TWISTED ALEXANDER POLYNOMIAL OF LINKS IN THE PROJECTIVE SPACE

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ABSTRACT. We use Reidemeister torsion to study a twisted Alexander polynomial, as defined by Turaev, for links in the projective space. Using sign-refined torsion we derive a skein relation for a normalized form of this polynomial.

1. INTRODUCTION

The study of polynomial invariants for links in the projective space $\mathbb{RP}^3$ was initiated in 1990 by Drobotukhina [Dro90]. She provided a set of Reidemeister moves for links in $\mathbb{RP}^3$, and constructed an analogue of the Jones polynomial using Kauffman's approach involving state sum and the Kauffman bracket. Later she composed a table of links in $\mathbb{RP}^3$ up to six crossings, using the method of Conway's tangles [Dro94]. More recently Mroczkowski [Mro04] defined the Homflypt and Kauffman polynomials using an inductive argument on descending diagrams similar to the one for S^3.

The twisted Alexander polynomial of a link associated to a representation of the fundamental group of the link's complement to $\mathrm{GL}(n;\mathbb{F})$ is a generalization of the Alexander polynomial and has been studied since the early 1990s. In some circumstances the twisted polynomial is more powerful than the usual one: It could distinguish some pairs of knots which the usual polynomial could not, and it also provides more information on fiberedness and sliceness of knots.

For a link in $\mathbb{RP}^3$, the Alexander polynomial will not detect information coming from the torsion part of the first homology group of the link's complement. We will study a version of the twisted Alexander polynomial defined by Turaev which takes the torsion part of the first homology group into account.

In his 1986 paper Turaev [Tur86] extensively studied the Alexander polynomial using the method of Reidemeister torsion. By introducing a refinement of Reidemeister torsion – the sign-refined torsion – he was able to normalize the Alexander polynomial and derive a skein relation for it. Since then the sign-refined torsion

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has played important roles in such works as on the Casson invariant [Les96] and the Seiberg-Witten invariant [Tur01].

In this paper, following Turaev's method, we first identify our twisted Alexander polynomial with a corresponding Reidemeister torsion (Theorem 4.4). Using torsion we derive a skein relation for the polynomial with a certain indeterminacy (Theorem 5.5). Then by introducing sign-refined torsion we normalized the twisted Alexander polynomial and provide a skein relation without indeterminacies (Theorem 5.7). Finally we study relationships between the twisted Alexander polynomial of a link and the Alexander polynomial of the link's lift to S^3 (Theorem 6.3), also using Reidemeister torsion. Although many of Turaev's arguments carry to our case, for the sake of completeness we still provide them in details.

In our view the interest here lies primarily on the 3-dimensional nature of the method. Skein relations for link polynomial invariants are usually studied diagrammatically on two-dimensional link projections. Here we study skein relations through three-dimensional topology, using a classical yet contemporary topological invariant – the Reidemeister torsion.

2. Diagrams for links in $\mathbb{RP}^3$ and the fundamental group

2.1. **Diagrams.** Throughout if L is a link in $\mathbb{RP}^3$ then we let $X = \mathbb{RP}^3 \setminus \overset{\circ}{N}(L)$ be its complement, where $N(L)$ is a tubular neighborhood of L, a collection of solid tori. We write $\pi = \pi_1(X)$ and $H = H_1(X)$.

We follow the terminology of Drobotukhina in [Dro90]. Consider the standard model of $\mathbb{RP}^3$ as a ball B^3 with antipodal points on the boundary sphere ∂B^3 identified. In this way $\mathbb{RP}^3 = \mathbb{RP}^2 \cup B^3$. Let N and S be respectively the North Pole and the South Pole of ∂B^3. Given a link L in $\mathbb{RP}^3$, let $\widetilde{L}$ be its inverse image in B^3 under the quotient map. Isotope L a bit so that $\widetilde{L}$ does not pass through N or S. Define a projection map p from $\widetilde{L}$ to the equator disk D^2 so that a point x is mapped to the point $p(x)$ which is the intersection between the disk D^2 and the semicircle passing through the three points N, S and x, see Fig. 1.

We can always isotope L so that $\widetilde{L}$ satisfies the following conditions of general position:

(1) $\widetilde{L}$ intersects the boundary sphere ∂B^3 transversally, no two points of $\widetilde{L}$ lie on the same half of a great circle joining N and S (i.e. $p(\widetilde{L})$ has no double point on the boundary circle ∂D^2).
(2) The projection $p(\widetilde{L})$ contains no cusps, no points of tangency, and no triple points.

At each double point P of $p(\widetilde{L})$, the inverse image $p^{-1}(P)$ consists of two points in $\widetilde{L}$ which are on the same semicircle joining N and S; the one nearer to N is called the upper point, the other one is called the lower point. The projection of a small arc of $\widetilde{L}$ around an upper point is called an overpass, similarly, the projection of

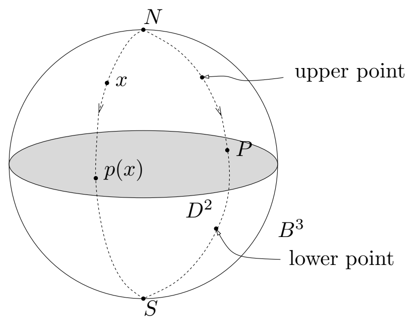


FIGURE 1. Standard model.

a small arc of $\widetilde{L}$ around a lower point is called an underpass. The projection $p(\widetilde{L})$ together with information about overpasses and underpasses is called a diagram of the link L. Figure 4 is an example of a diagram of a link.

2.2. **A Wirtinger-type presentation for the fundamental group.** Let D be a diagram of a link L (we always consider a knot as a link having one component). Choose an orientation for D. Label the upper arcs of D, each of which connecting two underpasses, as $a_1, a_2, \ldots, a_q$, $q \geq 0$, in arbitrary order (in case of an unknotting component which does not cross under, consider the whole component as an upper arc). Let $2p$, $p \geq 0$ be the number of intersections between D and the boundary circle of the projection disk. Label the intersection point counterclockwise as $b_1, b_2, \ldots, b_{2p}$, starting from any point. To each b_i, associate a number ϵ_i as follows. At the point b_i, if D is entering the boundary then let $\epsilon_i = 1$, and let $\epsilon_i = -1$ in the other case.

Similar to the case of links in S^3, an application of the van Kampen gives us the following presentation for the fundamental group.

Theorem 2.1. *With the above notations π has a presentation with generators $a_1, a_2, \ldots, a_q, b_1, b_2, \ldots, b_{2p}, c$; and relations:*

$$b_{p+i} = c^{-1} b_1^{\epsilon_1} b_2^{\epsilon_2} \cdots b_{i-1}^{\epsilon_{i-1}} b_i b_{i-1}^{-\epsilon_{i-1}} b_{i-2}^{-\epsilon_{i-2}} \cdots b_1^{-\epsilon_1} c, \ 1 \leq i \leq p;$$

$$b_1^{\epsilon_1} b_2^{\epsilon_2} \cdots b_p^{\epsilon_p} = c^2$$

together with Wirtinger-type relations involving a_i's and b_j's at each crossing; and if there is an upper arc connecting b_i and b_j then there is a relation $b_i = b_j$.

Remark 2.2. In [Huy05] it is shown that if the diagram contains more than one crossing then a relation at a crossing can be deduced from the remaining relations. As a consequence if there is no affine unknot component then in the presentation of Theorem 2.1 one may choose to omit one Wirtinger-type relation so that the number of generators is one more than the number of relations.

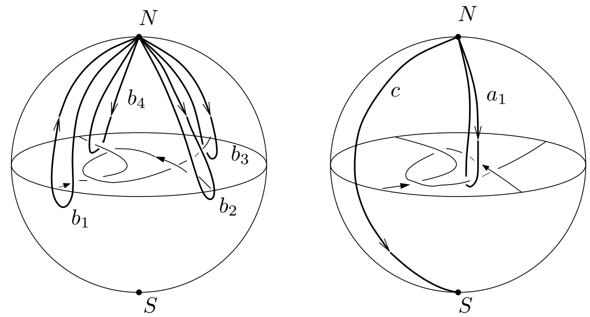


FIGURE 2. Generators.

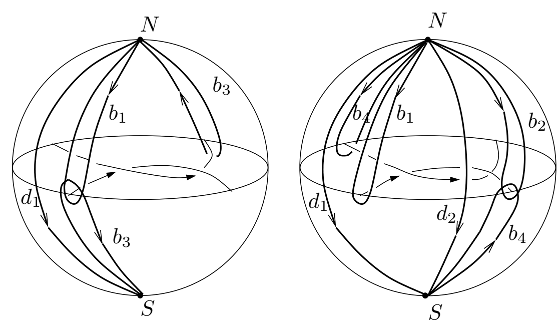


FIGURE 3. Relations.

2.3. **The first homology group.**

Corollary 2.3. *Let L be a link with v components, and let S be the $\mathbb{R}\mathrm{P}^2$ surface represented by the boundary sphere in our model. If there exists one component of L whose number of intersection points with S is odd then this component represents the non-trivial first homology class of $\mathbb{R}\mathrm{P}^3$, and $H \cong \mathbb{Z}^v$. In the other case, L represents the trivial homology class of $\mathbb{R}\mathrm{P}^3$ and $H \cong \mathbb{Z}^v \oplus \mathbb{Z}_2$.*

Proof. As a result of the abelianization, the Wirtinger-type relations and the relation

$$b_{p+i} = c^{-1} b_1^{\epsilon_1} b_2^{\epsilon_2} \cdots b_{i-1}^{\epsilon_{i-1}} b_i b_{i-1}^{-\epsilon_{i-1}} b_{i-2}^{-\epsilon_{i-2}} \cdots b_1^{-\epsilon_1} c, 1 \leq i \leq p$$

would identify all the b_i and a_j corresponding to the same k-th component of L as an element $t_k \in H$, and would also identify b_i and b_{p+i}. Thus

$$H = \langle c, t_1, t_2, \ldots, t_v / ct_i = t_i c, t_i t_j = t_j t_i, \prod_{i=1}^{v} t_i^{\delta_i} = c^2 \rangle;$$

where δ_i is the sum of all ϵ_k, $0 \leq k \leq p$, such that b_k corresponds to the i-th component, $1 \leq i \leq v$.

There are two cases:

Case 1: All δ_i are even. Write $\delta_i = 2k_i$, $k_i \in \mathbb{Z}$, $1 \leq i \leq v$. In this case $t_1^{2k_1} t_2^{2k_2} \cdots t_v^{2k_v} = c^2$, so $(ct_1^{-k_1} t_2^{-k_2} \cdots t_v^{-k_v})^2 = 1$. Let $u = ct_1^{-k_1} t_2^{-k_2} \cdots t_v^{-k_v}$. Then $u^2 = 1$ and $c = ut_1^{k_1} t_2^{t_2} \cdots t_v^{k_v}$, so

$$H = \langle t_1, t_2, \dots, t_v, u / t_i u = u t_i, t_i t_j = t_j t_i, u^2 = 1 \rangle \cong \mathbb{Z}^v \oplus \mathbb{Z}_2.$$

Case 2: There is a δ_i that is odd. Let $I = \{i, 1 \leq i \leq v / \delta_i = 2k_i + 1\}$ and $J = \{i, 1 \leq i \leq v\} \setminus I$. Let $i_0 = \min\{i / i \in I\}$. Then $\prod_{i \in I} t_i^{2k_i+1} \prod_{j \in J} t_j^{2k_j} = c^2$, so $\prod_{i \in I} t_i = c^2 (\prod_{i \in I} t_i^{-k_i})^2 (\prod_{j \in J} t_j^{-k_j})^2 = (c \prod_{1 \leq i \leq v} t_i^{-k_i})^2$. Let $u = c \prod_{1 \leq i \leq v} t_i^{-k_i}$. Then $\prod_{i \in I} t_i = u^2$. Since $i_0 \in I$ we have $t_{i_0} \prod_{i \in I \setminus \{i_0\}} t_i = u^2$, which implies that $t_{i_0} = u^2 \prod_{i \in I \setminus \{i_0\}} t_i^{-1}$. Also $c = u \prod_{1 \leq i \leq v} t_i^{k_i} = t^{1+2k_{i_0}} \prod_{i \in I \setminus \{i_0\}} t_i^{k_i - k_{i_0}} \prod_{j \in J} t_j^{k_j}$. So

$$H = \langle t_1, t_2, \dots, \hat{t_{i_0}}, \dots, t_v, u / t_i u = u t_i, t_i t_j = t_j t_i \rangle \cong \mathbb{Z}^v$$

(a hat over an item indicates that the item is omitted).

