)$  is central in the algebra  $\mathcal{S}_\xi(\Sigma)$ .

Bonahon and Wong [BW2] showed that if  $\xi$  is root of unity of order  $2N$ , then  $T_N(z)$  is central, where  $T_N(z)$  is the Chebyshev polynomial of type 1 defined recursively by

$$T_0(z) = 2, T_1(z) = 1, T_n(z) = zT_{n-1}(z) - T_{n-2}(z), \forall n \geq 2.$$

We will prove a stronger version, using a different method.

**Theorem 1.** A non-constant polynomial  $p(z) \in \mathbb{C}[z]$  is central at  $\xi \in \mathbb{C}^\times$  if and only if  $\xi$  is a root of unity and  $p(z) \in \mathbb{C}[p_N(z)]$ , i.e.  $p$  is a  $\mathbb{C}$ -polynomial in  $T_N(z)$ , where  $N$  is the order of  $\xi^2$ .

In the proof we also find a version of “skew-centrality” when  $\xi^{2N} = -1$  (see Section 2), which will be useful in this paper and elsewhere.

A remarkable result of Bonahon and Wong is the following.

**Theorem 2** (Bonahon-Wong [BW2]). Let  $M$  be an oriented 3-manifold, possibly with boundary. Suppose  $\xi^4$  is a root of unity of order  $N$ . Let  $\varepsilon = \xi^{N^2}$ . There is a unique  $\mathbb{C}$ -linear map  $\mathbf{Ch} : \mathcal{S}_\varepsilon(M) \rightarrow \mathcal{S}_\xi(M)$  such that for any framed link  $L \subset M$ ,  $\mathbf{Ch}(L) = T_n(L)$ .

If  $M = \Sigma \times [-1, 1]$ , then the map  $\mathbf{Ch}$  is an algebra homomorphism. Actually Bonahon and Wong only consider the case of  $\mathcal{S}(\Sigma)$ , but their proof works also in the case of skein modules of 3-manifolds. In their proof, Bonahon and Wong used the theory of quantum Teichmüller space of Chekhov and Fock [CF] and Kashaev [Kas], and the quantum trace homomorphism developed in earlier work [BW1]. Bonahon and Wong asked for a proof using elementary skein theory. We will present here a proof of Theorem 2, using only elementary skein theory. The main idea is to use central properties (in more general setting) and several operators and filtrations on the skein modules defined by arcs.

In general, the calculation of  $\mathcal{S}(M)$  is difficult. For some results on knot and link complements in  $S^3$ , see [Le, LT, Mar]. Note that if  $\xi^{4N} = 1$ , then  $\varepsilon = \xi^{N^2}$  is a 4-th root of 1. In this case the  $\mathcal{S}_\varepsilon(M)$  is well-known and is related to character variety of  $M$ . This makes Theorem 2 interesting. At  $t = -1$ ,  $\mathcal{S}_{-1}(M)$  has an algebra structure and, modulo its nilradical, is equal to the ring of regular functions on the  $SL_2(\mathbb{C})$ -character variety of  $M$ , see [Bul, PS, BFK]. For the case when  $\varepsilon$  is a primitive 4-th root of 1, see [Si].

**0.3. Plan of the paper.** Section 1 is preliminaries on Chebyshev polynomials and relative skein modules. Section 2 contains the proof of Theorem 1. Section 3 introduces the filtrations and operators on skein modules, and Sections 4 and 5 contain some calculations which are used in Section 6, where the main technical lemma about the skein module of the twice punctured torus is proved. Theorem 2 is proved in Section 7.

**0.4. Acknowledgements.** The author would like to thank F. Bonahon, whose talk at the conference “Geometric Topology at Columbia University (August 12-16, 2013)” has prompted the author to work on this project. The author also thanks C. Frohman, A. Sikora, and H. Wong for helpful discussions. The work is supported in part by NSF.

## 1. GROUND RING, CHEBYSHEV POLYNOMIALS, AND RELATIVE SKEIN MODULES

**1.1. Ground ring.** Let  $R = \mathbb{C}[t^{\pm 1}]$ , which is a principal ideal domain. For an  $R$ -module and a non-zero complex number  $\xi \in \mathbb{C}^\times$  let  $V_\xi$  be the  $R$ -module  $V/(t - \xi)$ . Then  $R_\xi \cong \mathbb{C}$  as  $\mathbb{C}$ -modules, and  $V_\xi$  has a natural structure of an  $R_\xi$ -module.

We will often use the constants

$$(3) \quad \lambda_k := -(t^{2k+2} + t^{-2k-2}) \in R.$$

For example,  $\lambda_0$  is the value of the unknot  $U$  as a skein element.

**1.2. Chebyshev polynomials.** Recall that the Chebyshev polynomials of type 1  $T_n(z)$  and type 2  $S_n(z)$  are defined by

$$\begin{aligned} T_0 &= 2, T_1(z) = z, T_n(z) = zT_{n-1}(z) - T_{n-2}(z) \\ S_0 &= 1, S_1(z) = z, S_n(z) = zS_{n-1}(z) - S_{n-2}(z). \end{aligned}$$

Here are some well-known facts. We drop the easy proofs.

**Lemma 1.1.** (a) *One has*

$$(4) \quad T_n(u + u^{-1}) = u^n + u^{-n}$$

$$(5) \quad T_n = S_n - S_{n-2}$$

(b) *For a fixed positive integer  $N$ , the  $\mathbb{C}$ -span of  $\{T_{Nj}, j \geq 0\}$ , is  $\mathbb{C}[T_N(z)]$ , the ring of all  $\mathbb{C}$ -polynomials in  $T_N(z)$ .*

Since  $T_n(z)$  has leading term  $z^n$ ,  $\{T_n(z), n \geq 0\}$  is a  $\mathbb{C}$ -basis of  $\mathbb{C}[z]$ .

**1.3. Skein module of a surface.** Suppose  $\Sigma$  is a compact connected orientable 2-dimensional manifold with boundary. A knot in  $\Sigma$  is *trivial* if it bounds a disk in  $\Sigma$ . Recall that  $\mathcal{S}(\Sigma)$  is the skein module  $\mathcal{S}(\Sigma \times [-1, 1])$ . If  $\partial\Sigma \neq \emptyset$ , then  $\mathcal{S}(\Sigma)$  is a free  $R$ -module with basis the set of all links in  $\Sigma$  without trivial components, including the empty link, see [PS]. Here a link in  $\Sigma$  is considered as a framed link in  $\Sigma \times [-1, 1]$  by identifying  $\Sigma$  with  $\Sigma \times 0$ , and the framing at every point  $P \in \Sigma \times 0$  is vertical, i.e. given by the unit positive tangent vector of  $P \times [-1, 1] \subset \Sigma \times [-1, 1]$ .

The  $R$ -module  $\mathcal{S}(\Sigma)$  has a natural  $R$ -algebra structure, where  $L_1 L_2$  is obtained by placing  $L_1$  on top of  $L_2$ .

It might happen that  $\Sigma_1 \times [-1, 1] \cong \Sigma_2 \otimes [-1, 1]$  with  $\Sigma_1 \not\cong \Sigma_2$ . In that case,  $\mathcal{S}(\Sigma_1)$  and  $\mathcal{S}(\Sigma_2)$  are the same as  $R$ -modules, but the algebra structures may be different.

**1.4. Example: The annulus.** Let  $\mathbb{A} \subset \mathbb{R}^2$  be the annulus  $\mathbb{A} = \{\vec{x} \in \mathbb{R}^2, 1 \leq |\vec{x}| \leq 2\}$ . Let  $z \in \mathcal{S}(\mathbb{A})$  be the core of the annulus,  $z = \{\vec{x}, |\vec{x}| = 3/2\}$ . Then  $\mathcal{S}(\mathbb{A}) = R[z]$ .

**1.5. Relative skein modules.** A *marked surface*  $(\Sigma, \mathcal{P})$  is a surface  $\Sigma$  together with a finite set  $\mathcal{P}$  of points on its boundary  $\partial\Sigma$ . For such a marked surface, a *relative framed link* is a 1-dimensional compact framed submanifold  $X$  in  $\Sigma \times [-1, 1]$  such that  $\partial X = \mathcal{P} = X \cap \partial(\Sigma \times [-1, 1])$ ,  $X$  is perpendicular to  $\partial(\Sigma \times [-1, 1])$ , and the framing at each point  $P \in \mathcal{P} = \partial X$  is vertical. The relative skein module  $\mathcal{S}(\Sigma, \mathcal{P})$  is defined as the  $R$ -module spanned by isotopy class of relative framed links modulo the same skein relations (1) and (2). We will use the following fact.

