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A unified quantum $SO(3)$ invariant for rational homology 3-spheres

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Abstract Given a rational homology 3-sphere M with $|H_1(M,\mathbb{Z})| = b$ and a link L inside M, colored by odd numbers, we construct a unified invariant $I_{M,L}$ belonging to a modification of the Habiro ring where b is inverted. Our unified invariant dominates the whole set of the $SO(3)$ Witten–Reshetikhin–Turaev invariants of the pair (M, L). If $b = 1$ and $L = \emptyset$, I_M coincides with Habiro's invariant of integral homology 3-spheres. For $b > 1$, the unified invariant defined by the third author is determined by I_M. Important applications are the new Ohtsuki series (perturbative expansions of I_M) dominating quantum $SO(3)$ invariants at roots of unity whose order is not a power of a prime. These series are not known to be determined by the LMO invariant.

1 Introduction

1.1 Background

In the 25 years after the discovery of the Jones polynomial, knot theory experienced the transformation from an esoteric branch of pure mathematics

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to a modern dynamic research field with deep connections to mathematical physics, the theory of integrable and dynamic systems, von Neumann algebras, representation theory, homological algebra, algebraic geometry, etc. The main stones of this development were constructions of the finite type invariants, Kontsevich integral and Khovanov homology.

By the Kirby theorem, closed compact orientable 3-manifolds are in bijection with framed links modulo two Kirby moves. This rises the question whether the recent achievements in knot theory can be lifted to the theory of 3-manifolds. This paper is a step towards this goal.

The lift of the (colored) Jones polynomial is given by the Witten–Reshetikhin–Turaev (WRT) invariant which associates with any closed oriented 3-manifold, a semi-simple Lie algebra and a root of unity a complex number [28]. The Kontsevich integral was extended to 3-manifolds by the third author, Murakami and Ohtsuki and is known as LMO invariant [18]. The relationship between LMO and WRT invariants was known only in the case when the 3-manifold is a rational homology sphere and the order of the root of unity is a prime number bigger than the order of the torsion group. In this case the perturbative expansion of the WRT invariants given by the Ohtsuki series [25] coincides, on one side, with the LMO composed with the sl_2 weight system and, on the other side, is determined modulo a big prime p by the WRT invariant at a p-th root of unity. In the 13 years after Ohtsuki's work was published, no perturbative expansion of WRT invariants at not prime roots of unity was constructed. This is because Ohtsuki's techniques heavily rely on the fact that the order of the root is prime and can not be extended to other roots.

Also the related question of integrality for the WRT invariants, though intensively studied (see [6, 21, 23] and the references therein), was accessible for prime roots of unity only. Note that a conceptual solution of the integrality problem is of primary importance for any attempt of categorification of the WRT invariants (compare [10]).

In this paper, the theory of perturbative 3-manifold invariants finds its incarnation. For any rational homology 3-sphere M, with $|H_1(M,\mathbb{Z})| = b$, we construct series of perturbative invariants dominating WRT invariants of M at all roots of unity. More precisely, let us fix a divisor c of b and put $e_c := \exp(2\pi I/c)$, then our power series in $(q - e_c)$ with coefficients in $\mathbb{Z}[1/b][e_c]$ dominates the WRT invariants at roots of unity whose order has the greatest common divisor c with b. It is a challenging open problem to decide whether all these new perturbative invariants can be extracted from LMO or capture more information from the Chern–Simons theory than just the contribution of flat connections.

The way to the new Ohtsuki series goes through the unification of WRT invariants. This approach led already to the full solution of the integrality problem for quantum $SO(3)$ invariants in [3]. There the first and the third authors showed that $\tau_M(\xi) \in \mathbb{Z}[\xi]$ for any 3-manifold M and any root of unity ξ of odd order. By $\tau_M(\xi)$ we mean here the $SO(3)$ version of the WRT invariant introduced by Kirby and Melvin [12] for roots of unity ξ of odd order only.