Consider any component K of L. According to Poincaré Duality there is a non-degenerate bilinear form $H_1(\mathbb{R}\mathrm{P}^3; \mathbb{Z}_2) \times H_2(\mathbb{R}\mathrm{P}^3; \mathbb{Z}_2) \to \mathbb{Z}_2$ where $\langle [K], [S] \rangle$ is exactly the mod 2 intersection number between the curve K and the surface S, which is δ_i mod 2. When $\langle [K], [S] \rangle = 1$ we would have $[K]$ is non trivial in $H_1(\mathbb{R}\mathrm{P}^3; \mathbb{Z}_2) \cong \mathbb{Z}_2$, and $[S]$ is non trivial in $H_2(\mathbb{R}\mathrm{P}^3; \mathbb{Z}_2) \cong \mathbb{Z}_2$. On the other hand $\langle [K], [S] \rangle = 0$ would imply that $[K]$ is trivial in $H_1(\mathbb{R}\mathrm{P}^3; \mathbb{Z}_2)$. □

2.3.1. *Terminology.* We will call a link a *nontorsion link* if each of its component is null-homologous (in its diagram the number of intersection points of each component with the boundary circle is a multiple of four). The first homology group of its complement is isomorphic to $\mathbb{Z}^v \oplus \mathbb{Z}_2$. The other links are called *torsion links*. From now on we will fix the splitting of H as in Corollary 2.3. In this splitting if a link is nontorsion then the free part of the first homology group is generated by the meridians.

3. Twisted Alexander polynomial

3.1. **The twisted homomorphism from $\mathbb{Z}[H]$ to $\mathbb{Z}[G]$.** Fix a splitting of H as a product $H = G \times \operatorname{Tors} H$ of the torsion part $\operatorname{Tors} H = \langle u \rangle$ and a free part $G \cong H / \operatorname{Tors} H$. Consider a representation (a character) φ from $\operatorname{Tors} H = \langle u \rangle$ to $\operatorname{Aut}_{\mathbb{C}}(\mathbb{C}) \cong \mathbb{C}^*$. If $|\operatorname{Tors} H| = 1$ let $\varphi(u) = 1$; if $|\operatorname{Tors} H| = 2$, let $\varphi(u) = -1$, i.e. $\varphi(u) = (-1)^{|\operatorname{Tors} H|+1}$. The map φ then induces a ring homomorphism, called the *twisted homomorphism* from $\mathbb{Z}[H]$ to $\mathbb{Z}[G]$ by defining $\varphi(gu) = g\varphi(u)$. In the case $|\operatorname{Tors} H| = 1$, φ is exactly the canonical projection from $\mathbb{Z}[H]$ to $\mathbb{Z}[G]$. The composition of φ and the canonical projection $\mathbb{Z}[\pi] \to \mathbb{Z}[H]$ gives us a ring homomorphism from $\mathbb{Z}[\pi]$ to $\mathbb{Z}[G]$.

Let F be the free group generated by the generators of π, and let *pr* be the canonical projection $\mathbb{Z}[F] \to \mathbb{Z}[\pi] \to \mathbb{Z}[H]$. From now on for simplicity of notation depending on the context we use the letter φ for the twisted map above, either from $\mathbb{Z}[F]$ to $\mathbb{Z}[G]$, or from $\mathbb{Z}[\pi]$ to $\mathbb{Z}[G]$, or from $\mathbb{Z}[H]$ to $\mathbb{Z}[G]$.

3.2. **Twisted Alexander polynomial.** Given a presentation $\pi = \langle x_1, \dots, x_n / r_1, \dots, r_m \rangle$ with $m = n-1$ we construct an $m \times n$ matrix, the Alexander–Fox matrix, $[pr(\partial r_i/\partial x_j)]_{i,j}$, whose entries are elements of $\mathbb{Z}[H]$. Denote by $E(\pi)$ the ideal of $\mathbb{Z}[H]$ generated by the $(n-1) \times (n-1)$-minors of the Alexander–Fox matrix. It is known that $E(\pi)$ does not depend on a presentation of π.

Definition 3.1. The twisted Alexander polynomial of L is defined as $\Delta^{\varphi}(L) = \gcd \varphi(E(\pi)) \in \mathbb{Z}[G]$.

Note that in the unique factorization domain $\mathbb{Z}[G]$ the greatest common divisor is only defined up to units, which are elements of $\pm G$.

Remark 3.2. If we replace the twisted map φ by the canonical projection $\mathbb{Z}[H] \to \mathbb{Z}[G]$ (the torsion part of H is sent to 1) then we would get the usual Alexander polynomial $\Delta(L)$. Also, if the link L is a torsion link then the twisted Alexander polynomial $\Delta^{\varphi}(L)$ is exactly the Alexander polynomial $\Delta(L)$.

Example 3.3. Let K be the knot 2_1 in Drobotukhina's table. Its fundamental

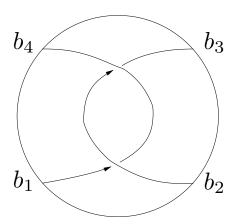


FIGURE 4. The knot 2_1.

group has a presentation

$$\pi = \langle b_1, b_2, b_3, b_4, c / b_2 b_1 = b_4 b_2 = b_3 b_4, b_3 = c^{-1} b_1 c, b_4 = c^{-1} b_1^{-1} b_2 b_1 c, b_1^{-1} b_2^{-1} = c^2 \rangle$$
$$= \langle b_1, c / c = b_1 c^3 b_1 \rangle,$$

the only relator is $r = c^{-1} b_1 c^3 b_1$. Its first homology group is $H = \langle t, c / (ct)^2 = 1, ct = tc \rangle$, where t is the projection of the meridian b_1. Let $u = ct$, then $H = \langle u, t / u^2 = 1, tu = ut \rangle \cong \mathbb{Z} \oplus \mathbb{Z}_2$. The twisted homomorphism $\varphi : \mathbb{Z}[\pi] \to \mathbb{Z}[t^{\pm 1}]$ is determined by $\varphi(b_1) = t$ and $\varphi(c) = \varphi(u) t^{-1} = -t^{-1}$. So $\Delta_K^{\varphi}(t) = \gcd\{\varphi(\partial r/\partial b_1), \varphi(\partial r/\partial c)\} = \gcd\{-t^{-2}(t^2-1), -t^{-1}(t-1)^2\} = t-1$. On the other hand $\Delta_K(t) = \gcd\{-t^{-2}(t^2+1), -t^{-1}(t^2+1)\} = t^2+1$.

Example 3.4. Suppose that L is an *affine link*, i.e. L can be isotoped so that it is contained inside a 3-ball in $\mathbb{R}\mathrm{P}^3$, and so L is a nontorsion link. Its fundamental group is generated by $a_1, a_2, \dots, a_q, c$, where q is the number of crossings; together with $q-1$ Wirtinger relations r_j involving the a_i's, and the relation $c^2=1$. Note that the $(q-1)\times q$ matrix $[pr(\partial r_i/\partial a_j)]_{i,j}$ is exactly the Alexander–Fox matrix of L viewed as a link in S^3. Then it is immediate that the twisted Alexander polynomial of L is equal to $\Delta^{\varphi}(L)=\varphi(\partial c^2/\partial c)=\varphi(1+c)=0$. On the other hand $\Delta(L)$ – the Alexander polynomial of L viewed as a link in $\mathbb{RP}^3$ – will be twice the Alexander polynomial of L viewed as a link in S^3. This supports the result that the value of the Alexander polynomial of a knot complement evaluated at 1 is exactly the cardinality of the torsion part of the homology group (see [Tur86, p. 133], [Nic03, p. 69]).

4. Twisted Alexander polynomial and Reidemeister torsion

4.1. **Background on Reidemeister torsion.** Two very readable references for this section are [Mil66] and [Tur01].

4.1.1. *Torsion of a chain complex.* Let $\mathbb{F}$ be a field, V be a k-dimensional vector space over $\mathbb{F}$. Suppose that $b=(b_1,b_2,\dots,b_k)$ and $c=(c_1,c_2,\dots,c_k)$ are two bases of V then there is a non-singular $k\times k$ matrix (a_{ij}) such that $c_j=\sum_{i=1}^k a_{ij}b_i$. We write $[c/b]=\det(a_{ij})\in\mathbb{F}^*$. Two bases b and c are said to have the same orientation if $[b/c]>0$, and to be equivalent if $[b/c]=1$.

Let $0\to C\overset{\alpha}{\hookrightarrow} D\overset{\beta}{\twoheadrightarrow} E\to 0$ be a short exact sequence of vector spaces. Let $c=(c_1,c_2,\dots,c_k)$ be a basis for C and $e=(e_1,e_2,\dots,e_l)$ be a basis for E. Since β is surjective we can lift e_i to a vector $\tilde{e}_i$ in D. Then $ce=(c_1,\dots,c_k,\tilde{e}_1,\dots,\tilde{e}_l)$ is a basis for D and its equivalence class depends not on the choice of $\tilde{e}_i$ but only on the equivalence classes of c and e.

The finite chain complex $(C,\partial)=(0\to C_m\xrightarrow{\partial_m}C_{m-1}\xrightarrow{\partial_{m-1}}\cdots\xrightarrow{\partial_2}C_1\xrightarrow{\partial_1}C_0\to 0)$ of finite-dimensional vector spaces over $\mathbb{F}$ is called *acyclic* if it is exact. The chain is called *based* if for each C_i a basis is chosen.

Assume that (C,∂) is acyclic and based with basis c. Choose a basis b_i for $B_i=\operatorname{Im}\partial_{i+1}=\ker\partial_i$. From the short exact sequence $0\to B_i\hookrightarrow C_i\twoheadrightarrow B_{i-1}\to 0$ we get a basis b_ib_{i-1} for C_i.

Definition 4.1. The torsion of the acyclic and based chain complex C is defined to be $\tau(C)=\prod_{i=0}^m[b_ib_{i-1}/c_i]^{(-1)^{i+1}}\in\mathbb{F}^*$. If C is not acyclic then $\tau(C)$ is defined to be 0.

Note that this torsion (Turaev's version) is the inverse of Milnor's version.

The torsion $\tau(C)$ depends on c but does not depend on the choice of b_i's. If a basis c_i' is used instead of c_i then the torsion is multiplied with $[c_i/c_i']^{(-1)^{i+1}}$.

4.1.2. *Torsion of a CW-complex.* Let X be a finite connected CW-complex and let $\pi=\pi_1(X)$. The universal cover $\widetilde{X}$ of X has a canonical CW-complex structure obtained by lifting the cells of X. If $\{e_i^k, 1\le i\le n_k\}$ is an ordered set of oriented

k-cells of X and $\tilde{e}_i^k$ is any lift of e_i^k then the ordered set $\{\tilde{e}_i^k, 1 \leq i \leq n_k\}$ is a basis of the $\mathbb{Z}[\pi]$-module $C_i(\widetilde{X})$.

If $\mathbb{Z}[\pi] \xrightarrow{\varphi} \mathbb{F}$ is a ring homomorphism then by the change of rings construction $\mathbb{F} \otimes_\varphi C_*(\widetilde{X})$ is a chain complex of finite dimensional vector spaces over $\mathbb{F}$. If this chain complex is acyclic then we can define its torsion $\tau(\mathbb{F} \otimes_\varphi C_*(\widetilde{X})) \in \mathbb{F}^*$. However $\tau(\mathbb{F} \otimes_\varphi C_*(\widetilde{X}))$ depends on the chosen basis for $C_*(\widetilde{X})$, that is on the choices of lifting cells $\{\tilde{e}_i^k, 1 \leq i \leq n_k\}$. If we fix a choice of a set of lifting cells as a basis for the $\mathbb{Z}[\pi]$-module $C_i(\widetilde{X})$ but change the order of the cells in the basis then $\tau(\mathbb{F} \otimes_\varphi C_*(\widetilde{X}))$ is multiplied with ± 1. If we change the orientations of the cells then torsion is also multiplied with ± 1. If we choose a different lifting cell for e_i^k – by an action $h \cdot \tilde{e}_i^k$ of a covering transformation $h \in \pi$ – then the torsion is multiplied with $\varphi(h)^{\pm 1}$.

Definition 4.2. The Reidemeister torsion $\tau^\varphi(X)$ of the CW-complex X is defined to be the image of $\tau(\mathbb{F} \otimes_\varphi C_*(\widetilde{X}))$ under the quotient map $\mathbb{F} \to \mathbb{F}/\pm\varphi(\pi)$.

Torsion is a simple homotopy invariant and a topological invariant of compact connected CW-complexes. In dimensions three or less, where our interests are, each topological manifold has a unique piecewise-linear structure, so the torsion of a manifold can be defined.