**Proposition 1.2.** (See [PS, Theorem 5.2]) *The  $R$ -module  $\mathcal{S}(\Sigma, \mathcal{P})$  is free with basis the set of isotopy classes of relative links embedded in  $\Sigma$  without trivial components.*

## 2. ANNULUS WITH TWO MARKED POINTS AND CENTRAL ELEMENTS

**2.1. Marked annulus.** Recall that  $\mathbb{A} \subset \mathbb{R}^2$  is the annulus  $\mathbb{A} = \{\vec{x} \in \mathbb{R}^2 \mid 1 \leq |\vec{x}| \leq 2\}$ . Let  $\mathbb{A}_{io}$  be the marked surface  $(\mathbb{A}, \{P_1, P_2\})$ , with two marked points  $P_1 = (0, 1)$ ,  $P_2 = (0, 2)$ , which are on different boundary components. See Figure 2, which also depicts the arcs  $e$ ,  $u$ ,  $u^{-1}$ .

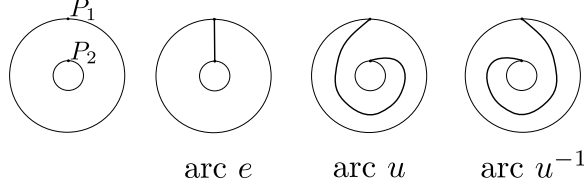


FIGURE 2. The marked annulus  $\mathbb{A}_{io}$ , and the arcs  $e$ ,  $u$ ,  $u^{-1}$

For  $L_1, L_2 \in \mathcal{S}(\mathbb{A}_{io})$  define the product  $L_1 L_2$  by placing  $L_1$  inside  $L_2$ . Formally this means we first shrink  $\mathbb{A}_{io} \supset L_1$  by factor  $1/2$ , we get  $(\frac{1}{2}\mathbb{A}_{io}) \supset (\frac{1}{2}L_1)$ , where  $\frac{1}{2}\mathbb{A}_{io}$  is an annulus on the plane whose outer circle is the inner circle of  $\mathbb{A}_{io}$ . Then  $L_1 L_2$  is  $(\frac{1}{2}L_1) \cup L_2 \subset (\frac{1}{2}\mathbb{A}_{io}) \cup \mathbb{A}_{io}$ . The identity of  $\mathcal{S}(A_2)$  is presented by  $e$ , and  $u^{-1}u = e = uu^{-1}$ .

**Proposition 2.1.** *The Kauffman bracket skein modules of  $\mathbb{A}_{io}$  is  $\mathcal{S}(A_2) = R[\mathbf{u}^{\pm 1}]$ , the ring of Laurent  $R$ -polynomial in one variable  $\mathbf{u}$ . In particular,  $\mathcal{S}(\mathbb{A}_{io})$  is commutative.*

*Proof.* Using Proposition 1.2 one can easily show that the set  $\{\mathbf{u}^k, k \in \mathbb{Z}\}$  is a free  $R$ -basis of  $\mathcal{S}(\mathbb{A}_{io})$ .  $\square$

**2.2. Passing through  $T_k$ .** Recall that  $\mathcal{S}(\mathbb{A}) = R[z]$ . One defines a left action and a right action of  $\mathcal{S}(\mathbb{A})$  on  $\mathcal{S}(\mathbb{A}_{io})$  as follows. For  $L \in \mathcal{S}(\mathbb{A}), K \in \mathcal{S}(\mathbb{A}_{io})$  let  $L \bullet K$  be the element in  $\mathcal{S}(\mathbb{A}_{io})$  obtained by placing  $L$  above  $K$ , and  $K \bullet L \in \mathcal{S}(\mathbb{A}_{io})$  be the element in  $\mathcal{S}(\mathbb{A}_{io})$  obtained by placing  $K$  above  $L$ . For example,

$$e \bullet z = \text{diagram}, \quad z \bullet e = \text{diagram}$$

**Proposition 2.2.** *One has*

$$(6) \quad T_k(z) \bullet e = t^k \mathbf{u}^k + t^{-k} \mathbf{u}^{-k}$$

$$(7) \quad e \bullet T_k(z) = t^k \mathbf{u}^{-k} + t^{-k} \mathbf{u}^k$$

$$(8) \quad T_k(z) \bullet e - e \bullet T_k(z) = (t^k - t^{-k})(\mathbf{u}^k - \mathbf{u}^{-k}).$$

*Proof.* It is important to note that the map  $f : \mathcal{S}(\mathbb{A}) \rightarrow \mathcal{S}(\mathbb{A}_{io})$  given by  $f(L) = L \bullet e$  is an algebra homomorphism.

Resolve the only crossing point, we have

$$z \bullet e = \text{diagram} = t \text{diagram} + t^{-1} \text{diagram} = t\mathbf{u} + t^{-1}\mathbf{u}^{-1}.$$

Hence,

$$\begin{aligned} T_k(z) \bullet e &= T_k(t\mathbf{u} + t^{-1}\mathbf{u}^{-1}) \quad \text{because } f \text{ is an algebra homomorphism} \\ &= t^k \mathbf{u}^k + t^{-k} \mathbf{u}^{-k} \quad \text{by (4)}. \end{aligned}$$

This proves (6). The proof of (7) is similar, while (8) follows from (6) and (7).  $\square$

**Corollary 2.3.** *Suppose  $\xi^{2N} = 1$ . Then  $T_N(z)$  is central at  $\xi$ .*

*Proof.* We have  $\xi^N = \xi^{-N}$  since  $\xi^{2N} = 1$ . Then (8) shows that  $T_N(z) \bullet e = e \bullet T_N(z)$ , which easily implies the centrality of  $T_N(z)$ .  $\square$

**Remark 2.4.** The corollary was first proved by Bonahon and Wong [BW2] using another method.

**2.3. Transparent elements.** We say that  $p(z) \in \mathbb{C}[z]$  is *transparent* at  $\xi$  if for any 3 disjoint framed knots  $K, K_1, K_2$  in any oriented 3-manifold  $M$ ,  $p(K) \cup K_1 = p(K) \cup K_2$  in  $\mathcal{S}_\xi(M)$ , provided that  $K_1$  and  $K_2$  are isotopic in  $M$ .

**Proposition 2.5.** *The following are equivalent*

- (i)  $p(z) \bullet e = e \bullet p(z)$  in  $\mathcal{S}_\xi(\mathbb{A}_{io})$ .
- (ii)  $p(z)$  is transparent at  $\xi$ .
- (iii)  $p(z)$  is central at  $\xi$ .

*Proof.* It is clear that (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii). Let us prove (iii)  $\Rightarrow$  (i).

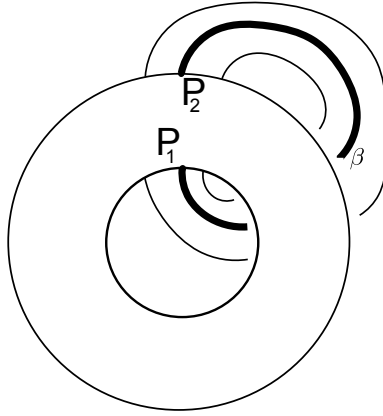


FIGURE 3. The core  $\beta$  connects  $P_1$  and  $P_2$  in  $\mathbb{T}_{\text{punc}}$

By gluing a 1-handle to  $\mathbb{A}$  we get a punctured torus  $\mathbb{T}_{\text{punc}}$  as in Figure 3. Here the base of the 1-handle is glued to a small neighborhood of  $\{P_1 \cup P_2\}$  in  $\partial\mathbb{A}$ , and the core of the 1-handle is an arc  $\beta$  connecting  $P_1$  and  $P_2$ . Let  $\iota : \mathcal{S}(\mathbb{A}_{io}) \rightarrow \mathcal{S}(\mathbb{T}_{\text{punc}})$  be  $R$ -map which is the closure by  $\beta$ , i.e.  $\iota(K) = K \cup \beta$ . Then  $\iota(\mathbf{u}^k)$  is a knot in  $\mathbb{T}_{\text{punc}}$  for every  $k \in \mathbb{Z}$ , and  $\iota(\mathbf{u}^k)$  is not isotopic to  $\iota(\mathbf{u}^l)$  if  $k \neq l$ . Since  $\{\mathbf{u}^k, k \in \mathbb{Z}\}$  is an  $R$ -basis of  $\mathcal{S}(\mathbb{A}_{io})$  and the isotopy classes of links in  $\mathbb{T}_{\text{punc}}$  form an  $R$ -basis of  $\mathcal{S}(T\mathfrak{p})$ ,  $\iota$  is injective.

