FINITE TYPE INVARIANTS OF 3-MANIFOLDS

THANG T. Q. LÊ

Abstract. This is a survey article on finite type invariants of 3-manifolds written for the *Encyclopedia of Mathematical Physics* to be published by Elsevier.

1. Introduction

1.1. **Physics background and motivation.** Suppose G is a semi-simple compact Lie group G and M a closed oriented 3-manifold. Witten [Wit] defined quantum invariants by the path integral over all G-connections A:

$$Z(M,G;k) := \int \exp(\sqrt{-1}\,k\ \mathrm{CS}(A))\,\mathcal{D}A,$$

where k is an integer and $\mathrm{CS}(A)$ is the Chern-Simons functional,

$$\mathrm{CS}(A) = \frac{1}{4\pi}\int_M Tr(A\wedge dA + \frac{2}{3}A^3).$$

The path integral is not mathematically rigorous. According to the stationary phase approximation in quantum field theory, in the limit $k\to\infty$ the path integral decomposes as a sum of contributions from the flat connections:

$$Z(M,G;k) \sim \sum_{\text{flat connections } f} Z^{(f)}(M,K;k) \qquad \text{as } k\to\infty.$$

Each contribution is $\exp(2\pi\sqrt{-1}\,k\ \mathrm{CS}(f))$ times a power series in $1/k$. The contribution from the trivial connection is important, especially for rational homology 3-spheres, and the coefficients of the powers $(1/k)^n$, calculated using $(n+1)$-loop Feynman diagrams by quantum field theory techniques, are known as perturbative invariants.

1.2. **Mathematical theories.** A mathematically rigorous theory of quantum invariants $Z(M,G;k)$ was pioneered by Reshetikhin and Turaev in 1990, see [Tu]. A number-theoretical expansion of the quantum invariants into power series that should correspond to the perturbative invariants was given by Ohtsuki (in the case of sl_2, and general simple Lie algebras by the author) in 1994 that lead him to introducing finite type invariant theory for 3-manifolds. A universal perturbative invariant was constructed by Le-Murakami-Ohtsuki (the LMO invariant) in 1995; it is universal for both finite type invariants and quantum invariants, at least for homology 3-spheres. Rozansky in 1996 defined perturbative invariants using Gaussian integral, very close in the spirit to the original physics point of view. Later Habiro (for sl_2 and Habiro and the author for all simple Lie algebras) found a finer expansion of quantum invariants, known as the cyclotomic expansion, but no physics origin is known for the cyclotomic expansion. The cyclotomic expansion helps to show that the LMO invariant dominates all quantum invariants for homology 3-spheres.

The purpose of this article is to give an overview of the mathematical theory of finite type and perturbative invariants of 3-manifolds.

1.3. **Conventions and notations.** All vector spaces are assumed to be over the ground field $\mathbb{Q}$ of rational numbers, unless otherwise stated. For a graded space A, let $\mathrm{Gr}_n A$ be the subspace of grading n and $\mathrm{Gr}_{\le n} A$ the subspace of grading less than or equal to n. For $x \in A$ let $\mathrm{Gr}_n x$ and $\mathrm{Gr}_{\le n} x$ be the projections of x onto respectively $\mathrm{Gr}_n A$ and $\mathrm{Gr}_{\le n} A$.

All 3-manifolds are supposed to be closed and oriented. A 3-manifold M is an integral homology 3-sphere ($\mathbb{Z}$HS) if $H_1(M,\mathbb{Z}) = 0$; it is a rational homology 3-sphere ($\mathbb{Q}$HS) if $H_1(M,\mathbb{Q}) = 0$. For a framed link L in a 3-manifold M denote M_L the 3-manifold obtained from M by surgery along L, see for example [Tu].

2. Finite type invariants

After its introduction by Ohtsuki in 1994, the theory of finite type invariants (FTI) of 3-manifolds has been developed rapidly by many authors. Later Goussarov and Habiro independently introduced clasper calculus, or Y-surgery, which provides a powerful geometric technique and deep insight in the theory. Y-surgery, corresponding to the commutator in group theory, naturally gives rise to 3-valent graphs.

2.1. Generality on finite type invariants.

2.1.1. *Decreasing filtration.* In a theory of FTI, one considers a class of objects, and a "good" decreasing filtration $\mathcal{F}_0 \supset \mathcal{F}_1 \supset \mathcal{F}_2 \supset \dots$ on the vector space $\mathcal{F} = \mathcal{F}_0$ spanned by these objects. An invariant of the objects with values in a vector space is *of order less than or equal to* n if its restriction to $\mathcal{F}_{n+1}$ is 0; it is of finite type if it is of order less than or equal to n for some n. An invariant has order n if it is of order $\le n$ but not $\le n-1$. Good here means at least the space of FTI of each order is finite-dimensional. It is desirable to have an algorithm of polynomial time to calculate every FTI. Also one wants the set of FTI invariants to separate the objects (completeness).

The space of invariants of order $\le n$ can be identified with the dual space of $\mathcal{F}_0/\mathcal{F}_{n+1}$; its subspace $\mathcal{F}_n/\mathcal{F}_{n+1}$ is isomorphic to the space of invariants of order $\le n$ modulo the space of invariants of order $\le n-1$. Informally one can say that $\mathcal{F}_n/\mathcal{F}_{n+1}$ is more or less the set of invariants of order n.