4.1.3. *Torsion with homological bases.* Here we consider the case when the chain complex is not acyclic, following [Mil66, p. 158]. Suppose that $(C, \partial) = (0 \to C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0)$ is a chain complex of based finite-dimensional vector spaces, not necessarily acyclic. Let $Z_i = \ker \partial_i$ and $B_i = \operatorname{Im} \partial_{i+1}$. Let $H_i(C) = Z_i/B_i$ and h_i be its chosen basis. There is a short exact sequence $0 \to B_i \hookrightarrow Z_i \twoheadrightarrow H_i \to 0$. This combined with the short exact sequence $0 \to Z_i \hookrightarrow C_i \twoheadrightarrow B_{i-1} \to 0$ show that $(b_i h_i) b_{i-1}$ is a basis for C_i (and is defined up to equivalence bases). We can define torsion in a similar manner: $\tau(C, c, h) = \prod_{i=0}^m [b_i h_i b_{i-1}/c_i]^{(-1)^{i+1}} \in \mathbb{F}^*$. It depends on c and h but does not depend on the choice of the bases b_i's.

4.1.4. *Symmetry of torsion.* Let M be a compact connected orientable three-manifold. Suppose that in the field $\mathbb{F}$ there is a certain "bar" operation so that for all $\alpha \in \pi$, $\overline{\varphi(\alpha)} = \varphi(\alpha^{-1})$. If ∂M consists of tori then we have $\tau^\varphi(M) = \overline{\tau^\varphi(M)}$. For more details see [Tur01, p. 70].

4.1.5. *Sign-refined torsion.* This was introduced by Turaev [Tur86] to remove the sign ambiguity of torsion. Let C be a finite based chain complex of vector spaces over $\mathbb{F}$. Let $\beta_i(C) = \sum_{j=0}^i \dim(H_j(C)) \mod 2$, $\gamma_i(C) = \sum_{j=0}^i \dim(C_j) \mod 2$, and $N(C) = \sum \beta_i(C)\gamma_i(C) \mod 2$. Let c be a basis for C and h be a basis for $H_*(C)$. Define $\check{\tau}(C, c, h) = (-1)^{N(C)}\tau(C, c, h) \in \mathbb{F}$. Thus $\check{\tau}(C, c, h)$ is $\tau(C, c, h)$ up to a sign, and they are the same when C is acyclic.

A *homological orientation* for a finite CW-complex X is an orientation of the finite dimensional vector space $\oplus_i H_i(X;\mathbb{R})$. Let h be a basis for $H_*(X;\mathbb{R})$ representating a homological orientation, i.e. h is a positive basis, and let c be a basis for $C_*(X;\mathbb{Z})$ arising from an ordered set of oriented cells of X, which gives rise to a basis for $C_*(X;\mathbb{R})$. We call a lift $\tilde{c}$ of c to the universal cover $\widetilde{X}$ a *fundamental family of cells*. Let

$$\tau_0^{\varphi}(X,\tilde{c},h)=\operatorname{sign}(\breve{\tau}(C_*(X;\mathbb{R}),c,h))\tau^{\varphi}(X,\tilde{c}). \tag{4.1}$$

Definition 4.3. The sign-refined torsion $\tau_0^{\varphi}(X,h)$ is the image of $\tau_0^{\varphi}(X,\tilde{c},h)$ under the projection $\mathbb{F}\to\mathbb{F}/\varphi(\pi_1(X))$.

This torsion has no sign ambiguity. It depends on the homological orientation but not on the order or the orientations of the cells of X, since the signs of the two terms in the product change simultaneously. The choice of the number $N(C)$ is due to a change of base formula, with it the sign-refined torsion is invariant under simple homotopy equivalences preserving homological orientations [Tur01, p. 98].

4.1.6. *Product formulas for unrefined torsion.* Suppose that $0\to C'\to C\to C''\to 0$ is a short exact sequence of finite acyclic chain complexes of vector spaces. Suppose that the bases of C, C' and C'' are *compatible*, in the sense that c_i is equivalent to $c'_i c''_i$, then

$$\tau(C)=\pm\tau(C')\tau(C''). \tag{4.2}$$

When the chains are not acyclic there is also a product formula for torsion with homological bases. Let h, h' and h'' be the bases for $H_*(C)$, $H_*(C')$ and $H_*(C'')$ respectively. The short exact sequence involving C, C', C'' above gives rise to a finite long exact sequence of homology groups $\mathcal{H}=(\cdots\to H_i(C')\to H_i(C)\to H_i(C'')\to H_{i-1}(C')\to\cdots\to H_0(C')\to H_0(C)\to H_0(C'')\to 0)$. Since these vector spaces are based the chain $\mathcal{H}$ has a well-defined torsion $\tau(\mathcal{H})$, which depends on h, h' and h''. Suppose that the bases of C, C' and C'' are compatible, then

$$\tau(C,h)=\pm\tau(C',h')\tau(C'',h'')\tau(\mathcal{H}). \tag{4.3}$$

4.1.7. *Product formulas for sign-refined torsion.* The work of keeping track of the shuffling of the bases has been done (cf. [Tur86, Lemma 3.4.2]) in the following formula:

$$\breve{\tau}(C,c'c'',h)=(-1)^{\mu+\nu}\breve{\tau}(C',c',h')\breve{\tau}(C'',c'',h'')\tau(\mathcal{H}), \tag{4.4}$$

in other words

$$\tau(C,c'c'',h)=(-1)^{\mu+\nu+N(C)+N(C')+N(C'')}\tau(C',c',h')\tau(C'',c'',h'')\tau(\mathcal{H}), \tag{4.5}$$

where $\mu=\sum[(\beta_i(C)+1)(\beta_i(C')+\beta_i(C''))+\beta_{i-1}(C')\beta_i(C'')] \mod 2$; and $\nu=\sum_{i=0}^{m}\gamma_i(C'')\gamma_{i-1}(C') \mod 2$.

4.1.8. *Homological orientations of oriented link complements.* Let L be an oriented link in an oriented rational homology three-sphere M, and let X be the link's complement. Let $U = N(L) = \cup_{1\le i\le v} U_i$ and let m_i and l_i be the meridian and the longitude of the torus boundary component ∂U_i. The canonical homological orientation of L is the orientation of the vector space $H_*(X;\mathbb{R})$ represented by the basis $([pt],[m_1],\dots,[m_v],[\partial U_1],\dots,[\partial U_{v-1}])$. The classes $[m_i]$ depend on the orientation of L and so does the homological orientation.

4.1.9. *Reidemeister torsion associated with representations to* $\mathrm{SL}(n;\mathbb{C})$. The Reidemeister torsion associated with a representation to $\mathrm{O}(n)$ was considered by Milnor [Mil66, p. 180], see also Kitano [Kit96]. Let X be a finite connected CW-complex and let $\widetilde{X}$ be its universal cover. Let $\rho : \pi \to \mathrm{SL}(n;\mathbb{C})$ be a representation of the fundamental group. Since there is a natural action of $\mathrm{SL}(n;\mathbb{C})$ on $\mathbb{C}^n$, which is the right multiplication of a matrix with a vector, by using ρ we can view $\mathbb{C}^n$ as a right $\mathbb{Z}[\pi]$-module. Thus we can form the tensor product $C_i^\rho(X) = \mathbb{C}^n \otimes_{\mathbb{Z}[\pi],\rho} C_i(\widetilde{X})$, which is a vector space over $\mathbb{C}$. If the induced chain complex $C_*^\rho(X)$ is acyclic then we can define the torsion $\tau^\rho(X) = \tau(C_*^\rho(X)) \in \mathbb{C}$. Because $\rho(\pi) \subset \mathrm{SL}(n;\mathbb{C})$ the determinant computations will destroy some ambiguities about the choice of representing cells, so that $\tau^\rho(X)$ is defined up to ± 1.

4.2. **Reidemeister torsion of link complements in** $\mathbb{R}\mathrm{P}^3$**.** Let L be a link in $\mathbb{R}\mathrm{P}^3$. In terms of the Euler characteristic, noting $0 = \chi(\mathbb{R}\mathrm{P}^3) = \chi(X \cup N(L)) = \chi(X) + \chi(N(L)) - \chi(X \cap N(L))$, it follows that $\chi(X) = 0$. The complement X is simple homotopic to a 2-dimensional cell complex Y which has one 0-cell σ^0; n 1-cells $\sigma_1^1,\dots,\sigma_n^1$; and m 2-cells $\sigma_1^2,\dots,\sigma_m^2$, where $m = n-1$. The boundary maps are $\partial_1 = 0$ and $\partial_2(\sigma_i^2) = r_i$, where r_i is a word in σ_j^1, giving a presentation of the fundamental group as $\pi = \langle x_1, x_2, \dots, x_n / r_1, r_2, \dots, r_m\rangle$. This presentation is not necessarily the same as the one in Theorem 2.1, however.

Let $\widetilde{Y}$ be the maximal abelian cover of Y. Consider the cellular complexes of $\widetilde{Y}$ as modules over $\mathbb{Z}[H]$. We have a chain complex of $\mathbb{Z}[H]$-modules $C_2(\widetilde{Y}) \xrightarrow{\partial_2} C_1(\widetilde{Y}) \xrightarrow{\partial_1} C_0(\widetilde{Y}) \to 0$. The boundary maps are obtained using Fox's Free Differential Calculus: $\partial_1(\tilde\sigma_i^1) = pr(x_i - 1)\tilde\sigma^0$ and $\partial_2(\tilde\sigma_i^2) = \sum_{j=1}^n pr(\frac{\partial r_i}{\partial x_j})\tilde\sigma_j^1$, where the tilde sign denotes a lift of the cell to $\widetilde{Y}$.

Denote the quotient field $Q(\mathbb{Z}[G])$ of $\mathbb{Z}[G]$ by $\mathbb{Q}(G)$. Using the homomorphism $\varphi : \mathbb{Z}[H] \to \mathbb{Z}[G] \hookrightarrow \mathbb{Q}(G)$, construct the tensor $\mathbb{Q}(G) \otimes_{\mathbb{Z}[H]} C_i(\widetilde{Y})$, considered as a vector space over $\mathbb{Q}(G)$. We have a chain complex of vector spaces over $\mathbb{Q}(G)$:

$$C = (\mathbb{Q}(G) \otimes_{\mathbb{Z}[H],\varphi} C_2(\widetilde{Y}) \xrightarrow{\partial_2} \mathbb{Q}(G) \otimes_{\mathbb{Z}[H],\varphi} C_1(\widetilde{Y}) \xrightarrow{\partial_1} \mathbb{Q}(G) \otimes_{\mathbb{Z}[H],\varphi} C_0(\widetilde{Y}) \to 0).$$

The boundary maps are $[\partial_1]_i = \varphi(x_i) - 1$, and $[\partial_2]_{i,j} = \varphi(\frac{\partial r_j}{\partial x_i})$, $1 \le i \le n$, $1 \le j \le n-1$. Let $A = [\partial_2]^t$.

Denote the columns of A by u_i, $1 \le i \le n$, and denote the $(n-1)\times(n-1)$ matrix obtained from A by omitting the column u_i by A_i. Since C is a chain we have

$0=\partial_1(\partial_2(\tilde\sigma_i^2))=(\sum_{j=1}^n \varphi(\frac{\partial r_i}{\partial x_j})(\varphi(x_j)-1))\tilde\sigma^0$, thus $\sum_{j=1}^n \varphi(\frac{\partial r_i}{\partial x_j})(\varphi(x_j)-1)=0$. This means $\sum_{j=1}^n(\varphi(x_j)-1)u_j=0$. For any $i>j$ we have

$$\begin{aligned}(\varphi(x_j)-1)\det A_i &= \det[u_1,\dots,u_{j-1},(\varphi(x_j)-1)u_j,u_{j+1},\dots,\hat{u}_i,\dots,u_n]\\ &= \det[u_1,\dots,u_{j-1},-\sum_{k\neq j}(\varphi(x_k)-1)u_k,u_{j+1},\dots,\hat{u}_i,\dots,u_n]\\ &= (-1)^{i-j+1}(\varphi(x_i)-1)\det A_j.\end{aligned}$$

Thus for any i and j,

$$(\varphi(x_i)-1)\det A_j=\pm(\varphi(x_j)-1)\det A_i. \tag{4.6}$$

Because H has at least one free generator (Corollary 2.3), the image $\varphi(\pi)$ cannot be $\{1\}$, thus there is at least one x_i such that $\varphi(x_i)\neq 1$. The property $\partial_1(\tilde\sigma_i^1)=(\varphi(x_i)-1)\tilde\sigma^0$ implies $\partial_1(\frac{1}{\varphi(x_i)-1}\tilde\sigma_i^1)=\tilde\sigma^0$, so ∂_1 is onto. Therefore the chain C is exact if and only if ∂_2 is injective, which means the rank of its matrix is exactly $n-1$. Thus C is acyclic if and only if A has a nonzero $(n-1)\times(n-1)$ minor.