Assume (iii). Then  $p(z)\iota(e) = \iota(e)p(z)$ , or  $\iota(p(z) \bullet e) = \iota(e \bullet p(z))$ . Since  $\iota$  is injective, we have  $p(z) \bullet e = e \bullet p(z)$ .  $\square$

**2.4. Proof of Theorem 1.** The “if” part has been proved, see Corollary 2.3. Let us prove the “only if part”. Assume that  $p(z)$  is central at  $\xi$  and having degree  $k \geq 1$ . Since  $\{T_j(z), j \geq 0\}$  is a basis of  $\mathbb{C}[z]$ , we can write

$$(9) \quad p(z) = \sum_{j=0}^k c_j T_j(z), \quad c_j \in \mathbb{C}, c_k \neq 0.$$

By Proposition 2.5,  $p(z) \bullet e - e \bullet p(z) = 0$ . Using expression (9) for  $p(z)$  and (8), we get

$$0 = p(z) \bullet e - e \bullet p(z) = \sum_{j=0}^k c_j (\xi^j - \xi^{-j}) (\mathbf{u}^j - \mathbf{u}^{-j}).$$

Because  $\{\mathbf{u}^j, j \in \mathbb{Z}\}$  is a basis of  $\mathcal{S}_\xi(\mathbb{A}_{io})$ , the coefficient of each  $\mathbf{u}^j$  on the right hand side is 0. This means,

$$(10) \quad c_j = 0 \quad \text{or} \quad \xi^{2j} = 1, \quad \forall j.$$

Since  $c_k \neq 0$ , we have  $\xi^{2k} = 1$ . Since  $k \geq 1$ , this shows  $\xi^2$  is a root of unity of some order  $N$ . Then (10) shows that  $c_j = 0$  unless  $N|j$ . Thus,  $p(z)$  is a  $\mathbb{C}$ -linear combination of  $T_j$  with  $N|j$ . This completes the proof of Theorem 1.

**2.5. Skew transparency.** One more consequence of Proposition 2.2 is the following.

**Corollary 2.6.** *Suppose  $\xi^{2N} = -1$ . Then in  $\mathcal{S}_\xi(\mathbb{A}_{io})$ ,*

$$T_N(z) \bullet e = -e \bullet T_N(z).$$

This means every time we move  $T_N(K)$  passed through a component of a link  $L$ , the value of the skein gets multiplied by  $-1$ . Following is a precise statement.

Suppose  $K_1$  and  $K_2$  are knots in a 3-manifold  $M$ . Recall that an isotopy between  $K_1$  and  $K_2$  is a smooth map  $H : S^1 \times [1, 2] \rightarrow M$  such that for each  $t \in [1, 2]$ , the map  $H_t : S^1 \rightarrow M$  is an embedding, and the image of  $H_t$  is  $K_t$  for  $t = 1, 2$ . Here  $H_t(x) = H(x, t)$ . For a knot  $K \subset M$  let  $I_2(H, K)$  be the mod 2 intersection number of  $H$  and  $K$ . Thus, if  $H$  is transversal to  $K$  then  $I_2(H, K)$  is the number of points in the finite set  $H^{-1}(K)$  modulo 2.

**Definition 2.** *Suppose  $\mu = \pm 1$ . A polynomial  $p(z) \in \mathbb{C}[z]$  is  $\mu$ -transparent at  $\xi \in \mathbb{C}^\times$  if for any 3 disjoint framed knots  $K, K_1, K_2$  in any oriented 3-manifold  $M$ , with  $K_1$  and  $K_2$  connected by an isotopy  $H$ , one has the following equality in  $\mathcal{S}_\xi(M)$ :*

$$p(K) \cup K_1 = \mu^{I_2(H, K)} [p(K) \cup K_2].$$

From Corollary 2.6 we have

**Corollary 2.7.** *Assume  $\xi^{4N} = 1$ . Then  $\mu := \xi^{2N} = \pm 1$ , and  $T_N(z)$  is  $\mu$ -transparent.*

A special case is the following. Suppose  $D \subset M$  is a disk in  $M$  with  $\partial D = K$ , and a framed link  $L \subset M$  is disjoint from  $K$ . Then, if  $\xi^{2N} = \mu = \pm 1$ , one has

$$(11) \quad K \sqcup T_N(L) = \mu^{I_2(D, L)} \lambda_0 T_N(L) \quad \text{in } \mathcal{S}_\xi(M).$$

Here  $\lambda_0 = -(\xi^2 + \xi^{-2})$  is the value of trivial knot in  $\mathcal{S}_\xi(M)$ .

### 3. FILTRATIONS OF SKEIN MODULES

Suppose  $\Phi$  is a link in  $\partial M$ . We define an  $R$ -map  $\Phi : \mathcal{S}(M) \rightarrow \mathcal{S}(M)$  by  $\Phi(L) = \Phi \cup L$ .

**3.1. Filtration by an arc.** Suppose  $\alpha$  is an arc properly embedded in a marked surface  $(\Sigma, \mathcal{P})$  with  $\partial\Sigma \neq \emptyset$ . Assume the two boundary points of  $\alpha$ , which are on the boundary of  $\Sigma$ , are disjoint from the marked points. Then  $\mathcal{D}_\alpha := \alpha \times [-1, 1]$  is a disk properly embedded in  $\Sigma \times [-1, 1]$ , with boundary  $\Phi_\alpha = \partial(\alpha \times [-1, 1]) = (\alpha \times \{-1, 1\}) \cup (\partial\alpha \times [-1, 1])$ .

Let  $\mathcal{F}_k^\alpha = \mathcal{F}_k^\alpha(\mathcal{S}(M))$  be the  $R$ -submodule of  $\mathcal{S}(M)$  spanned by all relative links which intersect with  $\mathcal{D}_\alpha$  at less than or equal to  $k$  points. For  $L \in \mathcal{S}(\Sigma)$ , we define  $\text{fil}_\alpha(L) = k$  if  $L \in \mathcal{F}_k^\alpha \setminus \mathcal{F}_{k-1}^\alpha$ . The filtration is compatible with the algebra structure, i.e.

$$\text{fil}_\alpha(L_1 L_2) \leq \text{fil}_\alpha(L_1) + \text{fil}_\alpha(L_2).$$

**Remark 3.1.** A similar filtration was used in [Mar] to calculate the skein module of torus knot complements.

A convenient way to count the number of intersection points of a link  $L$  with  $\mathcal{D}_\alpha$  is to count the intersection points of the diagram of  $L$  with  $\alpha$ : Let  $D$  be the vertical projection of  $L$  onto  $\Sigma$ . In general position  $D$  has only singular points of type double points, and we assume further that  $D$  is transversal to  $\alpha$ . In that case, the number of intersection points of  $L$  with  $\mathcal{D}_\alpha$  is equal to the number of intersections of  $D$  with  $\alpha$ , where each intersection point of  $\alpha$  and  $D$  at a double point of  $D$  is counted twice.

Recall that  $\Phi_\alpha(L) = L \cup \Phi_\alpha$ , where  $\Phi_\alpha$  is the boundary of the disk  $\mathcal{D}_\alpha = \alpha \times [-1, 1]$ . It is clear that  $\mathcal{F}_k^\alpha$  is  $\Phi_\alpha$  invariant, i.e.  $\Phi_\alpha(\mathcal{F}_k^\alpha) \subset \mathcal{F}_k^\alpha$ . It turns out that the action of  $\Phi_\alpha$  on the quotient  $\mathcal{F}_k^\alpha / \mathcal{F}_{k-1}^\alpha$  is very simple. Recall that  $\lambda_k = -(t^{2k+2} + t^{-2k-2})$ .

**Proposition 3.2.** *For  $k \geq 0$ , the action of  $\Phi_\alpha$  on  $\mathcal{F}_k^\alpha / \mathcal{F}_{k-1}^\alpha$  is  $\lambda_k$  times the identity.*

This is a consequence of Proposition 3.3 proved in the next subsection.

**3.2. The Temperley-Lieb algebra and the operator  $\Phi$ .** The well-known Temperley-Lieb algebra  $TL_k$  is the skein module of the disk with  $2k$  marked points on the boundary. We will present the disk as the square  $\text{Sq} = [0, 1] \times [0, 1]$  on the standard plane, with  $k$  marked points on the top side and  $k$ -marked points on the bottom side. The product  $L_1 L_2$  in  $TL_k$  is defined as the result of placing  $T_1$  on top of  $T_2$ . The unit  $\tilde{e}_k$  of  $TL_k$  is presented by  $k$  vertical straight arcs, see Figure 4.

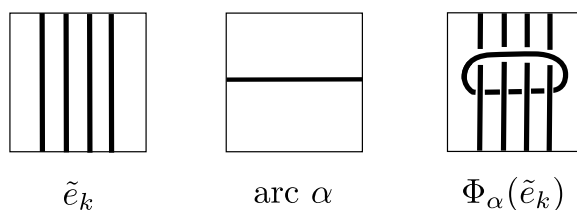


FIGURE 4. The unit  $\tilde{e}_k$ , the arc  $\alpha$ , and  $\Phi_\alpha(\tilde{e}_k)$ . Here  $k = 4$ .

Let  $\alpha \subset \text{Sq}$  be the horizontal arc  $[0, 1] \times (1/2)$ . The element  $\Phi_\alpha(\tilde{e}_k)$  is depicted in Figure 4. In general,  $\Phi_\alpha(L)$  is  $L$  encircled by one simple closed curve.

**Proposition 3.3.** *With the above notation, one has*

$$(12) \quad \Phi_\alpha(\tilde{e}_k) = \lambda_k \tilde{e}_k \pmod{\mathcal{F}_{k-1}^\alpha}.$$

*Proof.* A direct proof can be carried out as follows. Using the skein relation (1) one resolves all the crossing of the diagram of  $\Phi_\alpha(\tilde{e}_k)$ , and finds that only a few terms are not in  $\mathcal{F}_{k-1}^\alpha$ , and the sum of these terms is equal to  $\lambda_k \tilde{e}_k$ . This is a good exercise for the dedicated reader.