2.1.2. *Elementary moves, the knot case.* Usually the filtrations are defined using *independent elementary moves.* For the class of knots the elementary move is given by crossing change. Any two knots can be connected by a finite sequence of such moves. The idea is if $K, K' \in \mathcal{F}_n$, the n-th term of the filtration, then $K - K' \in \mathcal{F}_{n+1}$, where K' is obtained from K by an elementary move. Formal definition is as follows. Suppose S is a set of double points of a knot diagram D. Let

$$[D,S] = \sum_{S' \subset S} (-1)^{\#S'} D_{S'},$$

where the sum is over all subsets S' of S, including the empty set, $D_{S'}$ is the knot obtained by changing the crossing at every point in S', and $\#S'$ is the number of elements of S'. Then $\mathcal{F}_n$ is the vector space spanned by all elements of the form $[D, S]$ with $\#S = n$. For the knot case, the Kontsevich integral is an invariant that is universal for all FTI's, see [BN].

2.2. **Ohtsuki's definition of finite type invariants for** $\mathbb{Z}$HS. An elementary move here is a surgery along a knot: $M \to M_K$, where K is a framed knot in a $\mathbb{Z}$HS M. A collection of moves corresponds to surgery on a framed link. To always remain in the class of $\mathbb{Z}$HS we need to restrict ourselves to *unit-framed and algebraically split links*, i.e. framed links in $\mathbb{Z}$HS each component of which has framing ± 1 and the linking number of every two components is 0. It is easy to prove that a link L in a $\mathbb{Z}$HS M is unit-framed and algebraically split if and only if $M_{L'}$ is a $\mathbb{Z}$HS for every sublink L' of L. For a unit-framed, algebraically split link L in a $\mathbb{Z}$HS M define

$$[M, L] = \sum_{L' \subset L} (-1)^{|\#L'|} M_{L'},$$

which is an element in the vector space $\mathcal{M}$ freely spanned by $\mathbb{Z}$HS.

For a non-negative integer n let $\mathcal{F}_n^{AS}$ be the subspace of $\mathcal{M}$ spanned by $[M, L]$ with $\#L = n$. Then the descending filtration $\mathcal{M} = \mathcal{F}_0^{AS} \supset \mathcal{F}_1^{AS} \supset \mathcal{F}_2^{AS} \dots$ defines a theory of FTI on the class of $\mathbb{Z}$HS.

Theorem 1. *a)* (Ohtsuki) *The dimension of $\mathcal{F}_n(\mathcal{M})$ is finite for every n.*
b) (Garoufalidis-Ohtsuki) *One has $\mathcal{F}_{3n+1}(\mathcal{M}) = \mathcal{F}_{3n+2}(\mathcal{M}) = \mathcal{F}_{3n+3}(\mathcal{M})$.*

The orders of FTI's in this theory are multiples of 3. The first non-trivial invariant, which is the only (up to scalar) invariant of degree 3, is the Casson invariant.

2.3. **The Goussarov-Habiro definition.**

2.3.1. *Y-surgery, or clasper surgery.* Consider the *standard Y-graph* Y and a small neighborhood $N(Y)$ of it in the standard $\mathbb{R}^2$, see Figure 1. Denote by $L(Y)$ the 6 component framed link diagram in $N(Y) \subset \mathbb{R}^3$, each components of which has framing 0 in $\mathbb{R}^3$, see Figure 1.

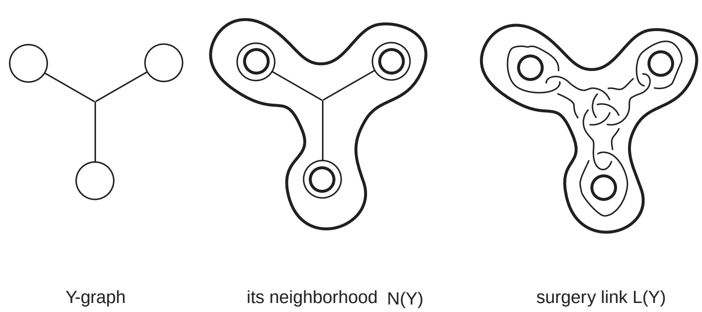

FIGURE 1.

A *framed Y-graph* C in a 3-manifold M is the image of an embedding of $N(Y)$ into M. The surgery of M along the image of the six-component link $L(Y)$ is called a *Y-surgery along C*, denoted by M_C. If one of the leaves bounds a disk in M whose interior is disjoint from the graph, then M_C is homeomorphic to M.

Matveev in 1987 proved that two 3-manifolds M and M' are related by a finite sequence of Y-surgeries if and only if there is an isomorphism from $H_1(M, \mathbb{Z})$ onto

$H_1(M', \mathbb{Z})$ preserving the linking form on the torsion group. It is natural to partition the class of 3-manifolds into subclasses of the same H_1 and the same linking form.

2.3.2. *Goussarov-Habiro Filtrations.* For a 3-manifold M denote by $\mathcal{M}(M)$ the vector space spanned by all 3-manifolds with the same H_1 and linking form. Define, for a set S of Y-graphs in M, $[M, S] = \sum_{S' \subset S} (-1)^{\# S'} M_{S'}$, and $\mathcal{F}^Y_n \mathcal{M}(M)$ the vector space spanned by all $[N, S]$ such that N is in $\mathcal{M}(M)$ and $\# S = n$. The following theorem of Goussarov and Habiro ([Gou, GGP, Ha1]) shows that the FTI theory based on Y-surgery is the same as the one of Ohtsuki in the case of $\mathbb{Z}$HS.