The Reidemeister torsion of C with respect to φ is the torsion $\tau^\varphi(Y)$ of Y, and since torsion is a simple homotopy invariant, it is also the torsion $\tau^\varphi(X)$ of X.

For a moment, assume that C is acyclic. Take the standard bases of $\mathbb{Q}(G)\otimes_{\mathbb{Z}[H],\varphi} C_i(\widetilde{Y})$ given by $\tilde\sigma_j^i$ as above. A lift of $c_0=\{\tilde\sigma_0\}$ is $\{\frac{1}{\varphi(x_i)-1}\tilde\sigma_i^1\}$. Then

$$\begin{aligned}\tau^\varphi(X) &= [(\sum_{j=1}^n \varphi(\frac{\partial r_1}{\partial x_j})\tilde\sigma_j^1,\dots,\sum_{j=1}^n \varphi(\frac{\partial r_{n-1}}{\partial x_j})\tilde\sigma_j^1,\frac{1}{\varphi(x_i)-1}\tilde\sigma_i^1)/(\tilde\sigma_1^1,\dots,\tilde\sigma_n^1)]\\ &= \frac{(-1)^{i+n}}{\varphi(x_i)-1}\det A_i.\end{aligned}$$

Thus if $\varphi(x_i)\neq 1$ then $\tau^\varphi(X)=\pm\det A_i/(\varphi(x_i)-1)$. By Eq. (4.6) if $\varphi(x_j)=1$ then $\det(A_j)=0$, hence the following formula is correct for all i, whether C is acyclic or not:

$$(\varphi(x_i)-1)\tau^\varphi(X)=\pm\det A_i\in\mathbb{Q}(G)/\pm G. \tag{4.7}$$

Theorem 4.4. *The Reidemeister torsion and the twisted Alexander polynomial of the complement of a nontorsion link are the same.*

Proof. According to Definition 3.1 and Formula (4.7), we have

$$\Delta^\varphi(X)=\gcd\{\det A_1,\dots,\det A_n\}=\gcd\{(\varphi(x_1)-1)\tau^\varphi(X),\dots,(\varphi(x_n)-1)\tau^\varphi(X)\}.$$

Thus we will get $\Delta^\varphi(X)=\tau^\varphi(X)$ immediately from the following claim.

Claim. $\gcd\{\varphi(x_1)-1,\varphi(x_2)-1,\dots,\varphi(x_n)-1\}=1\in\mathbb{Z}[G]/\pm G$.

To prove the claim we consider two cases.

Case 1: L has one component. In this case $H=\langle t,u/tu=ut,u^2=1\rangle$, $pr(x_i)=t^{m_i}u^{n_i}$, and $\varphi(x_i)=t^{m_i}(-1)^{n_i}$. Let $d=\gcd\{\varphi(x_1)-1,\varphi(x_2)-1,\dots,\varphi(x_n)-1\}\in\mathbb{Z}[t^{\pm1}]$. The following two identities :

$$(t^{m_i}(-1)^{n_i}-1)+t^{m_i}(-1)^{n_i}(t^{m_j}(-1)^{n_j}-1)=t^{m_i+m_j}(-1)^{n_i+n_j}-1,$$

$$(t^{m_i}(-1)^{n_i}-1)-t^{m_i-m_j}(-1)^{n_i-n_j}(t^{m_j}(-1)^{n_j}-1)=t^{m_i-m_j}(-1)^{n_i-n_j}-1,$$

(compare [Lic97, p. 117]) imply that $d|(t^{\sum_{i=1}^n \alpha_i m_i}(-1)^{\sum_{i=1}^n \alpha_i n_i}-1)$ for any $\alpha_i \in \mathbb{Z}$.

Since $t \in pr(\pi)$ there are $\alpha_i \in \mathbb{Z}$ such that $t=\prod_{i=1}^n pr(x_i^{\alpha_i}) = t^{\sum_{i=1}^n \alpha_i m_i} u^{\sum_{i=1}^n \alpha_i}$, which implies $\sum_{i=1}^n \alpha_i m_i = 1$ and $\sum_{i=1}^n \alpha_i$ is even. Thus $d|(t-1)$, hence either $d=1$ or $d=t-1$, up to $\pm t^k, k \in \mathbb{Z}$. Since $u \in pr(\pi)$ there is at least an i_0 such that n_{i_0} is odd, so that $\varphi(x_{i_0})-1=-t^{m_{i_0}}-1$. Since $\gcd\{t-1,-t^{m_{i_0}}-1\}=1$, we conclude that $d=1$.

Case 2: L has at least two components. Let $v \geq 2$ be the number of components. Now $pr(x_i)=t_1^{m_i^1}t_2^{m_i^2}\cdots t_v^{m_i^v}u^{n_i}$ and $\varphi(x_i)=t_1^{m_i^1}t_2^{m_i^2}\cdots t_v^{m_i^v}(-1)^{n_i}$. Letting $t_2=t_3=\cdots=t_v=1$ and applying the argument in Case 1 to t_1 we have the result. □

Theorem 4.5. *If L is a torsion knot and t is the generator of the first homology group then $\tau_L^\varphi(t)=\Delta_L^\varphi(t)/(t-1) \in \mathbb{Z}[t^{\pm 1},(t-1)^{-1}]$. If L is a torsion link with a least two components then the Reidemeister torsion and the twisted Alexander polynomial are the same.*

Proof. The proof is similar to the proof of Theorem 4.4.

Case 1: L has one component. In this case $H=\langle t\rangle$ and $\varphi(x_i)=t^{m_i}$. Using the two identities: $t^{m_i}+t^{m_i}(t^{m_j}-1)=t^{m_i+m_j}-1$, and $(t^{m_i}-1)-t^{m_i-m_j}(t^{m_j}-1)=t^{m_i-m_j}-1$, we get $\gcd\{\varphi(x_i)-1, 1\leq i\leq n\}=t-1$, thus $\Delta^\varphi(X)=(t-1)\tau^\varphi(X)$.

Case 2: L has at least two components. Now H is generated by $t_1,t_2,\ldots,t_v$; $v\geq 2$, and $\varphi(x_i)=t_1^{m_i^1}t_2^{m_i^2}\cdots t_v^{m_i^v}$. By subsequently letting $t_j=1$ for all $j\neq i$ and applying the argument in Case 1 to t_i we obtain $\gcd\{\varphi(x_1)-1,\varphi(x_2)-1,\ldots,\varphi(x_n)-1\}=\gcd\{t_1-1,t_2-1,\ldots,t_v-1\}=1$, hence $\Delta^\varphi(X)=\tau^\varphi(X)$. □

Remark 4.6. With a virtually identical proof, the statement of Theorem 4.5 is true for all links if we replace the twisted map φ by the canonical projection $\mathbb{Z}[H]\to\mathbb{Z}[G]$ and replace the twisted Alexander polynomial by the Alexander polynomial.

4.3. **Comparison with other twisted Alexander polynomials.** Among the first people who studied twisted Alexander polynomials were Lin [Lin01], Wada [Wad94], Kitano [Kit96], Kirk-Livingston [KL99]. Except for Lin's construction which used Seifert surfaces, other constructions were based on that of Wada. The twisted Alexander polynomial in the form considered in this paper was defined first by Turaev in [Tur02a] and was discussed further in [Tur02b, p. 27]. It receives attention recently in [HP05]. We outline Wada's construction to show its relationship with our polynomial.

Suppose $\pi=\langle x_1,\ldots,x_m/r_1,\ldots,r_{m-1}\rangle$ is a presentation of deficiency one, and let $\rho:\pi\to \mathrm{GL}(n;\mathbb{F})$ be a representation of π. Let $\alpha:\pi\to G\cong\langle t_1,t_2,\ldots,t_v/t_it_j=t_jt_i\rangle\cong\mathbb{Z}^v$ be a surjective group homomorphism. Define a ring homomorphism $\phi:\mathbb{Z}[\pi]\to M(n;\mathbb{F}[G])$ by letting $\phi(x)=\alpha(x)\rho(x)$ for $x\in\pi$ then extend linearly (or equivalently one first extends α linearly to a ring homomorphism $\tilde{\alpha}:\mathbb{Z}[\pi]\to\mathbb{Z}[G]$, and extends ρ linearly to a ring homomorphism $\tilde{\rho} : \mathbb{Z}[\pi] \to M(n;\mathbb{F})$, then let $\phi = \tilde{\alpha} \otimes \tilde{\rho}$).

Consider the $(m-1) \times m$ matrix M whose the (i,j) entry is $\phi(\partial r_i/\partial x_j) \in M(n;\mathbb{F}[G])$. Let M_j be the $(m-1)\times(m-1)$ matrix obtained from M by removing the j-th column. View M_j as an $n(m-1) \times n(m-1)$ matrix whose entries are in $\mathbb{F}[G]$. Supposing that $\phi(x_j) \neq I$, we define the twisted Alexander polynomial as $\Delta^{\rho}(X) = \det M_j / \det \phi(1-x_j) \in \mathbb{F}(G) = Q(\mathbb{F}[G])$. Wada proved that this polynomial is independent of the choice of j and the choice of a presentation of π, and is defined up to a factor in $\pm G$.

Let us compare Wada's polynomial with that of Turaev. Fix a splitting $H = G \times \operatorname{Tors} H$. Suppose that $\varphi \in \operatorname{Hom}(\operatorname{Tors} H, \mathbb{C}^*)$ is given. Let α be as above, and ρ be the composition of the maps $\pi \to H \xrightarrow{\beta} \operatorname{Tors} H \xrightarrow{\varphi} \{\pm 1\} \subset \mathrm{GL}(1;\mathbb{C})$; here the first arrow is the canonical projection map, and β maps an element $gh \in H$ where $g \in G$ and $h \in \operatorname{Tors} H$ to h. Then $\phi = \tilde{\alpha} \otimes \tilde{\rho}$ is exactly the twisted map in Section 3.1. Thus , in view of Formula (4.7) $\Delta^{\rho}(X)$ here is exactly the torsion $\tau^{\varphi}(X)$, and its relationships with Turaev's polynomial are provided in Theorems 4.4 and 4.5. Unlike the general case Turaev's invariant is still abelian.

Remark 4.7. Milnor proved in [Mil62] the identification between Alexander polynomial and Reidemeister torsion for knot complements in S^3. Kitano [Kit96] proved the identification between Wada's twisted Alexander polynomial and Reidemeister torsion, also for knot complements in S^3. Kirk–Livingston [KL99] generalized this result to CW-complex, but considered only a one variable twisted Alexander polynomial associated with an infinite cyclic cover of the complex. Turaev [Tur02b, p. 28] has also studied this problem. The elementary proof above of Theorem 4.4 is close to Milnor's original one.

5. A skein relation for the twisted Alexander polynomial

5.1. **The one variable twisted Alexander polynomial.** Let L be a link with v components, and let φ' be the composition of φ with the canonical projection from $\mathbb{Z}[t_1^{\pm 1}, \dots, t_v^{\pm 1}]$ to $\mathbb{Z}[t^{\pm 1}]$. The twisted Alexander polynomial $\Delta^{\varphi}(X)$ of the complement X is a polynomial in v variables $t_1, t_2, \dots, t_v$. The one variable polynomial $\Delta^{\varphi'}(X)$ is obtained from $\Delta^{\varphi}(X)$ by replacing φ by φ'.

We write $\mathbb{Q}(t) = Q(\mathbb{Z}[t, t^{-1}])$, $\Delta^{\varphi'}(X)$ as $\Delta_L^{\varphi'}(t)$, and $\tau^{\varphi'}(X)$ as $\tau_L^{\varphi'}(t)$.

The proofs of the following two theorems are identical to the proofs for the cases of knots of Theorems 4.4 and 4.5.

Theorem 5.1. *If L is a nontorsion link then the Reidemeister torsion $\tau_L^{\varphi'}(t)$ and the one variable twisted Alexander polynomial $\Delta_L^{\varphi'}(t)$ are the same.*

Theorem 5.2. *If L is a torsion link then the Reidemeister torsion and the one variable twisted Alexander polynomial are related by the formula* $\tau_L^{\varphi'}(t) = \Delta_L^{\varphi'}(t)/(t-1) \in \mathbb{Z}[t^{\pm 1}, (t-1)^{-1}]$.