Here is another proof using more advanced knowledge of the Temperley-Lieb algebra. First we extend the ground ring to field of fraction  $\mathbb{C}(t)$ . Then the Temperley-Lieb algebra contains a special element called the Jones-Wenzl idempotent  $f_k$  (see e.g. [Lic, Chapter 13]). We have  $f_k = \tilde{e}_k \pmod{\mathcal{F}_{k-1}^\alpha}$ , and  $f_k$  is an eigenvector of  $\Phi_\alpha$  with eigenvalue  $\lambda_k$ . Hence, we have (12).  $\square$

#### 4. ANOTHER ANNULUS WITH TWO MARKED POINTS

**4.1. Annulus with two marked points on the same boundary.** Let  $\mathbb{A}_{oo}$  be the annulus  $A$  with two marked points  $Q_1, Q_2$  on the outer boundary as in Figure 5. Let  $u_0, u_1$  be arcs connecting  $Q_1$  and  $Q_2$  in  $\mathbb{A}_{oo}$  as in Figure 5.

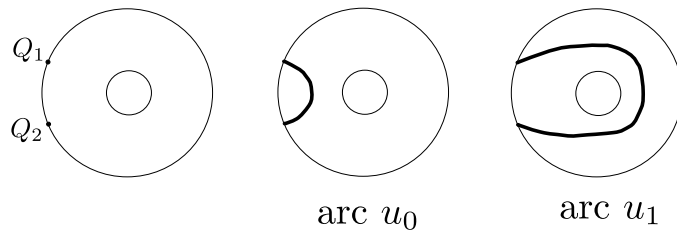


FIGURE 5. The marked annulus  $\mathbb{A}_{oo}$  and arcs  $u_0, u_1$

Define a left  $\mathcal{S}(\mathbb{A})$ -module and a right  $\mathcal{S}(\mathbb{A})$ -module on  $\mathcal{S}(\mathbb{A}_{oo})$  as follows. For  $K \in \mathcal{S}(\mathbb{A}_{oo})$  and  $L \in \mathcal{S}(\mathbb{A})$  let  $KL$  be the skein in  $\mathcal{S}(\mathbb{A}_{oo})$  obtained by placing  $K$  on top of  $L$ , and  $LK \in \mathcal{S}(\mathbb{A}_{oo})$  obtained by placing  $L$  on top of  $K$ . It is easy to see that  $KL = LK$ . Recall that  $\mathcal{S}(\mathbb{A}) = R[z]$ .

**Proposition 4.1.** *The module  $\mathcal{S}(\mathbb{A}_{oo})$  is a free- $\mathcal{S}(\mathbb{A})$ -module with basis  $\{u_0, u_1\}$ :*

$$\mathcal{S}(\mathbb{A}_{oo}) = R[z]u_0 \oplus R[z]u_1.$$

*Proof.* Any relative link in  $\mathbb{A}_{oo}$  is of the form  $u_i z^m$  with  $i = 0, 1$  and  $m \in \mathbb{Z}$ . The proposition now follows from Proposition 1.2.  $\square$

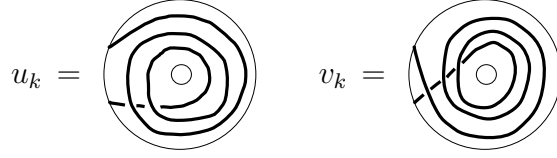
**4.2. Framing change and the unknot.** Recall that  $S_k$  is the  $k$ -th Chebyshev polynomial of type 2. The values of the unknot colored by  $S_k$  and the framing change are well-know (see e.g. [BHMV]): In  $\mathcal{S}(M)$ , where  $M$  is an oriented 3-manifold, one has

$$(13) \quad L \sqcup S_k(U) = (-1)^k \frac{t^{2k+2} - t^{-2k-2}}{t^2 - t^{-2}} L$$

$$(14) \quad S_k \left( \begin{array}{c} | \\ \text{⌢} \\ | \end{array} \right) = (-1)^k t^{k^2+2k} S_k \left( \begin{array}{c} | \\ | \\ | \end{array} \right).$$

Here in (13),  $U$  is the trivial knot lying in a ball disjoint from  $L$ .




 FIGURE 6.  $u_k$  and  $v_k$ , with  $k = 3$ 

4.3. **Some elements of  $\mathcal{S}(\mathbb{A}_{oo})$ .** Let  $u_k, v_k, k \geq 0$  are arcs in  $\mathbb{A}_{oo}$  depicted in Figure 6.

**Proposition 4.2.** *One has*

$$(15) \quad u_k = t^{k-1} S_{k-1}(z) u_1 + t^{k-3} S_{k-2}(z) u_0 \quad \forall k \geq 1$$

$$(16) \quad v_k = t^{2-k} S_{k-1}(z) u_1 + t^{-k} S_k(z) u_0 \quad \forall k \geq 0.$$

*Proof.* Suppose  $k \geq 3$ . Apply the skein relation to the innermost crossing of  $u_k$ , we get

$$u_k = \text{diagram} = t \text{diagram} + t^{-1} \text{diagram}$$

which, after an isotopy and removing a framing crossing, is

$$u_k = t u_{k-1} z - t^2 u_{k-2},$$

from which one can easily prove (15) by induction.

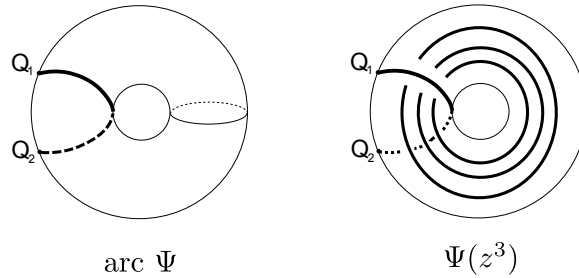
Similarly, using the skein relation to resolve the innermost crossing point of  $v_k$ , we get

$$v_k = t^{-1} v_{k-1} z - t^{-2} v_{k-2}, \quad \text{for } k \geq 2,$$

from which one can prove (16) by induction.  $\square$

**Remark 4.3.** Identity (15) does not hold for  $k = 0$ . This is due to a framing change.

4.4. **Operator  $\Psi$ .** Let  $\Psi$  be the arc in  $\partial\mathbb{A} \times [-1, 1]$  beginning at  $Q_1$  and ending at  $Q_2$ , as depicted in Figure 7. Here we draw  $\mathbb{A} \times [-1, 1]$  as a handlebody. For any element


 FIGURE 7. Arc  $\Psi$  connecting  $Q_1$  and  $Q_2$ , and  $\Psi(z^3)$ 

$\alpha \in \mathcal{S}(\mathbb{A})$  let  $\Psi(\alpha) \in \mathcal{S}(\mathbb{A}_{oo})$  be the skein  $\Psi \cup \alpha$ . For example,  $\Psi(z^3)$  is given in Figure 7.

**Proposition 4.4.** *For  $k \geq 1$ , one has*

$$(17) \quad \Psi(T_k(z)) = u_1 [t^2(t^{-2k} - t^{2k}) S_{k-1}(z)] + u_0 [t^{-2k} S_k(z) - t^{2k} S_{k-2}(z)].$$

*Proof.* Applying Proposition 2.2 to the part in the dashed box, we get

$$\Psi(T_k(z)) = \text{Diagram 1} = t^k \text{Diagram 2} + t^{-k} \text{Diagram 3}.$$

The positive framing crossing in the first term gives a factor  $-t^3$ , and

$$\Psi(T_k(z)) = -t^{k+3} u_k + t^{-k} v_k.$$

Plugging in the values of  $u_k, v_k$  given by Proposition 4.2, we get the results.  $\square$

**Remark 4.5.** One can use Proposition 4.4 to establish product-to-sum formulas similar to the ones in [FG].

## 5. TWICE PUNCTURED DISK

**5.1. Skein module of twice punctured disk.** Let  $\mathcal{D} \subset \mathbb{R}^2$  be the disk of radius 4 centered at the origin,  $\mathcal{D}_1 \subset \mathbb{R}^2$  the disk of radius 1 centered  $(-2, 0)$ , and  $\mathcal{D}_2$  the disk of radius 1 centered  $(2, 0)$ . We define  $\mathbb{D}$  to be  $\mathcal{D}$  with the interiors of  $\mathcal{D}_1$  and  $\mathcal{D}_2$  removed.