Theorem 2. *For the case $\mathcal{M} = \mathcal{M}(S^3)$, one has $\mathcal{F}^Y_{2n-1} = \mathcal{F}^Y_{2n} = \mathcal{F}^{AS}_{3n}$.*

2.4. **The fundamental theorem of finite type invariants of** $\mathbb{Z}$HS.

2.4.1. *Jacobi diagrams.* A *closed Jacobi diagram* is a vertex oriented trivalent graph, i.e. a graph for which the degree of each vertex equal to 3 and a cyclic order of the 3 half-edges at every vertex is fixed. Here multiple edges and self-loops are allowed. In pictures, the orientation at a vertex is the clockwise orientation, unless otherwise stated. The *degree* of Jacobi diagram is half the number of its vertices.

Let $\mathrm{Gr}_n \mathcal{A}(\emptyset), n \geq 0$, be the vector space spanned by all closed Jacobi diagrams of degree n, modulo the anti-symmetry (AS) and Jacobi (IHX) relations, see Figure 2.

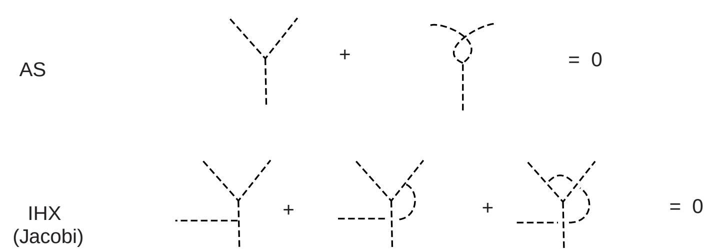

FIGURE 2.

2.4.2. *The universal weight map W.* Suppose D is a closed Jacobi diagram of degree n. Embedding D into $\mathbb{R}^3 \subset S^3$ arbitrarily and then projecting down onto $\mathbb{R}^2$ in general position, one can describe D by a diagram, with over/under-crossing information at every double point just like in the case of a link diagram. We can assume that the orientation at every vertex of D is given by clockwise cyclic order. From the image of D construct a set G of $2n$ Y-graphs as in Figure 3. Here only the cores of a Y-graph are drawn, with the convention that each framed Y-graph is a small neighborhood of its core in $\mathbb{R}^2$.

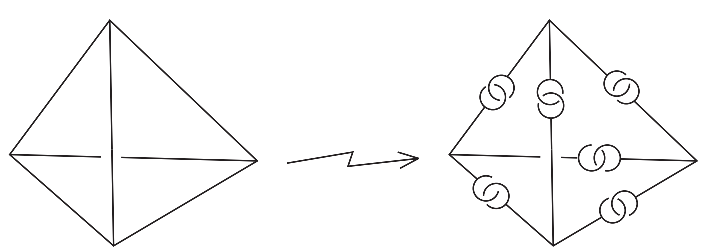

FIGURE 3.

If G' is a proper subset of G, then in G' there is a Y-graph one of the leave of which bounds a disk, hence $S^3_{G'} = S^3$. Thus $W(D) := [S^3, G] = S^3_G - S^3$. By definition, $W(D) \in \mathcal{F}^Y_{2n}$; it might depend on the embedding of D into $\mathbb{R}^3$, but one can show that $W(D)$ is well-defined in $\mathcal{F}^Y_{2n}/\mathcal{F}^Y_{2n+1}$. The map W was first constructed by Garoufalidis and Ohtsuki in the framework of $\mathcal{F}^{AS}$.

2.4.3. *Fundamental theorem.*

Theorem 3. [LMO, Le2] *The map W descends to a well-defined linear map $W : \mathrm{Gr}_n\mathcal{A}(\emptyset) \to \mathcal{F}^Y_{2n}/\mathcal{F}^Y_{2n+1}$ and moreover, is an isomorphism of vector spaces $\mathrm{Gr}_n\mathcal{A}(\emptyset)$ and $\mathcal{F}^Y_{2n}/\mathcal{F}^Y_{2n+1}$, for $\mathcal{M} = \mathcal{M}(S^3)$.*

The theorem essentially says that the set of invariants of degree $2n$ is dual to the space of closed Jacobi diagram $\mathrm{Gr}_n\mathcal{A}(\emptyset)$. The proof is based on the Le-Murakami-Ohtsuki invariant (see §3).

A $\mathbb{Q}$-valued invariant I of order $\leq 2n$ restricts to a linear map from $\mathcal{F}_{2n}/\mathcal{F}_{2n+1}$ to $\mathbb{Q}$. The composition of I and W is a functional on $\mathrm{Gr}_n\mathcal{A}(\emptyset)$ called the *weight system* of I. The theorem shows that every linear functional on $\mathrm{Gr}_{\leq n}\mathcal{A}(\emptyset)$ is the weight of an invariant of order $\leq 2n$.

2.4.4. *Relation to knot invariants.* Under the map that sends an (unframed) knot $K \subset S^3$ to the $\mathbb{Z}$HS obtained by surgery along K with framing 1, an invariant of degree $\leq 2n$ (in the $\mathcal{F}^Y$ theory) of $\mathbb{Z}$HS pulls back to an invariant of order $\leq 2n$ of knots. This was conjectured by Garoufalidis and proved by Habegger.