The unification of WRT invariants was initiated in 2006 by Habiro. For any integral homology 3-sphere M, Habiro [7] constructed a *unified invariant* J_M whose evaluation at any root of unity coincides with the value of the WRT invariant at that root. Habiro's unified invariant J_M is an element of the following ring (Habiro's ring)

$$\widehat{\mathbb{Z}[q]} := \lim_{\overleftarrow{k}} \frac{\mathbb{Z}[q]}{((q;q)_k)}, \quad \text{where } (q;q)_k = \prod_{j=1}^{k}(1-q^j).$$

Every element $f(q) \in \widehat{\mathbb{Z}[q]}$ can be written as an infinite sum

$$f(q) = \sum_{k\geq 0} f_k(q)\,(1-q)(1-q^2)\cdots(1-q^k),$$

with $f_k(q) \in \mathbb{Z}[q]$. When $q=\xi$, a root of unity, only a finite number of terms on the right hand side are not zero, hence the evaluation $\mathrm{ev}_\xi(f(q))$ is well-defined and is an algebraic integer.

The Habiro ring has beautiful arithmetic properties. Every element $f(q) \in \widehat{\mathbb{Z}[q]}$ can be considered as a function whose domain is the set of roots of unity. Moreover, there is a natural Taylor series for f at every root of unity. Two elements $f, g \in \widehat{\mathbb{Z}[q]}$ are the same if and only if their Taylor series at a root of unity coincide. In addition, each function $f(q) \in \widehat{\mathbb{Z}[q]}$ is totally determined by its values at, say, infinitely many roots of order 3^n, $n \in \mathbb{N}$. Due to these properties the Habiro ring is also called a ring of "analytic functions at roots of unity". Thus belonging to $\widehat{\mathbb{Z}[q]}$ means that the collection of the WRT invariants is far from a random collection of algebraic integers; together they form a nice function.

General properties of the Habiro ring imply that for any integral homology 3-sphere M, the Taylor expansion of the unified invariant J_M at $q=1$ coincides with the Ohtsuki series and dominates WRT invariants of M at all roots of unity (not only of prime order).

Recently, Habiro ring found an application in analytic geometry for constructing varieties over the non-existing field of one element [20].

In this paper, we give a full generalization of the Habiro theory to rational homology 3-spheres. This requires the use of completely new techniques coming from number theory, commutative algebra, quantum group and knot theory. Let us explain this in more details.

Assume M is a rational homology 3-sphere with $|H_1(M,\mathbb{Z})| = b$. Then our unified invariant I_M belongs to a modification of a Habiro ring where

b is inverted. Unlike the case $b = 1$, the modified Habiro ring is not an integral domain, but a product of different subrings, where each factor is determined by its proper Taylor expansion at some root of unity. In particular, $I_M = \prod_{c|b} I_{M,c}$, where $I_{M,c}$ dominates $\{\tau_M(\xi)|(\mathrm{ord}(\xi), b) = c\}$. The unified invariant constructed in [14] can be identified with $I_{M,1}$. We develop a general theory of such cyclotomic completions. The main breakthrough here is the construction of the b-th root of q in our modified Habiro ring.

This is important, since we use the Laplace transform method [2, 14], to eliminate the dependence of τ_M on ξ. The image of the Laplace transform contains the b-th root of q. Furthermore, to show that the image of the Laplace transform belongs to our ring we apply a difficult number-theoretic identity of Andrews [1], generalizing those of Rogers–Ramanujan.

Another challenging problem we had to solve is the following. In all previous constructions, the existence of I_M relies on a deep result of Habiro about cyclotomic integrality of the Jones polynomial of an algebraically split link. To diagonalize the linking matrix for a given surgery presentation of a 3-manifold, the usual trick consists of adding lens spaces and using multiplicativity of WRT invariants with respect to the connected sum (compare [25] or [14]). It does not work in our case, since if the order of the root of unity and b are not coprime, the invariants of lens spaces are often zero. The solution was to add links to lens spaces and to generalize Habiro's integrality result to algebraically split links together with arbitrary odd colored components. To do so, we had to use the whole machinery for universal invariants of bottom tangles developed in [8].