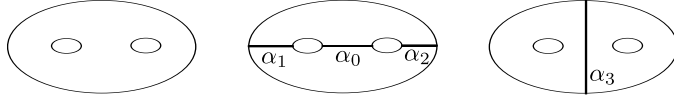


FIGURE 8. The twice punctured disk  $\mathbb{D}$  and the arcs  $\alpha_1, \alpha_0, \alpha_2, \alpha_3$

The horizontal axis intersects  $\mathbb{D}$  at 3 arcs denoted from left to right by  $\alpha_1, \alpha_0, \alpha_2$ , see Figure 8. The vertical axis of  $\mathbb{R}^2$  intersects  $\mathbb{D}$  at an arc denoted by  $\alpha_3$ . The corresponding curve  $\Phi_{\alpha_i}$  on  $\partial\mathbb{D} \times [-1, 1]$  will be denoted simply by  $\Phi_i$ , for  $i = 0, 1, 2, 3$ . If  $\mathbb{D} \times [-1, 1]$  is presented as the handlebody  $\mathcal{H}$ , which is a thickening of  $\mathbb{D}$  in  $\mathbb{R}^3$ , then the curves  $\Phi_1, \Phi_0, \Phi_2, \Phi_3$  look like in Figure 9.

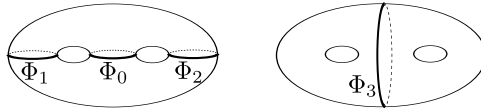


FIGURE 9. The curves  $\Phi_1, \Phi_0, \Phi_2, \Phi_3$  on the boundary of the handlebody

Let  $x_1, x_2$ , and  $y$  be the closed curves in  $\mathbb{D}$ :

$$x_1 = \text{Diagram 1}, \quad x_2 = \text{Diagram 2}, \quad y = \text{Diagram 3}$$

It is known that  $\mathcal{S}(\mathbb{D}) = R[x_1, x_2, y]$ , the  $R$ -polynomial in the variables  $x_1, x_2, y$ , see [BP]. In particular,  $\mathcal{S}(\mathbb{D})$  is commutative.

Let  $\sigma$  be the rotation about the origin of  $\mathbb{R}^2$  by  $180^\circ$ . Then  $\sigma(\mathbb{D}) = \mathbb{D}$ . Hence  $\sigma$  induces an automorphism of  $\mathcal{S}(\mathbb{D}) = R[x_1, x_2, y]$ , which is an algebra automorphism. One has  $\sigma(y) = y, \sigma(x_1) = x_2, \sigma(x_2) = x_1$ .

5.2. **Degrees on  $\mathcal{S}(\mathbb{D}) = R[x_1, x_2, y]$ .** Define the left degree, right degree, and double degree on  $R[x_1, y, x_2]$  as follows. For a monomial  $\mathbf{m} = x_1^{a_1} y^b x_2^{a_2}$  define its left degree  $\deg_l(\mathbf{m}) = a_1 + b$ , right degree  $\deg_r(\mathbf{m}) = a_2 + b$ , double degree  $\deg_{lr}(\mathbf{m}) = \deg_l(\mathbf{m}) + \deg_r(\mathbf{m}) = a_1 + a_2 + 2b$ . One readily finds that

$$\deg_l(\mathbf{m}) = \text{fil}_{\alpha_1}(\mathbf{m}), \quad \deg_r(\mathbf{m}) = \text{fil}_{\alpha_2}(\mathbf{m}),$$

where  $\text{fil}_\alpha$  is defined in Section 3.1. Using the definition of  $\text{fil}_\alpha$  involving the numbers of intersection points we get the following.

**Lemma 5.1.** *Suppose  $L$  is an embedded link in  $\mathbb{D}$  and  $L$  intersects transversally the arc  $\alpha_i$  at  $k_i$  points for  $i = 1, 2, 3$ . Then, as an element of  $\mathcal{S}(D_2)$ ,  $L = x_1^{a_1} x_2^{a_2} y^b$ , where  $2b \leq k_3$  and*

$$\begin{aligned} \deg_l(L) &\leq k_1, & \deg_l(L) &\equiv k_1 \pmod{2} \\ \deg_r(L) &\leq k_2, & \deg_r(L) &\equiv k_2 \pmod{2}. \end{aligned}$$

Consequently,  $\deg_{lr}(L) \leq k_1 + k_2$  and  $\deg_{lr}(L) \equiv k_1 + k_2 \pmod{2}$ .

*Proof.* If  $L = L_1 \sqcup L_2$  is the union of 2 disjoint sub-links, and the statement holds for each of  $L_i$ , then it holds for  $L$ . Hence we assume  $L$  has one component, i.e.  $L$  is an embedded loop in  $D_2 \subset \mathbb{R}^2$ . Then  $L$  is isotopic to either a trivial loop, or  $x_1$ , or  $x_2$ , or  $y$ . In each case, the statement can be verified easily. For example, suppose  $L = x_1$ . For the mod 2 intersection numbers,  $I_2(L, \alpha_1) = I_2(x_1, \alpha_1) = 1$ . Hence  $k_1$ , the geometric intersection number between  $L$  and  $\alpha_1$ , must be odd and bigger than or equal to 1. Hence, we have  $\deg_l(L) \leq k_1$  and  $\deg_l(L) \equiv k_1 \pmod{2}$ .  $\square$

**Corollary 5.2.** *Suppose  $L$  is a link diagram on  $\mathbb{D}$  which intersects transversally the arc  $\alpha_i$  at  $k_i$  points for  $i = 1, 2, 3$ . Then, as an element in  $\mathcal{S}(\mathbb{D})$ ,*

$$\deg_l(L) \leq k_1, \quad \deg_r(L) \leq k_2, \quad 2 \deg_y(L) \leq k_3,$$

and  $L$  is a linear  $R$ -combination of monomials whose double degrees are equal to  $k_1 + k_2$  modulo 2.

5.3. **The  $R$ -module  $V_n$  and the skein  $\gamma$ .** Let  $\gamma$  and  $\bar{\gamma}$  be the following link diagrams on  $\mathbb{D}$ ,

$$(18) \quad \gamma = \langle \text{link diagram} \rangle, \quad \bar{\gamma} = \langle \text{link diagram} \rangle.$$

Let

$$V_n = \{p \in R[x_1, x_2, y] \mid \deg_l(p) \leq n, \deg_r(p) \leq n, \deg_{lr}(p) \text{ even}\}.$$

In other words,  $V_n \subset R[x_1, x_2, y]$  is the  $R$ -submodule spanned by  $x_1^{a_1} x_2^{a_2} y^b$ , with  $a_i + b \leq n$  for  $i = 1, 2$ , and  $a_1 + a_2$  even.

**Lemma 5.3.** *One has  $T_n(\gamma), T_n(\bar{\gamma}) \in V_n$ .*

*Proof.* The diagram  $\gamma^k$  has  $k$  intersection points with each of  $\alpha_1$  and  $\alpha_2$ . By Corollary 5.2,  $\deg_l(\gamma^k) \leq k, \deg_r(\gamma^k) \leq k$ , and each monomial of  $\gamma^k$  has double degree  $\equiv k + k \equiv 0 \pmod{2}$ . This means  $\gamma^k \in V_k$  for every  $k \geq 0$ . Because  $T_n(\gamma)$  is  $\mathbb{Z}$ -linear combination of  $\gamma^k$  with  $k \leq n$ , we have  $T_n(\gamma) \in V_n$ . The proof for  $\bar{\gamma}$  is similar.  $\square$

**Remark 5.4.** It is an easy exercise to show that  $T_N(\bar{\gamma}) = T_N(\gamma)|_{t \rightarrow t^{-1}}$ .

## 6. SKEIN MODULE OF TWICE PUNCTURE DISK AT ROOT OF 1

Recall that  $\gamma$  and  $\bar{\gamma}$  are knot diagrams on  $\mathbb{D}$  defined by (18). The following was proved by Bonahon and Wong, using quantum Teichmüller algebras and their representations.

**Proposition 6.1.** *Suppose  $\xi^4$  is a root of 1 of order  $N$ . Then in  $\mathcal{S}_\xi(\mathbb{D})$  one has*

$$(19) \quad T_N(\gamma) = \xi^{-N^2} T_N(y) + \xi^{N^2} T_N(x_1) T_N(x_2)$$

$$(20) \quad T_N(\bar{\gamma}) = \xi^{N^2} T_N(y) + \xi^{-N^2} T_N(x_1) T_N(x_2).$$

As mentioned above, there was an urge to find a proof using elementary skein theory; one such is presented here. Our proof roughly goes as follows. Using the transparent property of  $T_N(\gamma)$ , we show that  $T_N(\gamma)$  is a common eigenvector of several operators. We then prove that the space of common eigenvectors has dimension at most 3, with a simple basis. We then fix coefficients of  $T_N(\gamma)$  in this basis using calculations in highest order. Then the result turns out to be the right hand side of (19).