2.4.5. *Other classes of rational homology 3-spheres.* Actually, the theorem was first proved in the framework of $\mathcal{F}^{AS}$. Clasper surgery theory allows Habiro [Ha1] to generalize the fundamental theorem to $\mathbb{Q}$HS: For M a $\mathbb{Q}$HS, the universal weight map $W : \mathrm{Gr}_n\mathcal{A}(\emptyset) \to \mathcal{F}_{2n}\mathcal{M}(M)/\mathcal{F}_{2n+1}\mathcal{M}(M)$, defined similarly as in the case of $\mathbb{Z}$HS, is an isomorphism, and $\mathcal{F}_{2n-1}\mathcal{M}(M) = \mathcal{F}_{2n}\mathcal{M}(M)$.

2.4.6. *Other filtrations and approaches.* Other equivalent filtrations were introduced (and compared) by Garoufalidis, Garoufalidis-Levine and Garoufalidis-Goussarov-Polyak [GL, GGP]. Of importance is the one using subgroups of mapping class groups in [GL]. A theory of n-equivalence was constructed by Goussarov and Habiro that encompasses many geometric aspects of FTI of 3-manifolds [Ha1, Gou]. Cochran and Melvin [CM] extended the original Ohtsuki definition to manifolds with homology, using algebraically split links, but the filtrations are different from those of Goussarov-Habiro.

3. The Le-Murakami-Ohtsuki invariant

3.1. **Jacobi diagrams.** An *open Jacobi diagram* is a vertex-oriented uni-trivalent graph, i.e., a graph with univalent and trivalent vertices together with a cyclic ordering of the edges incident to the trivalent vertices. A univalent vertex is also called *a leg*. The *degree* of an open Jacobi diagram is half the number of vertices (trivalent and univalent). A *Jacobi diagram based on X*, a compact oriented 1-manifold, is a graph D together with a decomposition $D = X \cup \Gamma$, such that D is the result of gluing all the legs of an open Jacobi diagram Γ to distinct interior points of X. The *degree* of D, by definition, is the degree of Γ. In picture X is depicted by bold lines. Let $\mathcal{A}^f(X)$ be the space of Jacobi diagrams based on X modulo the usual anti-symmetry, Jacobi and the new STU relations (see Figure 4). The completion of $\mathcal{A}^f(X)$ with respect to degree is denoted by $\mathcal{A}(X)$.

STU

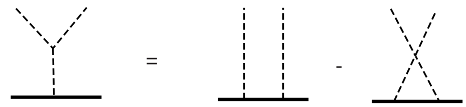

FIGURE 4.

When X is a set of m ordered oriented intervals, denote $\mathcal{A}(X)$ by $\mathcal{P}_m$, which has a natural algebra structure where the product DD' of 2 Jacobi diagrams is defined by stacking D on top of D' (concatenating the corresponding oriented intervals). When X is a set of m ordered oriented circles, denote $\mathcal{A}(X)$ by $\mathcal{A}_m$. By identifying the 2 end points of each interval one gets a map $pr : \mathcal{P}_m \to \mathcal{A}_m$, which is an isomorphism if $m = 1$, see [BN].

For $x \in \mathcal{A}_m$ and $y \in \mathcal{A}_1$, the connected sum is defined by $x\#_m y := pr((pr^{-1}x)\,(pr^{-1}y)^{\otimes m})$, where $(pr^{-1}y)^{\otimes m}$ is the element in $\mathcal{P}_m$ with $pr^{-1}y$ on each oriented interval.

3.1.1. *Symmetrization maps.* Let $\mathcal{B}_m$ be the vector space spanned by open Jacobi diagrams whose legs are labelled by elements of $\{1, 2, \ldots, m\}$, modulo the anti-symmetry and Jacobi relations. One can define an analog of the Poincare-Birkhoff-Witt isomorphism $\chi : \mathcal{B}_m \to \mathcal{P}_m$ as follows. For a diagram D, $\chi(D)$ is obtained by taking the average over all possible ways of ordering the legs labelled by j and attach them to the j-th oriented interval. It is known that χ is a vector space isomorphism [BN].

3.2. **The framed Kontsevich integral of links.** For an m-component framed link $L \subset \mathbb{R}^3$, the (framed version of the) Kontsevich integral $Z(L)$ is an invariant taking values in $\mathcal{A}_m$, see for example [Oht]. Let $\nu := Z(K)$, when K is the unknot with framing 0, and $\check{Z}(L) := Z(L)\#_m\nu$. An explicit formula for ν is given in [BLT].

3.3. **Removing solid loops: the maps ι_n.** Suppose $x \in \mathcal{B}_m$ is an open Jacobi diagram with legs labelled by $\{1, \ldots, m\}$. If the number of vertices of any label is different from $2n$, or if the degree of D is greater than $(m+1)n$, put $\iota_n(D) = 0$. Otherwise, partitioning the $2n$ vertices of each label into n pairs and identifying points in each pair, from x we get a trivalent graph which may contain some isolated loops (no vertices) and which depends on the partition. Replacing each isolated loop by a factor $-2n$, and summing up over all partitions, we get $\iota_n(D) \in \mathrm{Gr}_{\le n}\mathcal{A}(\emptyset)$.

For $x \in \mathcal{A}_m$, choose $y \in \mathcal{P}_m$ such that $pr(y) = x$. Using the isomorphism χ we pull back $\chi^{-1}y \in \mathcal{B}_m$. Define $\iota_n(x) := \iota_n(\chi^{-1}y)$. One can prove that $\iota_n(x)$ does not depend on the choice of the preimage y of x. Note that ι_n lowers the degree by nm.