As a consequence of the symmetry of torsion (Section 4.1.4), we have:

Theorem 5.3. *The Reidemeister torsion $\tau_L^{\varphi'}(t)$ is symmetric, that is $\tau_L^{\varphi'}(t^{-1}) = \tau_L^{\varphi'}(t)$ up to $\pm t^n, n \in \mathbb{Z}$, as elements in $\mathbb{Q}(t)$.*

From this we derive:

Theorem 5.4. *The one variable twisted Alexander polynomial is symmetric, that is $\Delta_L^{\varphi'}(t^{-1}) = \Delta_L^{\varphi'}(t)$ up to $\pm t^n, n \in \mathbb{Z}$, as elements in $\mathbb{Z}[t^{\pm 1}]$.*

Proof. If L is a nontorsion link then according to Theorem 5.1, $\Delta_L^{\varphi'}(t^{-1}) = \tau_L^{\varphi'}(t^{-1}) = \tau_L^{\varphi'}(t) = \Delta_L^{\varphi'}(t^{-1})$. If L is a torsion link then according to Theorem 5.2, $\Delta_L^{\varphi'}(t^{-1}) = (t^{-1}-1)\tau_L^{\varphi'}(t^{-1}) = t^{-1}(1-t)\tau_L^{\varphi'}(t) = \Delta_L^{\varphi'}(t)$, up to $\pm t^n, n \in \mathbb{Z}$. □

5.2. **A skein relation for torsion with indeterminacies.** Let L be an oriented link. Consider a crossing of L. Let B be an open 3-ball that encloses this crossing and intersects L at four points. Let $V = \mathbb{R}\mathrm{P}^3 \setminus (B \cup \overset{\circ}{N}(L))$, see Fig. 6.

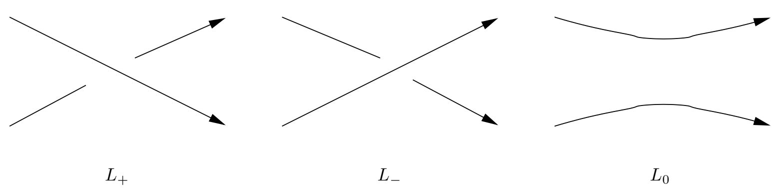


FIGURE 5. The links L_+, L_-, L_0 are identical except at one crossing.

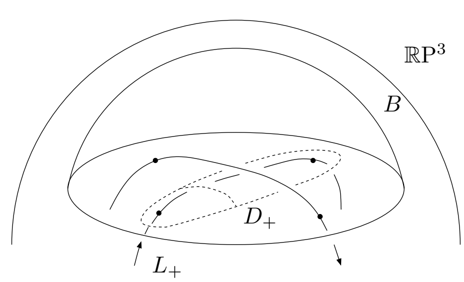


FIGURE 6. The disk D_+ at a crossing of the link L_+.

Take a triangulation of V. There is a deformation retraction (actually a simple homotopy) of the complement of L_α, $\alpha \in \{+, -, 0\}$, onto $X_\alpha = V \cup D_\alpha$, where D_α is a disk glued to ∂V along a simple loop ∂D_α circling two intersection points of B

and L_α as in Fig. 7 such that $V \cup D_\alpha$ has a cell decomposition consists of the cells of V plus the disk D_α. We can assume that the loops ∂D_α's have a common point.

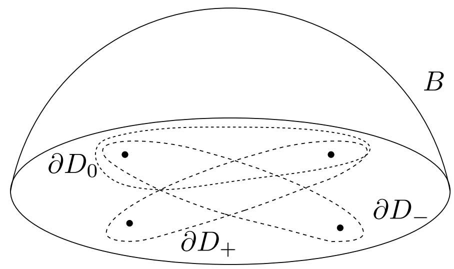


FIGURE 7. The curves ∂D_α's.

5.2.1. *Smoothing of crossings and torsion classes.* When the smoothing operation is done at a particular crossing the link L_0 may no longer be in the same torsion class with L_+ and L_- (recall Section 2.3.1). The three links L_+, L_- and L_0 are in the same torsion class in the following cases:

(1) There is one component of L_+ which is not involved at the crossing that is not null-homologous. In this case L_+, L_- and L_0 are all torsion links.
(2) The two strands of L_+ at the crossing come from one component, and after smoothing all components are null-homologous. In this case L_+, L_- and L_0 are all nontorsion links (cf. Fig. 9).
(3) The two strands of L_+ at the crossing come from two different components, and before smoothing all components are null-homologous. In this case L_+, L_- and L_0 are all nontorsion links (cf. Fig. 11).

In what follows we will need the condition that the links L_α's belong to the same torsion class, so that $\operatorname{Tors} H_1(X_\alpha)$'s are the same. Therefore throughout the rest of Section 5 we will assume that this condition is satisfied at the crossing under consideration.

5.2.2. *The chain complexes C_α and C.* Fix an $\alpha \in \{+,-,0\}$. Let $\widetilde{X}_\alpha$ be the $D = \mathbb{Z} \times \operatorname{Tors} H_1(X_\alpha)$ cover of X_α corresponding to the kernel of the map $proj_\alpha$: $\pi_1(X_\alpha) \to H_1(X_\alpha) \to G \times \operatorname{Tors} H_1(X_\alpha) \to \{t^m : m \in \mathbb{Z}\} \times \operatorname{Tors} H_1(X_\alpha)$. Let $\widetilde{V}$ be the inverse image of V under the covering map. The triangulation of V induces a CW-complex structure on $\widetilde{V}$.

Under the condition that L_α's are in the same torsion class we can construct $\widetilde{X}_\alpha$ in a different way as follows. Take $\widetilde{V}$ to be the cover of V corresponding to the kernel of the projection $\pi_1(V) \to H_1(V) \to G \times \operatorname{Tors} H_1(V) \to \{t^m : m \in \mathbb{Z}\} \times \operatorname{Tors} H_1(V)$. Noting that $\operatorname{Tors} H_1(V) = \operatorname{Tors} H_1(X_\alpha)$ for $\alpha = +,-,0$, we construct $\widetilde{X}_\alpha$ from $\widetilde{V}$ by gluing $|\mathbb{Z} \times \operatorname{Tors} H_1(X_\alpha)|$ copies of D_α along the lifts of $\partial D_\alpha \subset V$.

Consider the ring homomorphism $\varphi' : \mathbb{Z}[\{t^m : m \in \mathbb{Z}\} \times \operatorname{Tors} H_1(X_\alpha)] \to \mathbb{Z}[t^{\pm 1}] \hookrightarrow \mathbb{Q}(t)$, which does not depend on α. Let $C_\alpha = \mathbb{Q}(t) \otimes_{\mathbb{Z}[\mathbb{Z}\times \operatorname{Tors} H_1(X_\alpha)],\varphi'} C_*(\widetilde{X}_\alpha;\mathbb{Z})$ and let $C = \mathbb{Q}(t) \otimes_{\mathbb{Z}[\mathbb{Z}\times \operatorname{Tors} H_1(X_\alpha)],\varphi'} C_*(\widetilde{V};\mathbb{Z})$, both considered as chain complexes of $\mathbb{Q}(t)$-vector spaces. Note that C does not depend on α.

5.2.3. *Relations among $\tau(C_\alpha)$'s.* Since $C_i(\widetilde{V};\mathbb{Z}) \hookrightarrow C_i(\widetilde{X}_\alpha;\mathbb{Z})$ is an inclusion, the induced map $\mathbb{Q}(t) \otimes_{\varphi'} C_*(\widetilde{V};\mathbb{Z}) \hookrightarrow \mathbb{Q}(t) \otimes_{\varphi'} C_*(\widetilde{X}_\alpha;\mathbb{Z})$ is injective, and we have the short exact sequence of chain complexes of $\mathbb{Q}(t)$-vector spaces

$$0 \to C \to C_\alpha \to C_\alpha/C \to 0. \tag{5.1}$$

Choose a fundamental family of cells for $\widetilde{V}$ providing a basis for the chain C. A fundamental family of cells of $\widetilde{X}_\alpha$ is obtained from the one of $\widetilde{V}$ by adding a lift of D_α. We can choose these lifts D_α so that the loops $\partial \widetilde{D}_\alpha$ have a common point in $\widetilde{V}$, which is a lift of the common point of ∂D_α in V. Recalling that X_α is the result of gluing the disk D_α to V, we observe that only the second homology group of C_α/C is non-trivial, and the torsion of C_α/C with homology bases is $\tau(C_\alpha/C, h) = 1$ up to a sign.

Suppose that the chain complex C_α is acyclic. The product formula for torsion (4.3) applied to the short exact sequence (5.1) gives:

$$\tau(C_\alpha) = \pm\tau(C,h)\tau(C_\alpha/C,h)\tau(\mathcal{H}_\alpha) = \pm\tau(C,h)\tau(\mathcal{H}_\alpha), \tag{5.2}$$

where $\mathcal{H}_\alpha$ denotes the long exact homological sequence of the pair (C_α, C), with a chosen basis: $\mathcal{H}_\alpha = (\cdots \to H_i(C) \to H_i(C_\alpha) \to H_i(C_\alpha/C) \to H_{i-1}(C) \to \cdots \to H_0(C) \to H_0(C_\alpha) \to H_0(C_\alpha/C) \to 0)$. Since C_α is exact the sequence $\mathcal{H}_\alpha$ is reduced to $0 \to H_2(C_\alpha/C) \xrightarrow{\partial} H_1(C) \to 0 \to 0 \to 0 \to 0 \to 0 \to 0$, so $H_1(C) \cong H_2(C_\alpha/C) \cong \mathbb{Q}(t)$ and $\tau(\mathcal{H}_\alpha) = \det(\partial)$. Let y be the chosen basis of the one-dimensional $\mathbb{Q}(t)$-vector space $H_1(C)$. Then $\partial[\widetilde{D}_\alpha] = [\partial\widetilde{D}_\alpha] = \gamma_\alpha y$ for some $\gamma_\alpha \in \mathbb{Q}(t)$. Formula (5.2) now gives

$$\tau(C_\alpha) = \pm\gamma_\alpha\tau(C,h). \tag{5.3}$$

As can be seen from the proof the product formula in [Mil66, p. 160] or from the corresponding formula for sign-refined torsion (4.4), the sign $\pm$ in (5.3) above depends only on the ranks of the vector spaces in the chains C_α, C and $\mathcal{H}_\alpha$, thus does not depend on α (to the extent that C_α is assumed to be acyclic).

Under the assumption that there is at least one $\alpha_0 \in \{+,-,0\}$ such that C_{α_0} is acyclic, we show that (5.3) above still holds when C_α is not acyclic. When C_α is not acyclic, by definition $\tau(C_\alpha) = 0$. We will show that γ_α is zero, i.e. the boundary map $\partial : H_2(C_\alpha/C) \to H_1(C)$ is zero. Suppose the contrary, $\gamma_\alpha \neq 0$. Because $H_1(C) \cong H_2(C_{\alpha_0}/C) \cong \mathbb{Q}(t) \cong H_2(C_\alpha/C)$, if ∂ is not zero it must be a bijection. The long exact sequence $\mathcal{H}_\alpha$ shows that $H_1(C_\alpha) = 0$. Note that $\operatorname{rank}(\mathbb{Q}(t) \otimes_{\mathbb{Z}[\mathbb{Z}\times \operatorname{Tors} H_1(X_\alpha)],\varphi'} C_i(\widetilde{X}_\alpha,\mathbb{Z}))$ is exactly the number of i-cells of X_α.

This implies that $0 = \chi(X_\alpha) = \chi(C_\alpha) = \text{rank}(H_0(C_\alpha)) + \text{rank}(H_2(C_\alpha))$. Thus $H_0(C_\alpha) = H_1(C_\alpha) = H_2(C_\alpha) = 0$ i.e. C_α is acyclic, a contradiction.