Throughout this section we fix a complex number  $\xi$  such that  $\xi^4$  is a root of unity of order  $N$ . Define  $\varepsilon = \xi^{N^2}$ . We will write  $V_{N,\xi}$  simply by  $V_N$  and  $\lambda_k$  for  $\lambda_k(\xi)$ . Thus, in the whole section,

$$\lambda_k = -(\xi^{2k+2} + \xi^{-2k-2}).$$

**6.1. Properties of  $\xi$  and  $\lambda_k$ .** Recall that  $\xi^4$  is a root of 1 of order  $N$ .

**Lemma 6.2.** *Suppose  $1 \leq k \leq N - 1$ . Then*

- (a)  $\lambda_{2k} = \lambda_0$  if and only if  $k = N - 1$ .
- (b)  $\lambda_k = \xi^{2N} \lambda_0$  implies that  $k = N - 2$ .
- (c) If  $N$  is even then  $\xi^{2N} = -1$ .
- (d) One has

$$(21) \quad \xi^{2N^2+2N} = (-1)^{N+1}.$$

*Proof.* (a) With  $\lambda_k = -(\xi^{2k+2} + \xi^{-2k-2})$ , we have

$$\lambda_{2k} - \lambda_0 = -\xi^{-2-4k}(\xi^{4k} - 1)(\xi^{4k+4} - 1).$$

Hence,  $\lambda_{2k} - \lambda_0 = 0$  if and only if either  $N|k$  or  $N|(k+1)$ . With  $0 \leq k \leq N - 1$ , this is equivalent to  $k = N - 1$ .

(b) We have

$$\lambda_k - \xi^{2N} \lambda_0 = -\xi^{-2N-2}(\xi^{2N-2k} - 1)(\xi^{2N+2k+4} - 1).$$

Either (i)  $\xi^{2N-2k} = 1$  or (ii)  $\xi^{2N+2k+4} = 1$ . Taking the squares of both identities, we see that either  $N|(N-k)$  or  $N|(k+2)$ . With  $1 \leq k \leq N - 1$ , we conclude that  $k = N - 2$ .

(c) Suppose  $N$  is even. Since  $\xi^4$  has order  $N$ , one has  $(\xi^4)^{N/2} = -1$ . Then  $\xi^{2N} = (\xi^4)^{N/2} = -1$ .

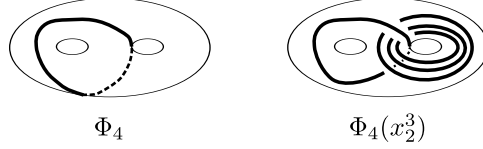
(d) is left for the reader.  $\square$

**6.2. Operators  $\Phi_i$  and the vector space  $W$ .** Recall that  $\Phi_i := \Phi_{\alpha_i}$ ,  $i = 0, 1, 2, 3$ , is defined in section 5.1. Then  $\Phi_i(V_N) \subset V_N$  for  $i = 0, 1, 2, 3$ .

Let  $\Phi_4$  be the curve on  $\partial\mathbb{D} \times [-1, 1]$  depicted in Figure 10. Here we draw  $\mathcal{H} = \mathbb{D} \times [-1, 1]$  as a handlebody. We also depict  $\Phi_4(x_2^3)$ .

We don't have  $\Phi_4(V_N) \subset V_N$ , since  $\Phi_4$  in general increases the double degree. By counting the intersection points with  $\alpha_1$  and  $\alpha_2$ , we have, for every  $E \in \mathcal{S}_\xi(\mathbb{D}) = \mathbb{C}[x_1, x_2, y]$ ,

$$(22) \quad \deg_{lr}(\Phi_4(E)) \leq \deg_{lr}(E) + 1.$$


 FIGURE 10. The curve  $\Phi_4$  and  $\Phi_4(x_2^3)$ 

**Proposition 6.3.** *If  $E$  is one of  $\{T_N(\gamma), T_N(\bar{\gamma}), T_N(y), T_N(x_1)T_N(x_2)\}$ , then one has*

$$(23) \quad \sigma(E) = E$$

$$(24) \quad \Phi_1(E) = \xi^{2N} \lambda_0 E$$

$$(25) \quad \Phi_i(E) = \lambda_0 E \quad \text{for } i = 0, 3,$$

$$(26) \quad \Phi_4(E) = \xi^{2N} x_1 E.$$

*Proof.* The first identity follows from the fact that each of  $\gamma, \bar{\gamma}, y, x_1 \cup x_2$  is invariant under  $\sigma$ . The remaining identities follows from the  $\xi^{2N}$ -transparent property of  $T_N(z)$ , Corollary 2.7.  $\square$

**Remark 6.4.** Note that  $\Phi_4$  is  $\mathbb{C}[x_1]$ -linear and (26) says that  $E$  is a  $\xi^{2N} x_1$ -eigenvector of  $\Phi_4$ .

Let  $W$  be the subspace of  $V_N$  consisting of elements satisfying (23)–(26). This means,  $W \subset V_N$  consists of elements which are at the same time 1-eigenvector of  $\sigma$ ,  $\xi^{2N} \lambda_0$ -eigenvector of  $\Phi_1$ ,  $\lambda_0$ -eigenvector of  $\Phi_0$  and  $\Phi_3$ , and  $\xi^{2N} x_1$ -eigenvector of  $\Phi_4$ .

We will show that  $W$  is spanned by  $T_N(y), T_N(x_1)T_N(x_2)$ , and possibly 1.

**6.3. Action of  $\Phi_3, \Phi_0$ , and  $\Phi_1$ .** For an element  $F \in \mathbb{C}[x_1, x_2, y]$  and a monomial  $\mathbf{m} = x_1^{a_1} x_2^{a_2} y^b$  let  $\text{coeff}(F, \mathbf{m})$  be the coefficient of  $\mathbf{m}$  in  $F$ .

**Lemma 6.5.** *Suppose  $E \in W$  and  $\text{coeff}(E, y^N) = 0$ . Then  $E \in \mathbb{C}[x_1, x_2]$ .*

*Proof.* Let  $k$  be the  $y$ -degree of  $E$ . Since  $E \in W$  and  $\text{coeff}(E, y^N) = 0$ , one has  $k \leq N - 1$ .

We need to show that  $k = 0$ . Suppose the contrary that  $1 \leq k$ . Then  $1 \leq k \leq N - 1$ .

First we will prove  $k = N - 1$ , using the fact that  $E$  is a  $\lambda_0$ -eigenvector of  $\Phi_3$  by (25).

Recall that  $\text{fil}_{\alpha_3}$  is twice the  $y$ -degree. One has  $\text{fil}_{\alpha_3}(E) = 2k$ . Thus  $E \neq 0 \in \mathcal{F}_{2k}^{\alpha_3} / \mathcal{F}_{2k-1}^{\alpha_3}$ . By Proposition 3.2, any non-zero element in  $\mathcal{F}_{2k}^{\alpha_3} / \mathcal{F}_{2k-1}^{\alpha_3}$  is an eigenvector of  $\Phi_3$  with eigenvalue  $\lambda_{2k}$ . But  $E$  is an eigenvector of  $\Phi_3$  with eigenvalue  $\lambda_0$ . It follows that  $\lambda_{2k} = \lambda_0$ . By Lemma 6.2, we have  $k = N - 1$ .

Because  $\text{deg}_{\text{ir}}(E)$  is even and  $\leq 2N$ , we must have

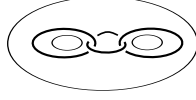
$$E = y^{N-1}(c_1 x_1 x_2 + c_2) + O(y^{N-2}), \quad c_1, c_2 \in \mathbb{C}.$$

We will prove  $c_1 = 0$  by showing that otherwise,  $\Phi_0$  will increase the  $y$ -degree. Note that  $\Phi_0$  can increase the  $y$ -degree by at most 1, and  $\Phi_0$  is  $\mathbb{C}[y]$ -linear. We have

$$(27) \quad \Phi_0(E) = y^{N-1}(c_1 \Phi_0(x_1 x_2) + c_2 \Phi_0(1)) + O(y^{N-1}).$$

The diagram of  $\Phi_0(x_1 x_2)$  has 4 crossings, see Figure 11. A simple calculation shows

$$\Phi_0(x_1 x_2) = (1 - t^4)(1 - t^{-4})y + O(y^0).$$

FIGURE 11.  $\Phi_0(x_1x_2)$ 

Plugging this value in (27), with  $\Phi_0(1) = \lambda_0 \in \mathbb{C}$ ,

$$(28) \quad \Phi_0(E) = y^N c_1 (1 - t^4)(1 - t^{-4}) + O(y^{N-1}).$$

If  $c_1 \neq 0$ , then the  $y$ -degree of  $\Phi_0(E)$  is  $N$ , strictly bigger than that of  $E$ , and  $E$  cannot be an eigenvector of  $\Phi_0$ . Thus  $c_1 = 0$ .