3.4. **Definition of the Le-Murakami-Ohtsuki invariant Z^{LMO}.** In $\mathcal{A}(\emptyset) := \prod_{n=0}^{\infty} \mathrm{Gr}_n\mathcal{A}(\emptyset)$ let the product of 2 Jacobi diagrams be their disjoint union. Also define the co-product $\Delta(D) = 1 \otimes D + D \otimes 1$ for D a *connected* Jacobi diagram. Then $\mathcal{A}(\emptyset)$ is a commutative co-commutative graded Hopf algebra.

For the unknot $U_\pm$ with framing ± 1, one has $\iota_n(\check{Z}(U_\pm)) = (\mp 1)^n +$(terms of degree ≥ 1), hence their inverses exist. Suppose the linking matrix of an oriented framed link

$L \subset \mathbb{R}^3$ has σ_+ positive eigenvalues and σ_- negative eigenvalues. Define

$$\Omega_n(L) = \frac{\iota_n(\check{Z}(L))}{(\iota_n(\check{Z}(U_+)))^{\sigma_+}(\iota_n(\check{Z}(U_-)))^{\sigma_-}} \in \mathrm{Grad}_{\leq n}(\mathcal{A}(\emptyset)). \tag{1}$$

Theorem 4. ([LMO]) *$\Omega_n(L)$ is an invariant of the 3-manifold $M = S^3_L$.*

We can combine all the Ω_n to get a better invariant:

$$Z^{LMO}(M) := 1 + \mathrm{Grad}_1(\Omega_1(M)) + \cdots + \mathrm{Grad}_n(\Omega_n(M)) + \cdots \in \mathcal{A}(\emptyset).$$

For M a $\mathbb{Q}$HS, we also define

$$\hat{Z}^{LMO}(M) := 1 + \frac{\mathrm{Grad}_1(\Omega_1(M))}{d(M)} + \cdots + \frac{\mathrm{Grad}_n(\Omega_n(M))}{d(M)^n} + \ldots,$$

where $d(M)$ is the cardinality of $H_1(M, \mathbb{Z})$.

Proposition 3.1. ([LMO]) *Both $Z^{LMO}(M)$ and $\hat{Z}^{LMO}(M)$ (when defined) are group-like elements, i.e.*

$$\begin{aligned}\Delta(Z^{LMO}(M)) &= Z^{LMO}(M) \otimes Z^{LMO}(M),\\ \Delta(\hat{Z}^{LMO}(M)) &= \hat{Z}^{LMO}(M) \otimes \hat{Z}^{LMO}(M).\end{aligned}$$

Moreover, $\hat{Z}^{LMO}(M_1 \# M_2) = \hat{Z}^{LMO}(M_1) \times \hat{Z}^{LMO}(M_2)$.

3.5. **Universality properties of the LMO invariant.** Let us restrict ourselves to the case of $\mathbb{Z}$HS.

Theorem 5. [Le2] *The less than or equal to n degree part $\mathrm{Gr}_{\leq n} Z^{LMO}$ is an invariant of degree $2n$. Any invariant of degree $\leq 2n$ is a composition $w(\mathrm{Gr}_{\leq n} Z^{LMO})$, where $w : \mathrm{Gr}_{\leq n}\mathcal{A}(\emptyset) \to \mathbb{Q}$ is a linear map.*

Clasper calculus (or Y-surgery) theory allows Habiro to extend the theorem to rational homology 3-spheres.

3.6. **The Arhus integral.** The Arhus integral (circa 1998) of Bar-Natan, Garoufalidis, Rozansky and Thurston, based on a theory of formal integration, calculates the LMO invariant of rational homology 3-spheres. The formal integration theory has a conceptual flavor and helps to relate the LMO invariant to perturbative expansions of quantum invariants. We give here the definition for the case when one does surgery on a knot K with *non-zero framing b*. The link case is similar, see [BGRT].

When K is a knot, $\check{Z}(K)$ is an element of $\mathcal{A}_1 \equiv \mathcal{P}_1 \equiv \mathcal{B}_1$. Note that $\mathcal{B}_1$ is an algebra where the product is the disjoint union $\sqcup$. Since the framing is b, one has

$$\check{Z}(K) = \exp_{\sqcup}(b\, w_1/2) \sqcup Y,$$

where w_1 is the "dashed interval" (the only connected open Jacobi diagram without trivalent vertex), and Y is an element in $\mathcal{B}$ every term of which must have at least one trivalent vertex. For uni-trivalent graphs $C, D \in \mathcal{B}_1$ let

$$\langle C, D\rangle = \begin{cases} 0 & \text{if the numbers of legs of } C, D \text{ are different} \\ \text{sum of all ways to glue legs of } C \text{ and } D \text{ together} \end{cases}$$

One defines $\int^{FG} \check{Z}(K) := \langle \exp_{\sqcup}(-w_1/2b), Y\rangle$. Then

$$\int^{FG} \check{Z}(K) = \sum_{n=0}^{\infty} \frac{\mathrm{Gr}_n(\iota_n \check{Z}(K))}{(-b)^n}.$$

And hence

$$\hat{Z}^{LMO}(S^3_K) = \frac{\int^{FG} \check{Z}(K)}{\int^{FG} \check{Z}(U_{\mathrm{sign}(b)})}.$$

3.7. **Other approaches.** Another construction of a universal perturbative invariant based on integrations over configuration spaces, closer to the original physics approach but harder to calculate because of the lack of a surgery formula, was developed by Axelrod and Singer, Kontsevich, Bott and Cattaneo, Kuperberg and Thurston, see [AS, BC].