5.2.4. *Relations among γ_α's.* In view of (5.3) to further study relations among $\tau(C_\alpha)$'s we now try to find a relation among γ_α's. Recall that $\gamma_\alpha \in H_1(C)$ is represented by the loop $\partial\widetilde{D}_\alpha$. Let a, b, c, d be simple meridian loops with a common base point, circling the four intersection points between L and B as in Fig. 8. The

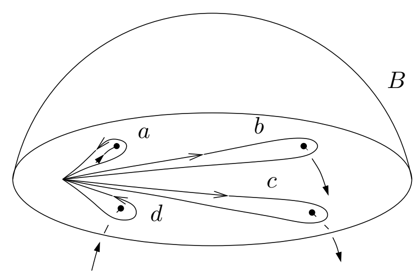


FIGURE 8. The loops a, b, c and d.

boundary of the disks D_α's are: $\partial D_+ = bd^{-1}$, $\partial D_- = a^{-1}c$, and $\partial D_0 = a^{-1}b$. Under the the map $proj_\alpha$ in Section 5.2.2, all of a, b, c, d are projected to t. Noting that $a^{-1}bcd^{-1} = 1$, we have $\tilde{d} = -t^{-1}\tilde{a} + t^{-1}\tilde{b} + \tilde{c}$. Hence $\gamma_0 y = \widetilde{a^{-1}b} = t^{-1}(-\tilde{a} + \tilde{b})$, $\gamma_- y = \widetilde{a^{-1}c} = t^{-1}(-\tilde{a} + \tilde{c})$, and $\gamma_+ y = \widetilde{bd^{-1}} = \tilde{b} - \tilde{c} + t^{-1}(\tilde{a} - \tilde{b}) = (t-1)\gamma_0 y - t\gamma_- y$. So in $\mathbb{Q}(t)$:

$$\gamma_+ + (1-t)\gamma_0 + t\gamma_- = 0. \tag{5.4}$$

Formulas (5.3) and (5.4) now give us, under the assumption that there is at least one $\alpha_0 \in \{+, -, 0\}$ such that C_{α_0} is acyclic, the formula $\tau(C_+) + (1-t)\tau(C_0) + t\tau(C_-) = 0$. But this formula is also trivially correct when none of the C_α are acyclic, since in that case all three torsions are zero. Thus we obtain the following theorem:

Theorem 5.5. *If L_+, L_- and L_0 belong to the same torsion class then*

$$\tau^{\varphi'}_{L_+}(t) + (1-t)\tau^{\varphi'}_{L_0}(t) + t\tau^{\varphi'}_{L_-}(t) = 0. \tag{5.5}$$

5.3. Sign-refined torsion and a normalized one variable twisted Alexander function.

5.3.1. *A skein relation for sign-refined torsion.* We consider sign-refined torsion, see Section 4.1.5. In all that follow the bases for the chain complexes are induced from the triangulations of the spaces as previously mentioned at the beginning of Section 5.2. There are two cases:

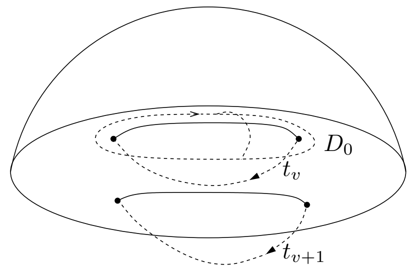


FIGURE 9. Case 1.

Case 1: The two strands of L_+ at the crossing come from the same component. See Fig. 9. Suppose that the crossing involves the v-th component of L_+. The bases h_α for $H_*(X_\alpha;\mathbb{R})$, $\alpha = +,-$ consist of $[pt]$, $t_1,\dots,t_v$, $q_1,\dots,q_{v-1}$, where q_i represents the i-th boundary component of L_+ and t_i represent the (oriented) meridian of this component. The basis for $H_*(X_0;\mathbb{R})$ consists of $[pt]$, $t_1,\dots,t_{v+1}$, $q_1,\dots,q_v$. The basis h_0 for $H_*(V;\mathbb{R})$ consists of $[pt]$, $t_1,\dots,t_{v+1}$, $q_1,\dots,q_{v-1}$.

We want to compare the terms $\check{\tau}(C_*(X_\alpha,\mathbb{R}),c_\alpha,h_\alpha))$. Consider the short exact sequence of chain complexes: $0\to C_*(V;\mathbb{R})\to C_*(X_\alpha;\mathbb{R})\to C_*(X_\alpha,V;\mathbb{R})\to 0$. Applying the product formula for sign-refined torsion (4.4) we obtain

$$\check{\tau}(C_*(X_\alpha;\mathbb{R})) = (-1)^{\mu_\alpha+\nu}\check{\tau}(C_*(V;\mathbb{R}))\check{\tau}(C_*(X_\alpha,V;\mathbb{R}))\tau(\mathcal{H}_\alpha),$$

where $\mathcal{H}_\alpha$ is the long exact homological sequence of the pair (X_α,V) with real coefficients, and

$$\mu_\alpha = \sum[(\beta_i(C_*(X_\alpha;\mathbb{R}))+1)(\beta_i(C_*(V;\mathbb{R}))+\beta_i(C_*(X_\alpha,V;\mathbb{R})))+ \\ +\beta_{i-1}(C_*(V;\mathbb{R}))\beta_i(C_*(X_\alpha,V;\mathbb{R}))] \mod 2$$

and $\nu = \sum_{i=0}^m \gamma_i(C_*(X_\alpha,V;\mathbb{R}))\gamma_{i-1}(C_*(V;\mathbb{R})) \mod 2$. Notice that ν does not depend on α.

Since the term $\check{\tau}(C_*(V;\mathbb{R}))\check{\tau}(C_*(X_\alpha,V;\mathbb{R}))$ does not depend on α we only need to compare the terms $(-1)^{\mu_\alpha}\operatorname{sign}(\tau(\mathcal{H}_\alpha))$. Straightforward calculations show that $\mu_+\equiv\mu_-\equiv\mu_0+v \pmod 2$. Because $H_1(X_\alpha,V;\mathbb{R})=0$, the chain complex $\mathcal{H}_\alpha$ has two portions: $0\to H_0(V;\mathbb{R})\to H_0(X_\alpha;\mathbb{R})\to H_0(X_\alpha,V;\mathbb{R})\to 0$, and

$$0\to H_2(V;\mathbb{R})\to H_2(X_\alpha;\mathbb{R})\to H_2(X_\alpha,V;\mathbb{R})\to H_1(V;\mathbb{R})\to H_1(X_\alpha;\mathbb{R})\to 0.$$

For the purpose of comparison we only need to look at the second portion.

When $\alpha=+$: Recalling that $\dim(H_2(V;\mathbb{R}))$ is the same as $\dim(H_2(X_\alpha;\mathbb{R}))$, we see that the torsion of $\mathcal{H}_+$ is the torsion of the chain $0\to H_2(X_\alpha,V;\mathbb{R})\xrightarrow{\partial} H_1(V;\mathbb{R})\to H_1(X_\alpha;\mathbb{R})\to 0$. Since $[\partial D_+]=[bd^{-1}]=t_v-t_{v+1}$, it follows that $\tau(\mathcal{H}_+)$ is the determinant of the change of bases matrix $[(t_v-t_{v+1},t_1,\dots,t_v)/(t_1,\dots,t_v,t_{v+1})]$, which is $(-1)^{v+1}$.

When $\alpha = -$: In this case $[\partial D_-] = [a^{-1}c] = t_{v+1} - t_v$, thus $\tau(\mathcal{H}_-) = (-1)^v$.

When $\alpha = 0$: The torsion $\tau(\mathcal{H}_0)$ is the torsion of the chain $0 \to H_2(V;\mathbb{R}) \xrightarrow{i_*} H_2(X_0;\mathbb{R}) \xrightarrow{j_*} H_2(X_0, V;\mathbb{R}) \to 0$. The map i_* is an injection, $i_*(q_i) = q_i,\ 1 \le i \le v-1$. The disk D_0 is a representative of a generator of $H_2(X_0, V;\mathbb{R})$. We need to take a lift of $[D_0]$ under the map j_*. The union of D_0 with part of the boundary of V constitutes either one of the two boundary components of X_0 corresponding to q_v and q_{v+1}. See Fig. 10, in which the solid two holes torus contains the ball B and the disk D_0, while V is outside. Because of the chosen orientation of ∂D_0

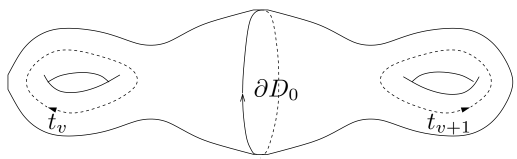


FIGURE 10. The disk D_0.

the two corresponding elements in $H_2(X_0)$, which are lifts of $[D_0]$ under j_*, are $-q_v$ and $q_{v+1} = -(q_1 + q_2 + \cdots + q_v)$. The choice of either lift would result that $\tau(\mathcal{H}_0) = [(q_1, \ldots, q_{v-1}, -q_v)/(q_1, \ldots, q_v)] = -1$.

Collecting the above computations and comparisons of μ_α and $\tau(\mathcal{H}_\alpha)$ we conclude that $\check{\tau}(C_*(X_+,\mathbb{R})) = -\check{\tau}(C_*(X_-,\mathbb{R})) = \check{\tau}(C_*(X_0,\mathbb{R}))$.

Case 2: The two strands of L_+ at the crossing come from different components. See Fig. 11. Similar to Case 1, the comparison of $\check{\tau}(C_*(X_\alpha;\mathbb{R}))$ is reduced to the

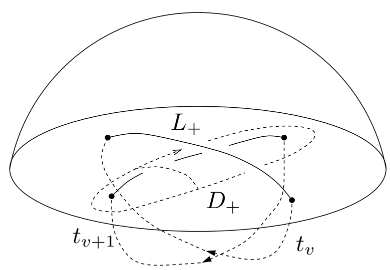


FIGURE 11. Case 2.

comparison of $(-1)^{\mu_\alpha}\mathrm{sign}(\tau(\mathcal{H}_\alpha))$. Straightforward calculations give that $\mu_+ \equiv \mu_- \equiv \mu_0 + v \pmod 2$. Again to study $\tau(\mathcal{H}_\alpha)$ we only need to pay attention to the exact chain complex

$$0 \to H_2(V;\mathbb{R}) \to H_2(X_\alpha;\mathbb{R}) \to H_2(X_\alpha, V;\mathbb{R}) \to H_1(V;\mathbb{R}) \to H_1(X_\alpha;\mathbb{R}) \to 0.$$

When $\alpha = +$: $\tau(\mathcal{H}_+)$ is the torsion of the chain $0 \to H_2(V;\mathbb{R}) \to H_2(X_+;\mathbb{R}) \to H_2(X_+, V;\mathbb{R}) \to 0$. The lift of $[D_+] \in H_2(X_+, V;\mathbb{R})$ to $H_2(X_+;\mathbb{R})$ is either q_v or $-q_{v+1}$. With either lift the we have $\tau(\mathcal{H}_+) = [(q_1, \ldots, q_{v-1}, q_v)/(q_1, \ldots, q_v)] = 1$.

When $\alpha = -$: Just as the case $\alpha = +$, except that now the lift of $[D_-]$ can be either $-q_v$ or q_{v+1}, so $\tau(\mathcal{H}_-) = -1$.

When $\alpha = 0$: $\tau(\mathcal{H}_0)$ is the torsion of the chain $0 \to H_2(X_0, V; \mathbb{R}) \xrightarrow{\partial} H_1(V; \mathbb{R}) \to H_1(X_0; \mathbb{R}) \to 0$. Since $[\partial D_0] = [a^{-1}b] = t_{v+1} - t_v \in H_1(V; \mathbb{R})$ we have $\tau(\mathcal{H}_0) = [(t_{v+1} - t_v, t_1, \ldots, t_v)/(t_1, \ldots, t_{v+1})] = (-1)^v$.

Thus as in Case 1, $\check{\tau}(C_*(X_+, \mathbb{R})) = -\check{\tau}(C_*(X_-, \mathbb{R})) = \check{\tau}(C_*(X_0, \mathbb{R}))$.

Now Formula (4.1) and the skein relation for unrefined torsion (5.5) give us a skein relation for sign-refined torsion:

$$\tau^{\varphi'}_{0,\ L_+}(t) + (1-t)\tau^{\varphi'}_{0,\ L_0}(t) - t\tau^{\varphi'}_{0,\ L_-}(t) = 0, \tag{5.6}$$

provided that L_+, L_- and L_0 belong to the same torsion class.

5.3.2. *Definition of the normalized one variable twisted Alexander function.* For a given link L the sign-refined torsion $\tau^{\varphi'}_{0,\ L}(t)$ is defined up to t^n, $n \in \mathbb{Z}$. Using Theorem 5.3, there is a number $r \in \mathbb{Z}$ arising from the symmetry of (un-refined) torsion such that $\tau^{\varphi'}_{0,\ L}(t^{-1}) = \pm t^r \tau^{\varphi'}_{0,\ L}(t)$ as elements in $\mathbb{Q}(t)$.