One has now

$$(29) \quad E = c_2 y^{N-1} + O(y^{N-2})$$

Since the  $y$ -degree of  $E$  is  $N - 1$ , one must have  $c_2 \neq 0$ . By counting the intersections with  $\alpha_3$ , we see that  $\Phi_1$  does not increase the  $y$ -degree. We have

$$\begin{aligned} \Phi_1(E) &= c_2 \Phi_1(y^{N-1}) + O(y^{N-2}) \\ &= c_2 \lambda_{N-1} y^{N-1} + O(y^{N-2}) \quad \text{by Proposition 3.2.} \end{aligned}$$

Comparing the above identity with (29) and using the fact that  $E$  is a  $\xi^{2N} \lambda_0$ -eigenvector of  $\Phi_1$ , we have

$$\lambda_{N-1} = \xi^{2N} \lambda_0,$$

which is impossible since Lemma 6.2 says that  $\lambda_k = \xi^{2N} \lambda_0$  only when  $k = N - 2$ . This completes the proof of the lemma.  $\square$

**6.4. Action of  $\Phi_4$ .** Recall that  $\Phi_4$  is the curve on the boundary of the handlebody  $\mathcal{H}$  (see Figure 10) which acts on  $\mathcal{S}_\xi(\mathbb{D}) = \mathbb{C}[x_1, x_2, y]$ . The action of  $\Phi_4$  is  $\mathbb{C}[x_1]$ -linear, and every element of  $W$  is a  $\xi^{2N} x_1$ -eigenvector of  $\Phi_4$ .

Recall that  $\deg_r = \text{fil}_{\alpha_2}$ , and  $\deg_r(x_1^{a_1} x_2^{a_2} y^b) = a_2 + b$ . Note that for  $F \in \mathbb{C}[x_1, x_2]$ ,  $\deg_r(F)$  is exactly the  $x_2$ -degree of  $F$ . By looking at the intersection with  $\alpha_2$ , we see that  $\Phi_4$  preserves the  $\alpha_2$ -filtration, i.e.  $\deg_r \Phi_4(F) \leq \deg_r(F)$ . We will study actions of  $\Phi_4$  on the associated graded spaces.

We will use the notation  $F + \deg_r\text{-l.o.t.}$  to mean  $F + F_1$ , where  $\deg_r(F_1) < \deg_r(F)$ .

**Lemma 6.6.** *Suppose  $1 \leq k \leq N - 1$ . One has*

$$(30) \quad \Phi_4(a(x_1) T_N(x_2)) = \xi^{2N} x_1 [a(x_1) T_N(x_2)]$$

$$(31) \quad \Phi_4(T_k(x_2)) = y [\xi^2 (\xi^{-2k} - \xi^{2k}) x_2^{k-1}] + \deg_r\text{-l.o.t.} \quad \text{mod } \mathbb{C}[x_1, x_2].$$

*Proof.* Identity (30) follows from the  $\xi^{2N}$ -transparency of  $T_N(z)$ .

Let us prove (31). Apply identity (17) to the dashed box below, we have

$$\begin{aligned} \Phi_4(T_k(x_2)) &= \text{Diagram} \\ &= y [\xi^2 (\xi^{-2k} - \xi^{2k}) S_{k-1}(x_2)] + x_1 [\xi^{-2k} S_k(x_2) - \xi^{2k} S_{k-2}(x_2)], \end{aligned}$$

which implies (31).  $\square$

### 6.5. The space $W \cap \mathbb{C}[x_1, x_2]$ .

**Lemma 6.7.** *Suppose  $E \in W \cap \mathbb{C}[x_1, x_2]$  and the coefficient of  $x_1^N x_2^N$  in  $E$  is 0. Then  $E \in \mathbb{C}$ .*

*Proof.* Since  $T_k(x_2)$  is a basis of  $\mathbb{C}[x_2]$ , we can write  $E$  uniquely as

$$E = \sum_{k=0}^N a_k(x_1) T_k(x_2), \quad a_k(x_1) \in \mathbb{C}[x_1].$$

Let  $j$  be the  $x_2$ -degree of  $E' := E - a_N(x_1) T_N(x_2)$ . Then  $j \leq N - 1$

First we will show that  $j = 0$ . Assume the contrary  $j \geq 1$ . Thus  $1 \leq j \leq N - 1$ . Note that  $E$ , by assumption, and  $a_N(x_1) T_N(x_2)$ , by (30), are eigenvectors of  $\Phi_4$  with eigenvalue  $\xi^{2N} x_1$ . It follows that  $E'$  is also an eigenvector of  $\Phi_4$  with eigenvalue  $\xi^{2N} x_1$ . We have

$$E' = \sum_{k=0}^j a_k(x_1) \Phi_4(T_k(x_2)) = a_j(x_1) T_j(x_2) + \text{deg}_r\text{-l.o.t.}$$

Using (31) and the fact that  $\Phi_4$  does not increase  $\text{deg}_r$ , we have

$$\Phi_4(E') = y [a_j(x_1) \xi^2 (\xi^{-2k} - \xi^{2k}) x_2^{k-1}] + \text{deg}_r\text{-l.o.t.} \quad \text{mod } \mathbb{C}[x_1, x_2].$$

When  $1 \leq j \leq n - 1$ , the coefficient of  $y$ , which is the element in the square bracket, is non-zero. Thus  $\Phi_4(E') \notin \mathbb{C}[x_1, x_2]$ , while  $E' \in \mathbb{C}[x_1, x_2]$ . This means  $E'$  can not be an eigenvector of  $\Phi_4$ , a contradiction. This proves  $j = 0$ .

So we have

$$E = a_N(x_1) T_N(x_2) + a_0(x_1).$$

Because the  $\text{deg}_{lr}(E) < 2N$ , the  $x_1$ -degree of  $a_N(x_1)$  is  $< n$ . Using the invariance under  $\sigma$ , one sees that  $E$  must be of the form

$$(32) \quad E = c_1(T_N(x_1) + T_N(x_2)) + c_2, \quad c_1, c_2 \in \mathbb{C}.$$

To finish the proof of the lemma, we need to show that  $c_1 = 0$ . Assume that  $c_1 \neq 0$ . Since  $E$  has even double degree,  $N$  is even. By Lemma 6.2(c),  $\xi^{2N} = -1$ .

Recall that  $E$  is  $\lambda_0$ -eigenvector of  $\Phi_0$ . Apply  $\Phi_0$  to (32),

$$\lambda_0 [c_1(T_N(x_1) + T_N(x_2)) + c_2] = \Phi_0(c_1(T_N(x_1) + T_N(x_2)) + c_2).$$

Both  $T_N(x_1)$  and  $T_N(x_2)$  are eigenvectors of  $\Phi_0$  with eigenvalues  $\xi^{2N} \lambda_0 = -\lambda_0$ , while  $\Phi_0(1) = \lambda_0$ . Hence we have

$$\lambda_0 [c_1(T_N(x_1) + T_N(x_2)) + c_2] = \lambda_0 [-c_1(T_N(x_1) - T_N(x_2)) + c_2].$$

which is impossible since  $c_1 \lambda_0 \neq 0$ .

Hence, we have  $c_1 = 0$ , and  $E \in \mathbb{C}$ . □

### 6.6. Some maximal degree parts of $T_N(\gamma)$ .

**Lemma 6.8.** *One has*

$$(33) \quad \text{coeff}(T_N(\gamma), y^N) = \xi^{-N^2}$$

$$(34) \quad \text{coeff}(T_N(\gamma), x_1^N x_2^N) = \xi^{N^2}.$$

*Proof.* Since  $T_N(\gamma) = \gamma^N + \text{deg}_{lr}\text{-l.o.t.}$ , we have

$$\text{coeff}(T_N(\gamma), y^N) = \text{coeff}(\gamma^N, y^N), \quad \text{coeff}(T_N(\gamma), x_1^N x_2^N) = \text{coeff}(\gamma^N, x_1^N x_2^N).$$

There are  $N^2$  crossing points in the diagram of  $\gamma^N$ . Each crossing can be smoothed in two ways. The positive smoothing acquires a factor  $t$  in the skein relation, and the negative smoothing acquires a factor  $t^{-1}$ . There are  $2^{N^2}$  smoothings of  $\gamma^N$ . Each smoothing  $s$  of all the  $N^2$  crossings gives rise to a link  $L_s$  embedded in  $\mathbb{D}$ . Then  $\gamma^N$  is a linear combination of all  $L_s$ . We will show that the only  $s$  for which  $L_s = y^N$  is the all negative smoothing.

Consider a crossing point  $C$  of  $\gamma^N$ . The vertical line passing through  $C$  intersect  $\mathbb{D}$  in an interval  $\alpha'_3$  which is isotopic to  $\alpha_3$ , and  $\text{fil}_{\alpha_3} = \text{fil}_{\alpha'_3}$ . For an embedded link  $L$  in  $\mathbb{D}$ , as an element of  $\mathcal{S}(\mathbb{D}) = R[x_1, x_2, y]$ ,  $L$  is a monomial whose  $y$ -degree is bounded above by half the number of intersection points of  $L$  with  $\alpha'_3$ . The diagram  $\gamma^N$  has exactly  $2N$  intersection points with  $\alpha'_3$ , with  $C$  contributing two (of the  $2N$  intersection points). If we positively smooth  $\gamma^N$  at  $C$ , the result is a link diagram with  $2N - 2$  intersection points with  $\alpha'_3$ , and no matter how we smooth other crossings, the resulting link will have less than or equal to  $2N - 2$  intersection points with  $\alpha'_3$ . Thus we cannot get  $y^N$  if any of the crossing is smoothed positively. The only smoothing which results in  $y^N$  is the all negative smoothing. The coefficient of this smoothing is  $\xi^{-N^2}$ .