4. Quantum invariants and perturbative expansion

Fix a simple (complex) Lie algebra $\mathfrak{g}$ of finite dimension. Using the quantized enveloping algebra of $\mathfrak{g}$ one can define quantum link and 3-manifold invariants. We recall here the definition, adapted for the case of root lattices (projective group case).

Here our q is equal to q^2 in the text book [Jan]. Fix a root system of $\mathfrak{g}$. Let X, X_+, Y denotes respectively the weight lattice, the set of dominant weights, and the root lattice. We normalize the invariant scalar product in the real vector space of the weight lattice so that the length of any short root is $\sqrt{2}$.

4.1. **Quantum link invariants.** Suppose L is a framed oriented link with m ordered components, then the quantum invariant $J_L(\lambda_1, \dots, \lambda_m)$ is a Laurent polynomial in $q^{1/2D}$, where $\lambda_1, \dots, \lambda_m$ are dominant weights, standing for the simple $\mathfrak{g}$-modules of highest weight $\lambda_1, \dots, \lambda_m$, and D is the determinant of the Cartan matrix of $\mathfrak{g}$, see for example [Tu, Le1]. The Jones polynomial is the case when $\mathfrak{g} = sl_2$ and all the λ_i's are the highest weight of the fundamental representation. For the unknot U with 0 framing one has (here ρ is the half-sum of all positive roots)

$$J_U(\lambda) = \prod_{\text{positive roots } \alpha} \frac{q^{(\lambda+\rho|\alpha)/2} - q^{-(\lambda+\rho|\alpha)/2}}{q^{(\rho|\alpha)/2} - q^{-(\rho|\alpha)/2}}$$

We will also use another normalization of the quantum invariant:

$$Q_L(\lambda_1, \dots, \lambda_m) := J_L(\lambda_1, \dots, \lambda_m) \times \prod_{j=1}^{m} J_U(\lambda_j).$$

This definition is good only for $\lambda_j \in X_+$. Note that each $\lambda \in X$ is either fixed by an element of the Weyl group under the dot action (see [Hu]) or can be moved to X_+ by the dot action. We define $Q_L(\lambda_1, \dots, \lambda_m)$ for arbitrary $\lambda_j \in X$ by requiring that $Q_L(\lambda_1, \dots, \lambda_m) = 0$ if one of the λ_j's is fixed by an element of the Weyl group, and that $Q_L(\lambda_1, \dots, \lambda_m)$ is component-wise invariant under the dot action of the Weyl group, i.e. for every $w_1, \dots, w_m$ in the Weyl group,

$$Q_L(w_1 \cdot \lambda_1, \dots, w_m \cdot \lambda_m) = Q_L(\lambda_1, \dots, \lambda_m).$$

Proposition 4.1. ([Le1]) *Suppose $\lambda_1, \dots, \lambda_m$ are in the root lattice Y.*

a) (Integrality) *Then $Q_L(\lambda_1, \dots \lambda_m) \in \mathbb{Z}[q^{\pm 1}]$, (no fractional power).*

b) (Periodicity) *If q is an r-th root of 1, then $Q_L(\lambda_1, \dots, \lambda_m)$ is invariant under the action of the lattice group rY, i.e. for $y_1, \dots, y_m \in Y$, $Q_L(\lambda_1, \dots, \lambda_m) = Q_L(\lambda_1 + ry_1, \dots, \lambda_m + ry_m)$.*

4.2. **Quantum 3-manifold invariants.** Although the infinite sum $\sum_{\lambda_j \in Y} Q_L(\lambda_1, \dots, \lambda_m)$ does not have a meaning, heuristic ideas show that it is invariant under the second Kirby move, and hence almost defines a 3-manifold invariant. The problem is to regularize the infinite sum. One solution is based on the fact that at r-th roots of unity, $Q_L(\lambda_1, \dots, \lambda_m)$ is periodic, so we should use the sum with λ_j's running over a fundamental set P_r of the action of rY, where

$$P_r := \{x = c_1\alpha_1 + \dots + c_\ell \alpha_\ell \mid 0 \le c_1, \dots, c_\ell < r\}.$$

Here $\alpha_1, \dots \alpha_\ell$ are basis roots. For a root ξ of unity of order r, let

$$F_L(\xi) = \sum_{\lambda_j \in (P_r \cap Y)} Q_L(\lambda_1, \dots, \lambda_m)|_{q=\xi}.$$

For an oriented framed link L let σ_+ and σ_- be respectively the number of positive and negative eigenvalues of the linking matrix of L. Let $U_\pm$ be the unknot with framing ± 1. If $F_{U_\pm}(\xi) \neq 0$, define

$$\tau_L(\xi) := \frac{F_L(\xi)}{(F_{U_+}(\xi))^{\sigma_+}(F_{U_-}(\xi))^{\sigma_-}}.$$

Recall that D is the determinant of the Cartan matrix. Let d be the maximum of the absolute values of entries of the Cartan matrix outside the diagonal.

Theorem 6. [Le3] *a) If the order r of ξ is co-prime with dD, then $F_{U_\pm}(\xi) \neq 0$.*

b) If $F_{U_\pm}(\xi) \neq 0$ then $\tau_M^{P\mathfrak{g}}(\xi) := \tau_L(\xi)$ is an invariant of the 3-manifold $M = S^3_L$.