Define the normalized twisted Alexander function of a link L to be

$$\nabla_L(t) = -t^r \tau^{\varphi'}_{0,\ L}(t^2). \tag{5.7}$$

Notice that $\nabla_L(t^{-1}) = -t^{-r}\tau^{\varphi'}_{0,\ L}(t^{-2}) = \pm t^{-r}t^{2r}\tau^{\varphi'}_{0,\ L}(t^2) = \pm t^r \tau^{\varphi'}_{0,\ L}(t^2) = \pm\nabla_L(t)$. Thus $\nabla_L(t)$ is symmetric, up to a sign. From Theorems 5.2 and 5.1, the function $\nabla_L(t)$ is an element of $\mathbb{Z}[t^{\pm 1}]$ (a Laurent polynomial) if L is nontorsion, and is an element of $\mathbb{Z}[t^{\pm 1}, (t-t^{-1})^{-1}]$ (a Laurent polynomial divided by $(t-t^{-1})^n$) if L is torsion.

Proposition 5.6. *The function $\nabla_L(t)$ does not depend on the choice of a representative of $\tau^{\varphi'}_{0,\ L}(t)$ and so is completely defined without indeterminacies.*

Proof. Suppose that τ and τ' are two representatives of the (sign-refined) torsion $\tau^{\varphi'}_{0,\ L}$. Then $\tau'(t) = t^m \tau(t)$ for some $m \in \mathbb{Z}$. This implies that there is an $n \in \mathbb{Z}$ such that $\nabla'(t) = t^n \nabla(t)$. Since $\nabla(t^{-1}) = \pm\nabla(t)$ and $\nabla'(t^{-1}) = \pm\nabla'(t)$ we must have $n = 0$, that is $\nabla'(t) = \nabla(t)$. □

5.4. **A skein relation for the normalized twisted Alexander function.**

Theorem 5.7. *If L_+, L_- and L_0 belong to the same torsion class then the normalized one variable twisted Alexander function satisfies the skein relation:*

$$\nabla_{L+}(t) - \nabla_{L_-}(t) = (t - t^{-1})\nabla_{L_0}(t). \tag{5.8}$$

Proof. Replacing t by t^2 in Eq. (5.6), and using Eq. (5.7) we have

$$t^{-r_+}\nabla_{L_+}(t) + (1-t^2)t^{-r_0}\nabla_{L_0}(t) - t^{2-r_-}\nabla_{L_-}(t) = 0,$$

that is

$$\nabla_{L_+}(t) = (t - t^{-1})t^{1+r_+-r_0}\nabla_{L_0}(t) + t^{2+r_+-r_-}\nabla_{L_-}(t).$$

Let $u = 2 + r_+ - r_-$ and $v = 1 + r_+ - r_0$ we get

$$\nabla_{L_+}(t) = (t - t^{-1})t^{v}\nabla_{L_0}(t) + t^{u}\nabla_{L_-}(t). \tag{5.9}$$

The purpose of the rest of the proof is to show that $u = v = 0$. The idea is to show that u and v are independent of the link. This is achieved by studying the numbers r_α's. Since these numbers arise from the symmetry of torsion, a study of duality of torsion is needed.

Topologically the complement X_α of L_α is the union of V and a 2-handle H_α glued to V along the loop ∂D_α. Assume that X_α is triangulated by a triangulation of V together with a compatible triangulation of H_α. Let $\widetilde{X}_\alpha$ be the $D = \mathbb{Z} \times \operatorname{Tors} H_1(X_\alpha)$ cover of X_α corresponding to the kernel of the map $proj_\alpha : \pi_1(X_\alpha) \to H_1(X_\alpha) \to G \times \operatorname{Tors} H_1(X_\alpha) \to \{t^m : m \in \mathbb{Z}\} \times \operatorname{Tors} H_1(X_\alpha)$. As in Section 5.2.2, $\widetilde{X}_\alpha$ can be constructed as $\widetilde{V} \cup_{t\in D} t\widetilde{H}_\alpha$, i.e. $\widetilde{V}$ with disjoint copies of H_α glued in along the lifts of ∂D_α. Because of our assumption that L_α's belong to the same torsion class, the deck transformation group D does not depend on α. An induced triangulation Y_α of $\widetilde{X}_\alpha$ is obtained, which is equivariant under the action of D. Let Y^*_α be its dual cell decomposition and ∂Y^*_α be the restriction of Y^*_α to the boundary $\partial\widetilde{X}_\alpha$.

Let $E_\alpha = \mathbb{Q}(t) \otimes_{\varphi'} C_*(Y_\alpha)$, $F_\alpha = \mathbb{Q}(t) \otimes_{\varphi'} C_*(Y^*_\alpha)$, $\partial F_\alpha = \mathbb{Q}(t) \otimes_{\varphi'} C_*(\partial Y^*_\alpha)$.

Choose a fundamental family of cells e_α for Y_α such that all the cells in e_α that cover a cell in H_α are contained in the same $\widetilde{H}_\alpha$. Denote by e^*_α the family of cells in Y^*_α that are dual to the simplexes in e_α.

The proof consists of the following steps.

Step 1: Studying $\tau(F_\alpha)$. The triangulation Y_α and its dual cell decomposition Y^*_α has a common cellular subdivision, namely the first barycentric subdivision Y'_α of Y_α. It is possible to choose two fundamental family of cells for $\widetilde{X}_\alpha$ corresponding to Y'_α. The first is a, consisting of the cells $a_1, a_2, \ldots, a_n$, each of which is contained in a cell in e_α. This provides a chosen basis for E_α. The second fundamental family of cells is b, consisting of the cells $b_1, b_2, \ldots, b_n$, each of which is contained in a cell in e^*_α, providing a chosen basis for F_α.

Using invariance of torsion under cellular subdivision (see [Tur86, Lemma 4.3.3 iii]) we have $\tau(E_\alpha, e_\alpha) = \pm\tau(\mathbb{Q}(t) \otimes_{\varphi'} C_*(Y'_\alpha), a)$ and $\tau(F_\alpha, e^*_\alpha) = \pm\tau(\mathbb{Q}(t) \otimes_{\varphi'} C_*(Y'_\alpha), b)$. Let us compare the torsion of the same chain complex $\mathbb{Q}(t) \otimes_{\varphi'} C_*(Y'_\alpha)$ with different bases a and b.

We have $\tau(\mathbb{Q}(t) \otimes_{\varphi'} C_*(Y'_\alpha), b) = \tau(\mathbb{Q}(t) \otimes_{\varphi'} C_*(Y'_\alpha), a)\varphi'([b/a])$, where $[b/a] \in D$ denotes the determinant of the change of base matrix. If two cells a_i and b_j cover the same cell in the 2-handle H_α then they must be contained in the same $\widetilde{H}_\alpha$

because of our choice for e_α above, and so a_i and b_j must be the same cell. This means that the correctional term $\varphi'([b/a])$ does not depend on α.

Thus there is $\beta \in \mathbb{Z}$ which does not depend on α such that

$$\tau(F_\alpha, e_\alpha^*) = \pm t^\beta \tau(E_\alpha, e_\alpha) = \pm t^\beta \tau^{\varphi'}_{L_\alpha}(t). \tag{5.10}$$

Step 2: Studying the chain ∂F_α. Consider the short exact sequence of chain complexes

$$0 \to \partial F_\alpha \to F_\alpha \to F_\alpha/\partial F_\alpha \to 0. \tag{5.11}$$

Note that ∂X_α is a collection of tori. It is simple to see that the chain ∂F_α is exact and its torsion – the torsion of a collection of tori – is 1 up to $\pm t^n$.

The long homological exact sequence associated with the short exact sequence (5.11) above shows that F_α is exact if and only if $F_\alpha/\partial F_\alpha$ is exact. Note that by the invariance of torsion under cellular subdivisions, F_α is exact if and only if E_α is exact, and in any case $\tau(F_\alpha) = \tau(E_\alpha)$ up to $\pm t^n$. The product formula for torsion of chain complexes applied to the short exact sequence (5.11) gives

$$\tau(F_\alpha) = \pm\tau(\partial F_\alpha)\tau(F_\alpha/\partial F_\alpha). \tag{5.12}$$

Both sides are zero when F_α is not exact.

Let R be the union of those tori of ∂X_α which do not involve the crossing, i.e. $R \cap B = \emptyset$, where B is the ball enclosing the crossing under scrutiny as in Fig. 6. Then $\partial X_\alpha \setminus R$ is a disjoint union of two tori if the two strands at the crossing belong to different components of the link or it is just a torus if the two strands belong to the same component.

Let $P = \mathbb{Q}(t) \otimes_{\varphi'} C_*(\partial Y^*|_R)$ and $Q_\alpha = \mathbb{Q}(t) \otimes_{\varphi'} C_*(\partial Y^*|_{\partial X_\alpha \setminus R})$. Then $\partial F_\alpha = P \oplus Q_\alpha$. Note that ∂F_α, P and Q_α are all acyclic chain complexes. The torsion of P does not depend on α and is 1 up to units: $\tau(P) = \pm t^p$ for some $p \in \mathbb{Z}$, on the other hand $\tau(Q_\alpha) = \pm t^{q_\alpha}$ for some $q_\alpha \in \mathbb{Z}$. The number q_α depends on how the lifting cells are chosen. It depends only on whether the two strands at the crossing under investigation belong to the same component or two different components of the link L_α. The product formula gives us $\tau(\partial F_\alpha) = \pm\tau(P)\tau(Q_\alpha) = \pm t^{p+q_\alpha}$.

Step 3: Studying $\tau(F_\alpha/\partial F_\alpha)$. By the symmetry of torsion (Section 4.1.4), $\tau(F_\alpha/\partial F_\alpha) = \overline{\tau(E_\alpha)} = \tau^{\varphi'}_{L_\alpha}(t^{-1}) = \pm t^{r_\alpha}\tau^{\varphi'}_{L_\alpha}(t)$. Note that this r_α is the one in Eq. (5.7).

Step 4: Skein relation for ∇. From Eq. (5.12), Step 2 and Step 3 we have $\tau(F_\alpha) = \pm t^{p+q_\alpha} t^{r_\alpha} \tau^{\varphi'}_{L_\alpha}(t)$. Comparing with Eq. (5.10) we get $\pm t^{p+q_\alpha+r_\alpha}\tau^{\varphi'}_{L_\alpha}(t) = \pm t^\beta \tau^{\varphi'}_{L_\alpha}(t)$. This gives us

$$\beta = \beta_\alpha = p + q_\alpha + r_\alpha. \tag{5.13}$$

Using Eq. (5.13) we have $u = 2 + r_+ - r_- = 2 + q_- - q_+$ and $v = 1 + r_+ - r_0 = 1 + q_0 - q_+$. Thus Eq. (5.9) depends on the links L_α's only to the extent that whether the two strands at the crossing under investigation belong to the same component or two different components of the link L_α. Equation (5.9) is satisfied with the same u and v for all link L_+ whose two strands at the crossing come from the same component, and is also satisfied with the same u and v for all link L_+ whose two strands at the crossing come from two different components. Thus in each case a particular example is enough to determine the values of u and v.

Case 1: The two strands of L_+ at the crossing come from one component. Consider the knot 3_1 and the particular crossing in Fig. 12. Direct computation

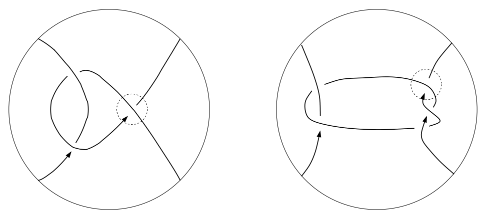


FIGURE 12. The knots 3_1 and 5_6.

gives that $\nabla_{L_+}(t) = \pm(t - t^{-1})$, $\nabla_{L_-}(t) = \pm(t - t^{-1})$, and $\nabla_{L_0}(t) = 0$, thus $u = 0$. Also consider the knot 5_6 in that figure. We have $\nabla_{L_+}(t) = \pm(t - t^{-1})$, $\nabla_{L_-}(t) = \pm(t - t^{-1})(t^2 - 1 + t^{-2})$, and $\nabla_{L_0}(t) = \pm(t - t^{-1})^2$, thus $v = 0$.

Case 2: The two strands of L_+ at the crossing come from two different components. Consider the link 4^2_2 in Fig. 13. At the first crossing in the figure,

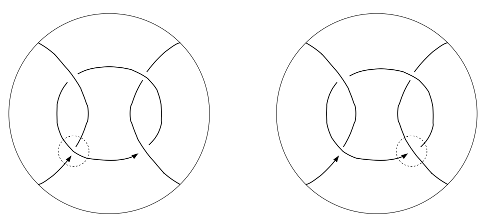


FIGURE 13. The link 4^2_2.

$\nabla_{L_+}(t) = \pm(t - t^{-1})^2$, $\nabla_{L_0}(t) = \pm(t - t^{-1})$, and $\nabla_{L_-}(t) = 0$, thus $v = 0$. On the other hand at the second crossing in the figure $\nabla_{L_-}(t) = \pm(t - t^{-1})^2$, $\nabla_{L_0}(t) = \pm(t - t^{-1})$, and $\nabla_{L_+}(t) = 0$, thus $u = 0$.