Similarly, one can prove that the only smoothing which results in  $x_1^N x_2^N$  is the all positive smoothing, whose coefficient is  $\xi^{N^2}$ .  $\square$

### 6.7. Proof of Proposition 6.1.

Let

$$E = T_N(\gamma) - \xi^{N^2} T_N(x_1) T_N(x_2) - \xi^{-N^2} T_N(y).$$

Then  $E \in W$ . Lemma 6.8 shows that  $\text{coeff}(E, y^N) = 0 = \text{coeff}(E, x_1^N x_2^N)$ . By Lemma 6.5,  $E \in \mathbb{C}[x_1, x_2]$ . Then, by Lemma 6.7, we have  $E \in \mathbb{C}$ , i.e.  $E$  is a constant.

We will show that  $E = 0$ . This is done by using the inclusion  $\mathcal{H}$  into  $\mathbb{R}^3$ , which gives a  $\mathbb{C}$ -linear map  $\iota : \mathcal{S}_\xi(\mathbb{D}) \rightarrow \mathcal{S}_\xi(\mathbb{R}^3) = \mathbb{C}$ . Under  $\iota$ , we have

$$(35) \quad E = \iota(T_N(\gamma)) - \iota(\xi^{N^2} T_N(x_1) T_N(x_2)) - \iota(\xi^{-N^2} T_N(y)).$$

The right hand side involves the trivial knot and the trivial knot with framing 1, and can be calculated explicitly as follows. Note that  $\iota(\gamma)$  is the unknot with framing 1, while  $\iota(x_1) = \iota(x_2) = \iota(y) = U$ , the trivial knot. With  $T_N = S_N - S_{N-2}$ , and the framing change given by (14), we find

$$(36) \quad T_N\left(\begin{array}{c} | \\ \bigcirc \\ | \end{array}\right) = (-1)^N \xi^{N^2+2N} T_N\left(\begin{array}{c} | \\ | \\ | \end{array}\right) = -\xi^{N^2} T_N\left(\begin{array}{c} | \\ | \\ | \end{array}\right),$$

where the second identity follows from (21). Similarly, using (13), we have

$$(37) \quad T_N(L \sqcup U) = 2(-1)^N \xi^{2N} T_N(L) = -(\xi^{2N^2} + \xi^{-2N^2}) T_N(L).$$

From (36) and (37), we calculate the right hand side of (35), and find that  $E = 0$ . This proves (19).

The proof of (20) is similar. Alternatively, one can get (20) from (19) by noticing that the mirror image map on  $R[x_1, x_2, y]$  is the  $\mathbb{C}$ -algebra map sending  $t$  to  $t^{-1}$ , leaving each of  $x_1, x_2, y$  fixed.

This completes the proof of Proposition 6.1.



## 7. PROOF OF THEOREM 2

Recall that  $\varepsilon = \xi^{N^2}$ , where  $\xi^4$  is a root of 1 of order  $N$ . Then  $\varepsilon^4 = 1$ . The map  $\mathcal{S}_\varepsilon(M) \rightarrow \mathcal{S}_\xi(M)$ , defined for framed links by  $L \rightarrow T_N(L)$ , is well-defined if and only if it preserves the skein relations (1) and (2), i.e. in  $\mathcal{S}_\xi(M)$ ,

$$(38) \quad T_N(L) = \varepsilon T_N(L_+) + \varepsilon^{-1} T_N(L_-)$$

$$(39) \quad T_N(L \sqcup U) = -(\varepsilon^2 + \varepsilon^{-2}) T_N(L).$$

Here, in (38),  $L, L_+, L_-$  are links appearing in the original skein relation (1), they are identical everywhere, except in a ball  $B$ , where they look like in Figure 12.

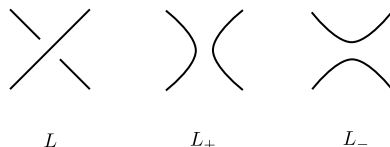


FIGURE 12. The links  $L, L_+$ , and  $L_-$

Identity (39) follows from (37). Let us prove (38).

Case 1: The two strands of  $L$  in the ball  $B$  belong to the same component. Then (38) follows from Proposition 6.1, applied to the handlebody which is the union of  $B$  and a tubular neighborhood of  $L$ .

Case 2: The two strands of  $L$  in  $B$  belongs to different component. Then the two strands of  $L_+$  belongs to the same component, and we can apply (38) to the case when the left hand side is  $L_+$ . We have

$$(40) \quad T_N(L_+) = T_N(\text{right}) \left( \text{left} \right) = T_N(\text{right}) \left( \text{right} \right)$$

$$(41) \quad = \varepsilon^{-1} T_N(\text{right}) \left( \text{right} \right) + \varepsilon T_N(\text{right}) \left( \text{left} \right)$$

$$(42) \quad = \varepsilon^{-1} T_N(L) - \varepsilon^2 T_N(L_-),$$

where (40) follows from Case 1, and (41) follows from the framing factor formula (36). Multiplying (42) by  $\varepsilon$  and using  $\varepsilon^3 = \varepsilon^{-1}$ , we get (38) in this case. This completes the proof of Theorem 2.

**Remark 7.1.** In [BW2], in order to prove Theorem 2, the authors proved, besides Proposition 6.1, a similar statement for links in the cylinder over a punctured torus. Here we bypass this extra statement by reducing the extra statement to Proposition 6.1. Essentially this is due to the fact that the cylinder over a punctured torus is the same as the cylinder over a twice punctured disk.

## REFERENCES

- [BHMV] C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel, *Three-manifold invariants derived from the Kauffman bracket*, *Topology* **31** (1992), no. 4, 685–699.

- [Bul] D. Bullock, *Rings of  $Sl_2(\mathbb{C})$ -characters and the Kauffman bracket skein module*, Comment. Math. Helv. **72** (1997), no. 4, 521–542.
- [BFK] D. Bullock, C. Frohman, and J. Kania-Bartoszyńska, *Understanding the Kauffman bracket skein module*, J. Knot Theory Ramifications **8** (1999), no. 3, 265–277.
- [BP] D. Bullock and J. Przytycki, *Multiplicative structure of Kauffman bracket skein module quantizations*, Proc. Amer. Math. Soc. **128** (2000), no. 3, 923–931.
- [BW1] F. Bonahon and H. Wong, *Quantum traces for representations of surface groups in  $SL_2(\mathbb{C})$* , Geom. Topol. **15** (2011), no. 3, 1569–1615.
- [BW2] F. Bonahon and H. Wong, *Representations of the Kauffman skein algebra I: invariants and miraculous cancellations*, preprint arXiv:1206.1638.
- [CF] L. Chekhov and V. Fock, *Quantum Teichmüller spaces* (Russian) Teoret. Mat. Fiz. **120** (1999), no. 3, 511–528; translation in Theoret. and Math. Phys. **120** (1999), no. 3, 1245–1259.
- [FG] C. Frohman and R. Gelca, *Skein modules and the noncommutative torus*, Trans. Amer. Math. Soc. **352** (2000), no. 10, 4877–4888.
- [Jo] V. Jones, *Polynomial invariants of knots via von Neumann algebras*, Bull. Amer. Math. Soc., **12** (1985), 103–111.
- [Kau] L. Kauffman, *States models and the Jones polynomial*, Topology, **26** (1987), 395–407.
- [Kas] R. Kashaev, *Quantization of Teichmüller spaces and the quantum dilogarithm*, Lett. Math. Phys. **43** (1998), no. 2, 105–115.
- [Le] T. T. Q. Lê, *The colored Jones polynomial and the  $A$ -polynomial of knots*, Adv. Math. **207** (2006), no. 2, 782–804.
- [Lic] W. B. Lickorish, *An introduction to knot theory*, Graduate Texts in Mathematics, **175**, Springer-Verlag, New York, 1997. x+201 pp.
- [LT] T. T. Q. Lê and A. Tran, *The Kauffman bracket skein module of two-bridge links*, to appear in the Proceedings of the AMS, arXiv:1111.0332.
- [Mar] J. Marche, *The skein module of torus knots*, Quantum Topol. **1** (2010), no. 4, 413–421.
- [Pr] J. Przytycki, *Fundamentals of Kauffman bracket skein modules*, Kobe J. Math. **16** (1999) 45–66.
- [PS] J. Przytycki and A. Sikora, *On the skein algebras and  $Sl_2(\mathbb{C})$ -character varieties*, Topology **39** (2000), 115–148.
- [Si] A. Sikora, *Skein modules at the 4th roots of unity*, J. Knot Theory Ramifications, **13** (2004), no. 5, 571–585.
- [Tu] V. Turaev, *Skein quantization of Poisson algebras of loops on surfaces*, Ann. Sci. Sc. Norm. Sup. (4) **24** (1991), no. 6, 635–704.

SCHOOL OF MATHEMATICS, 686 CHERRY STREET, GEORGIA TECH, ATLANTA, GA 30332, USA  
*E-mail address:* letu@math.gatech.edu