Remark 4.2. The version presented here corresponds to projective groups. It was defined by Kirby and Melvin for sl_2, Kohno and Takata for sl_n, and for arbitrary simple Lie algebra by the author [Le3]. When r is co-prime with dD, there is also an associated modular category that generates a Topological Quantum Field Theory. In most texts in literature, say [Ki, Tu], another version $\tau^{\mathfrak{g}}$ was defined. The reason we choose $\tau^{P\mathfrak{g}}$ is it has nice integrality and eventually a perturbative expansion. For relations between the version $\tau^{P\mathfrak{g}}$ and the usual $\tau^{\mathfrak{g}}$, see [Le3].

4.2.1. *Examples.* When M is the Poincare sphere and $\mathfrak{g} = sl_2$,

$$\tau_M^{Psl_2}(q) = \frac{1}{1-q} \sum_{n=0}^{\infty} q^n (1 - q^{n+1})(1 - q^{n+2}) \dots (1 - q^{2n+1}).$$

Here q is a root of unity, and the sum is easily seen to be finite.

4.2.2. *Integrality.* The following theorem was proved for $\mathfrak{g} = sl_2$ by H. Murakami [Mu] and $\mathfrak{g} = sl_n$ by Takata-Yokota and Masbaum-Wenzl (using ideas of J. Roberts) and for arbitrary simple Lie algebras by the author in [Le3].

Theorem 7. *Suppose the order r of ξ is a prime big enough, then $\tau_M^{P\mathfrak{g}}(\xi)$ is in $\mathbb{Z}[\xi] = \mathbb{Z}[\exp(2\pi i/r)]$.*

4.3. **Perturbative expansion.** Unlike the link case, quantum 3-manifold invariants can be defined only at certain roots of unity. In general, there is no analytic extension of the function $\tau_M^{P\mathfrak{g}}$ around $q = 1$. In perturbative theory, we want to expand the function $\tau_M^{\mathfrak{g}}$ around $q = 1$ into power series. For $\mathbb{Q}$HS, Ohtsuki (for $\mathfrak{g} = sl_2$) and then the author (for all other simple Lie algebras) showed that there is a number-theoretical expansion of $\tau_M^{P\mathfrak{g}}$ around $q = 1$ in the following sense.

Suppose r is a big enough prime, and $\xi = \exp(2\pi i/r)$. By the integrality (Theorem 7),

$$\tau_M^{P\mathfrak{g}}(\xi) \in \mathbb{Z}[\xi] = \mathbb{Z}[q]/(1 + q + q^2 + \cdots + q^{r-1}).$$

Choose a representative $f(q) \in \mathbb{Z}[q]$ of $\tau_M^{P\mathfrak{g}}(\xi)$. Formally substitute $q = (q-1)+1$ in $f(q)$:

$$f(q) = c_{r,0} + c_{r,1}(q-1) + \cdots + c_{r,n-2}(q-1)^{n-2}$$

The integers $c_{r,n}$ depend on r and the representative $f(q)$. It is easy to see that $c_{r,n} \pmod r$ does not depend on the representative $f(q)$ and hence is an invariant of $\mathbb{Q}$HS. The dependence on r is a big drawback. The theorem below says that there is a *rational* number c_n, not depending on r, such that $c_{r,n} \pmod r$ is the reduction of either c_n or $-c_n$ modulo r, for sufficiently large prime r. It is easy to see that if such c_n exists, it must be unique. Let s be the number of positive roots of $\mathfrak{g}$. Recall that ℓ is the rank of $\mathfrak{g}$.

Theorem 8. *For every $\mathbb{Q}$HS M, there is a sequence of numbers $c_n \in \mathbb{Z}[\frac{1}{(2n+2s)!|H_1(M,\mathbb{Z})|}]$, such that for sufficiently large prime r*

$$c_{r,n} \equiv \left(\frac{|H_1(M,\mathbb{Z})|}{r}\right)^{\ell} c_n \pmod r,$$

where $\left(\frac{|H_1(M,\mathbb{Z})|}{r}\right) = \pm 1$ is the Legendre symbol. Moreover, c_n is an invariant of order $\leq 2n$.

The series $\mathfrak{t}_M^{P\mathfrak{g}}(q-1) := \sum_{n=0}^{\infty} c_n(q-1)^n$, called the Ohtsuki series, can be considered as the perturbative expansion of the function $\tau_M^{P\mathfrak{g}}$ at $q = 1$. For actual calculation of $\mathfrak{t}_M^{P\mathfrak{g}}(q-1)$ see [Le3, Oht, Roz1].

4.3.1. *Recovery from the LMO invariant.* It is known that for any metrized Lie algebra $\mathfrak{g}$, there is a linear map $W_{\mathfrak{g}} : \mathrm{Gr}_n\mathcal{A}(\emptyset) \to \mathbb{Q}$, see [BN].

Theorem 9. *One has*

$$\sum_{n=0}^{\infty} W_{\mathfrak{g}}(\mathrm{Gr}_n Z^{LMO})\, h^n = \mathfrak{t}_M^{P\mathfrak{g}}(q-1)|_{q=e^h}.$$

This shows that the Ohtsuki series $\mathbf{t}_M^{P\mathfrak{g}}(q-1)$ can be recovered from, and hence totally determined by, the LMO invariant. The theorem was proved by Ohtsuki for sl_2. For other simple Lie algebras the theorem follows from the Arhus integral, see [BGRT, Oht].

4.4. **Rozansky's Gaussian integral.** Rozansky gave a definition of the Ohtsuki series using formal Gaussian integral in the important work [Roz1]. The work is only for sl_2, but can be generalized to other Lie algebras; it is closer to the original physics ideas of perturbative invariants.