In both cases $u = v = 0$, and the proof of Theorem 5.7 is completed. □

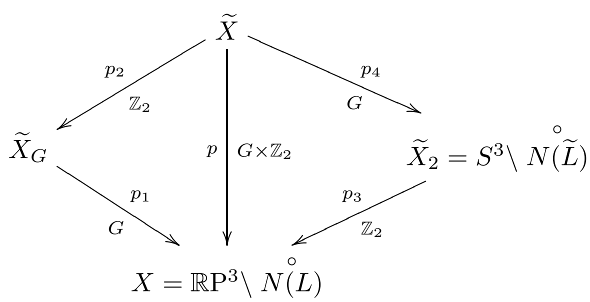
Remark 5.8. In general it is not possible to compute $\nabla_L(t)$ from the skein relation (5.8) alone because of the restriction of our theorem that the torsion classes do not change after a smoothing at a crossing.

6. Relationships among twisted and untwisted Alexander polynomials

Suppose that L is a nontorsion link in $\mathbb{R}\mathrm{P}^3$. Let $\widetilde{L}$ be the preimage of L under the canonical covering map from S^3 to $\mathbb{R}\mathrm{P}^3$. Because each component of L is null-homologous hence is null-homotopic in $\mathbb{R}\mathrm{P}^3$, its preimage in S^3 has two components. Thus $\widetilde{L}$ has an even number of components. A way to draw a diagram for $\widetilde{L}$ is to put a copy of a diagram D of L on the top disk of a cylinder. On the bottom disk put a diagram obtained from D by reflecting it through the center of the disk, then connect the corresponding boundary points on the boundary circles of the top and bottom disks by vertical lines. If furthermore we rotate the bottom disk an angle of 180° along a horizontal line passing through the disk center (i.e. flipping it, changing undercrossings to overcrossings and vice versa) then we obtain Drobotukhina's description in [Dro90, p. 616].

Consider the following diagram of coverings:

(6.1)



In the diagram $p : \widetilde{X} \to X$ corresponds to the kernel of the map $\pi \to H$; $p_1 : \widetilde{X}_G \to X$ corresponds to the kernel of the map $\pi \to H \to G$; $p_3 : \widetilde{X}_2 \to X$ corresponds to the kernel of the map $\pi \to H \to \mathbb{Z}_2$; and p_2 and p_4 are lifts of p. The diagram is commutative. The cellular structure of X induces cellular structures on the remaining spaces.

Let $C_i^+(\widetilde{X})$ be the subcomplex of $C_i(\widetilde{X})$ generated by chains of the form $\sigma + u\sigma$ where σ is an i-cell in $\widetilde{X}$. Similarly let $C_i^-(\widetilde{X})$ be the subcomplex generated by chains of the form $\sigma - u\sigma$. Consider $\mathbb{Q}(G) \otimes_{\mathbb{Z}[H],\varphi} C_i(\widetilde{X})$, where φ is the twisted map of Section 3.1.

Proposition 6.1. *We have the following isomorphisms of $\mathbb{Q}(G)$-vector spaces:*

a). $\mathbb{Q}(G) \otimes_{\mathbb{Z}[H]} C_i(\widetilde{X}) = (\mathbb{Q}(G) \otimes_{\mathbb{Z}[H]} C_i^+(\widetilde{X})) \oplus (\mathbb{Q}(G) \otimes_{\mathbb{Z}[H]} C_i^-(\widetilde{X}))$.

b). $\mathbb{Q}(G) \otimes_{\mathbb{Z}[H]} C_i(\widetilde{X}_G) \cong \mathbb{Q}(G) \otimes_{\mathbb{Z}[H]} C_i^+(\widetilde{X})$.

c). $\mathbb{Q}(G) \otimes_{\mathbb{Z}[H],\varphi} C_i(\widetilde{X}) \cong \mathbb{Q}(G) \otimes_{\mathbb{Z}[H]} C_i^-(\widetilde{X})$.

Proof. Here we are dealing with homology with local coefficients and the following proof is adapted from Hatcher [Hat01, p. 330].

a). Noting that $C_i^+(\widetilde{X}) \cap C_i^-(\widetilde{X}) = \{0\}$ and $\sigma = ((\sigma + u\sigma) + (\sigma - u\sigma))/2$, the result follows immediately.

b). A cell in $\widetilde{X}$ is a lift of a cell in $\widetilde{X}_G$. The isomorphism is induced from the map $\sigma \mapsto (\tilde{\sigma} + u\tilde{\sigma})$.

c). Consider the the projection $pr : \mathbb{Q}(G) \otimes_{\mathbb{Z}[H]} C_i(\widetilde{X}) \to \mathbb{Q}(G) \otimes_{\mathbb{Z}[H],\varphi} C_i(\widetilde{X})$ mapping $1 \otimes \sigma$ to $1 \otimes_\varphi \sigma$. We have $pr(1 \otimes (\sigma + u\sigma)) = 1 \otimes_\varphi \sigma + 1 \otimes_\varphi u\sigma = 1 \otimes_\varphi \sigma + 1 \cdot u \otimes_\varphi \sigma = 1 \otimes_\varphi \sigma + \varphi(u) \otimes_\varphi \sigma = 0$, since $\varphi(u) = -1$. This implies $\mathbb{Q}(G) \otimes_{\mathbb{Z}[H]} C_i^+(\widetilde{X}) \subset \ker(pr)$. By a similar argument we see that $(\mathbb{Q}(G) \otimes_{\mathbb{Z}[H]} C_i^-(\widetilde{X})) \cap \ker(pr) = \{0\}$. Thus $\mathbb{Q}(G) \otimes_{\mathbb{Z}[H]} C_i^+(\widetilde{X}) = \ker(pr)$ and using a) the result follows. □

Note that $\mathbb{Q}(G) \otimes_{\mathbb{Z}[H]} C_i(\widetilde{X}_G)$ is in fact $\mathbb{Q}(G) \otimes_{\mathbb{Z}[G]} C_i(\widetilde{X}_G)$. It follows from this proposition that we have the short exact sequence of chain complexes of $\mathbb{Q}(G)$-vector spaces:

$$(6.2)\quad 0 \to \mathbb{Q}(G) \otimes_{\mathbb{Z}[H],\varphi} C_*(\widetilde{X}) \to \mathbb{Q}(G) \otimes_{\mathbb{Z}[H]} C_*(\widetilde{X}) \to \mathbb{Q}(G) \otimes_{\mathbb{Z}[G]} C_*(\widetilde{X}_G) \to 0.$$

From this sequence we now derive a relationship among multi-variable Alexander polynomials. If L is a nontorsion link having v components then $\widetilde{L}$ has $2v$ components. We enumerate so that the i-th component and the $(v+i)$-th component of $\widetilde{L}$ are projected to the same i-th component of L. Let ψ be the homomorphism from $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_v^{\pm 1}, t_{v+1}^{\pm 1}, \ldots, t_{2v}^{\pm 1}]$ to $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_v^{\pm 1}]$ identifying t_{v+i} with t_i for all $1 \le i \le v$. Consider the multi-variable Alexander polynomial of $\widetilde{L}$, $\Delta_{\widetilde{L}}(t_1, t_2, \ldots, t_{2v})$. Let $\Delta'_{\widetilde{L}}(t_1, t_2, \ldots, t_v)$ be obtained from $\Delta_{\widetilde{L}}(t_1, t_2, \ldots, t_{2v})$ by identifying the t_i and t_{v+i} variables for all $1 \le i \le v$, that is $\Delta'(\widetilde{L}) = \psi(\Delta(\widetilde{L}))$. Recall from our fixed splitting of H in Section 3.1 that the free part G is generated by the meridians of the components of L, thus $\mathbb{Z}[G] = \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_v^{\pm 1}]$.

Example 6.2. Let K be the knot 2_1 in Drobotukhina's table (see Example 3.3). Then $\Delta_K^\varphi(t) = t - 1$ and $\Delta_K(t) = t^2 + 1$. The lift $\widetilde{K}$ of this knot is the link 4_2^1 in Rolfsen's table, $\Delta_{\widetilde{K}}(t_1, t_2) = t_1 t_2 + 1$, and $\Delta'_{\widetilde{K}}(t) = t^2 + 1$.

Theorem 6.3. *Let L be a nontorsion link. If L has one component then $(t-1)\Delta'(\widetilde{L}) = \Delta(L)\Delta^\varphi(L)$ as elements in $\mathbb{Z}[t^{\pm 1}]$. If L has at least two components then $\Delta'(\widetilde{L}) = \Delta(L)\Delta^\varphi(L)$ as elements in $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_v^{\pm 1}]$.*

Proof. Recall the diagram of covering spaces (6.1). The map p_4 corresponds to the kernel of the canonical projection $\pi_1(\widetilde{X}_2) \to H_1(\widetilde{X}_2) = \langle t_1, \ldots, t_{2v} / t_i t_j = t_j t_i \rangle \to G = \langle t_1, \ldots, t_v / t_i t_j = t_j t_i \rangle$, where the second projection identifies t_i and t_{v+i} for all $1 \le i \le v$. Thus $p_{4*}(\pi_1(\widetilde{X}))$ will be the subgroup of $\pi_1(\widetilde{X}_2)$ whose projection to $H_1(\widetilde{X}_2)$ is $\{t_1^{\alpha_1} \cdots t_{2v}^{\alpha_{2v}} / \alpha_i + \alpha_{v+i} = 0, 1 \le i \le v\}$. Then p_{3*} will send $p_{4*}(\pi_1(\widetilde{X}))$ to the subgroup of $\pi_1(X)$ whose projection to H is $\{t_1^{\alpha_1 + \alpha_{v+1}} \cdots t_v^{\alpha_v + \alpha_{2v}}\} = \{1\}$. So $(p_3 \circ p_4)_*$ sends $\pi_1(\widetilde{X})$ to the subgroup of π which vanishes in H, this is why $p_3 \circ p_4 = p$.

Now we look at the space $\widetilde{X}$ as the G-cover of $\widetilde{X}_2$ corresponding to p_4. Then there is an action of G on $C_i(\widetilde{X})$ turning it to a $\mathbb{Z}[G]$-module $C_i'(\widetilde{X})$, and so we can form the vector space $\mathbb{Q}(G)\otimes_{\mathbb{Z}[G]} C'_i(\widetilde{X})$. It can be seen that $\mathbb{Q}(G)\otimes_{\mathbb{Z}[G]} C'_i(\widetilde{X}) \cong \mathbb{Q}(G)\otimes_{\mathbb{Z}[H]} C_*(\widetilde{X})$. Thus the sequence (6.2) becomes

$$0 \to \mathbb{Q}(G)\otimes_{\mathbb{Z}[H],\varphi} C_*(\widetilde{X}) \to \mathbb{Q}(G)\otimes_{\mathbb{Z}[G]} C'_*(\widetilde{X}) \to \mathbb{Q}(G)\otimes_{\mathbb{Z}[G]} C_*(\widetilde{X}_G) \to 0.$$

Apply the product formula for torsion (4.2) to this short exact sequence we obtain

$$\tau(\mathbb{Q}(G)\otimes_{\mathbb{Z}[G]} C'_*(\widetilde{X})) = \pm\tau(\mathbb{Q}(G)\otimes_{\mathbb{Z}[G]} C_*(\widetilde{X}_G))\tau(\mathbb{Q}(G)\otimes_{\mathbb{Z}[H],\varphi} C_*(\widetilde{X})). \tag{6.3}$$

Theorem 4.4 says that $\tau(\mathbb{Q}(G)\otimes_{\mathbb{Z}[H],\varphi} C_*(\widetilde{X}))$ is $\Delta^\varphi(L)$; Remark 4.6 says $\tau(\mathbb{Q}(G)\otimes_{\mathbb{Z}[G]} C_*(\widetilde{X}_G))$ is $\Delta(L)$ if L has more than one component and is $\Delta(L)/(t-1)$ if L has one component. Finally the identification of torsion and Alexander polynomial for links in S^3 ([Mil62], [Tur01, p. 55]) says that $\tau(\mathbb{Q}(G)\otimes_{\mathbb{Z}[G]} C'_*(\widetilde{X}))$ is $\psi(\Delta(\widetilde{L}))$ if $\widetilde{L}$ has more than one component (here the functority of torsion [Tur01, Lemma 13.5] is used). The theorem then follows from (6.3). □

Remark 6.4. A similar result also holds true if we consider only one variable polynomials.

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