Assume M is the integral homology 3-sphere obtained by framing 1 surgery on the figure 8 knot. Then

$$I_M = \frac{q}{1-q}\sum_{k=0}^{\infty}(-1)^k q^{-(k+1)^2}(1-q^{k+1})(1-q^{k+2})\cdots(1-q^{2k+1}).$$

We expect that the categorification of WRT invariants will led to a homology theory with Euler characteristic given by I_M.

1.2 Results

The WRT or quantum $SO(3)$ invariant $\tau_{M,L}(\xi)$ is defined for a pair of a closed 3-manifold M and a link L in it, with link components colored by integers. Here ξ is a root of unity of odd order. We will recall the definitions in Sect. 2.

Suppose M is a rational homology 3-sphere, i.e. $|H_1(M,\mathbb{Z})| := \mathrm{card}\, H_1(M,\mathbb{Z}) < \infty$. There is a unique decomposition $H_1(M,\mathbb{Z}) = \bigoplus_i \mathbb{Z}/b_i\mathbb{Z}$, where each b_i is a prime power. We renormalize the $SO(3)$ WRT invariant of the pair (M, L) as follows:

$$\tau'_{M,L}(\xi) = \frac{\tau_{M,L}(\xi)}{\prod_i \tau_{L(b_i,1)}(\xi)}, \tag{1}$$

where $L(b,a)$ denotes the (b,a) lens space. We will see that $\tau_{L(b,1)}(\xi)$ is always nonzero.

For any positive integer b, we define the cyclotomic completion ring $\mathcal{R}_b$ to be

$$\mathcal{R}_b := \lim_{\overleftarrow{k}} \frac{\mathbb{Z}[1/b][q]}{((q;q^2)_k)}, \quad \text{where } (q;q^2)_k = (1-q)(1-q^3)\cdots(1-q^{2k-1}). \tag{2}$$

For any $f(q) \in \mathcal{R}_b$ and a root of unity ξ of *odd* order, the evaluation $\mathrm{ev}_\xi(f(q)) := f(\xi)$ is well-defined. Similarly, we put

$$\mathcal{S}_b := \lim_{\overleftarrow{k}} \frac{\mathbb{Z}[1/b][q]}{((q;q)_k)}.$$

Here the evaluation at any root of unity is well-defined. For odd b, there is a natural embedding $\mathcal{S}_b \to \mathcal{R}_b$, see Sect. 4.

Let us denote by $\mathcal{M}_b$ the set of rational homology 3-spheres such that $|H_1(M,\mathbb{Z})|$ divides b^n for some n. The main result of this paper is the following.

Theorem 1 *Suppose the components of a framed oriented link $L \subset M$ have odd colors, and $M \in \mathcal{M}_b$. Then there exists a unique invariant $I_{M,L} \in \mathcal{R}_b$, such that for any root of unity ξ of odd order*

$$\mathrm{ev}_\xi(I_{M,L}) = \tau'_{M,L}(\xi).$$

In addition, if b is odd, then $I_{M,L} \in \mathcal{S}_b$.

If $b = 1$ and L is the empty link, I_M coincides with Habiro's unified invariant J_M.

The proof of Theorem 1 uses the Laplace transform method and Andrew's identity. The new ingredients are

- Frobenius theory for cyclotomic completions of polynomial rings;
- computation of WRT invariants for lens spaces with links inside at all roots of unity;
- generalization of Habiro's integrality result to algebraically split bottom tangles with odd colored closed components.

These new techniques could be of separate interest for analytic geometry (compare [20]), quantum topology and representation theory.

The rings $\mathcal{R}_b$ and $\mathcal{S}_b$ have properties similar to those of the Habiro ring. An element $f(q) \in \mathcal{R}_b$ is totally determined by the values at many infinite sets of roots of unity (see Sect. 4), one special case is the following.