5. Cyclotomic Expansion

5.1. **The Habiro ring.** Let us define the Habiro ring $\widehat{\mathbb{Z}[q]}$ by

$$\widehat{\mathbb{Z}[q]} := \lim_{\leftarrow n} \mathbb{Z}[q]/((1-q)(1-q^2)\dots(1-q^n)).$$

Habiro [Ha2] called it the cyclotomic completion of $\mathbb{Z}[q]$. Formally, $\widehat{\mathbb{Z}[q]}$ is the set of all series of the form

$$f(q) = \sum_{n=0}^{\infty} f_n(q)\,(1-q)(1-q^2)\dots(1-q^n), \qquad \text{where} \qquad f_n(q) \in \mathbb{Z}[q].$$

Suppose U is the set of roots of 1. If $\xi \in U$ then $(1-\xi)(1-\xi^2)\dots(1-\xi^n) = 0$ if n is big enough, hence one can define $f(\xi)$ for $f \in \widehat{\mathbb{Z}[q]}$. One can consider every $f \in \widehat{\mathbb{Z}[q]}$ as a function with domain U. Note that $f(\xi) \in \mathbb{Z}[\xi]$ is always an algebraic integer. It turns out $\widehat{\mathbb{Z}[q]}$ has remarkable properties, and plays an important role in quantum topology.

Note that the formal derivative of $(1-q)(1-q^2)\dots(1-q^n)$ is divisible by $(1-q)(1-q^2)\dots(1-q^k)$ with k the integer part of $(n-1)/2$. This means every element $f \in \widehat{\mathbb{Z}[q]}$ has a derivative $f' \in \widehat{\mathbb{Z}[q]}$, and hence derivatives of all orders in $\widehat{\mathbb{Z}[q]}$. One can then associate to $f \in \widehat{\mathbb{Z}[q]}$ its Taylor series at a root ξ of 1:

$$T_\xi(f) := \sum_{n=0}^{\infty} \frac{f^{(n)}(\xi)}{n!}(q-\xi)^n,$$

which can also be obtained by noticing that $(1-q)(1-q^2)\dots(1-q^n)$ is divisible by $(q-\xi)^k$ if n is bigger than k times the order of ξ. Thus one has a map $T_\xi : \widehat{\mathbb{Z}[q]} \to \mathbb{Z}[\xi][[q-\xi]]$.

Theorem 10. [Ha2] *a) For each root of unity ξ, the map T_ξ is injective, i.e. a function in $\widehat{\mathbb{Z}[q]}$ is determined by its Taylor expansion at a point in the domain U.*

b) If $f(\xi) = g(\xi)$ at infinitely many roots ξ of prime power orders, then $f = g$ in $\widehat{\mathbb{Z}[q]}$.

One important consequence is that $\widehat{\mathbb{Z}[q]}$ is an integral domain, since we have the embedding $T_1 : \widehat{\mathbb{Z}[q]} \hookrightarrow \mathbb{Z}[[q-1]]$.

In general the Taylor series $T_1 f$ has 0 convergence radius. However, one can speak about p-adic convergence to $f(\xi)$ in the following sense. Suppose the order r of ξ is a power of prime, $r = p^k$. Then it's known that $(\xi-1)^n$ is divisible by p^m if $n > mk$.

Hence $T_1 f(\xi)$ converges in the p-adic topology, and it can be easily shown that the limit is exactly $f(\xi)$.

The above properties suggest to consider $\widehat{\mathbb{Z}[q]}$ as a class of "analytic functions" with domain U.

5.2. **Quantum invariants as an element of $\widehat{\mathbb{Z}[q]}$.** It was proved, by Habiro for sl_2 and by Habiro with the author for general simple Lie algebras, that quantum invariants of $\mathbb{Z}$HS's belong to $\widehat{\mathbb{Z}[q]}$ and thus have remarkable integrality properties:

Theorem 11. *a) For every $\mathbb{Z}$HS M, there is an invariant $I_M^{\mathfrak{g}} \in \widehat{\mathbb{Z}[q]}$ such that if ξ is a root of unity for which the quantum invariant $\tau_M^{P\mathfrak{g}}(\xi)$ can be defined, then $I_M^{\mathfrak{g}}(\xi) = \tau_M^{P\mathfrak{g}}(\xi)$.*

b) The Ohtsuki series is equal to the Taylor series of $I_M^{\mathfrak{g}}$ at 1.

Corollary 5.1. *Suppose M is a $\mathbb{Z}$HS.*

a) For every root of unity ξ, the quantum invariant at ξ is an algebraic integer, $\tau_M^{\mathfrak{g}}(\xi) \in \mathbb{Z}[\xi]$. (No restriction on the order of ξ is required).

b) The Ohtsuki series $\mathfrak{t}_M^{P\mathfrak{g}}(q-1)$ has integer coefficients. If ξ is a root of order $r = p^k$, where p is prime, then the Ohtsuki series at ξ converges p-adically to the quantum invariant at ξ.

c) The quantum invariant $\tau_M^{P\mathfrak{g}}$ is determined by values at infinitely many roots of prime power orders and also determined by its Ohtsuki series.

d) The LMO invariant totally determines the quantum invariants $\tau_M^{P\mathfrak{g}}$.

Part (b) was conjectured by R. Lawrence for sl_2 and first proved by Rozansky (also for sl_2 [Roz2]). Part (d) follows from the fact that the LMO invariant determines the Ohtsuki series; it exhibits another universality property of the LMO invariant.

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