Proposition 2 *Let p be an odd prime not dividing b and T the set of all integers of the form $p^k b'$ with $k \in \mathbb{N}$ and b' any odd divisor of b^n for some n. Any element $f(q) \in \mathcal{R}_b$, and hence also $\{\tau_M(\xi)\}$, is totally determined by the values at roots of unity with orders in T.*

Furthermore, any element of $\mathcal{R}_b$ is determined by an infinite collection of its Taylor expansions at different roots of unity. For example, if $b = p$ is prime, we will need Taylor expansions at p^k-th roots of unity, for $k = 0, 1, 2, \ldots$. The Ohtsuki series [15, 25], originally defined through some arithmetic congruence property of the $SO(3)$ invariant, can be identified with the Taylor expansion of I_M at $q = 1$ [7, 14]. The new power series at say c-th root of unity dominates $\{\tau_M(\xi) | (\mathrm{ord}(\xi), b) = c\}$ and satisfies congruence relations similar to the original definition of the Ohtsuki series.

An interesting open problem is to determine whether the coefficients of these new series are 3-manifold invariants of finite type.

1.3 Plan of the paper

In Sect. 2 we recall known results and definitions. In the next section we explain the strategy of our proof of Theorem 1. In Sects. 4 and 6, we develop properties of cyclotomic completions of polynomial rings. New Ohtsuki series are discussed in Sect. 5. The unified invariant of lens spaces, needed for the diagonalization, is defined in Sect. 7. The main technical result of the paper based on Andrew's identity is proved in Sect. 8. The Appendix is devoted to the proof of the generalization of Habiro's integrality theorem.

2 Quantum (WRT) invariants

2.1 Notations and conventions

We will consider $q^{1/4}$ as a free parameter. Let

$$\{n\} = q^{n/2} - q^{-n/2}, \qquad \{n\}! = \prod_{i=1}^{n} \{i\},$$

$$[n] = \frac{\{n\}}{\{1\}}, \qquad \begin{bmatrix} n \\ k \end{bmatrix} = \frac{\{n\}!}{\{k\}!\{n-k\}!}.$$

We denote the set $\{1, 2, 3, \ldots\}$ by $\mathbb{N}$. We also use the following notation from q-calculus:

$$(x; q)_n := \prod_{j=1}^{n}(1 - xq^{j-1}).$$

Throughout this paper, ξ will be a primitive root of unity of *odd* order r and $e_n := \exp(2\pi I/n)$.

All 3-manifolds in this paper are supposed to be closed and oriented. Every link in a 3-manifold is framed, oriented, and has components ordered.

In this paper, $L \sqcup L'$ denotes a framed link in S^3 with disjoint sublinks L and L', with m and l components, respectively. Surgery along the framed link L transforms (S^3, L') into (M, L'). We use the same notation L' to denote the link in S^3 and the corresponding one in M.

2.2 The colored Jones polynomial

Suppose L is a framed, oriented link in S^3 with m ordered components. For positive integers $n_1, \ldots, n_m$, called the colors of L, one can define the quantum invariant $J_L(n_1, \ldots, n_m) \in \mathbb{Z}[q^{\pm 1/4}]$, known as the colored Jones polynomial of L (see e.g. [22, 28]). Let us recall here a few well-known formulas. For the unknot U with 0 framing one has

$$J_U(n) = [n]. \tag{3}$$

If L_1 is obtained from L by increasing the framing of the ith component by 1, then

$$J_{L_1}(n_1, \ldots, n_m) = q^{(n_i^2-1)/4} J_L(n_1, \ldots, n_m). \tag{4}$$

If all the colors n_i are odd, then $J_L(n_1, \ldots, n_m) \in \mathbb{Z}[q^{\pm 1}]$.

2.3 Evaluation and Gauss sums

For each root of unity ξ of odd order r, we define the evaluation map ev_ξ by replacing q with ξ.

Suppose $f(q; n_1, \ldots, n_m)$ is a function of variables $q^{\pm 1}$ and integers $n_1, \ldots, n_m$. In quantum topology, the following sum plays an important role

$$\sum_{n_1,\ldots,n_m}^{\xi} f := \sum_{\substack{0<n_i<2r \\ n_i \text{ odd}}} \mathrm{ev}_\xi f(q; n_1, \ldots, n_m)$$

where in the sum all the n_i run over the set of *odd* numbers between 0 and $2r$.

In particular, the following variation of the Gauss sum

$$\gamma_b(\xi) := \sum_n^{\xi} q^{b\frac{n^2-1}{4}}$$

is well-defined, since for odd n, $4 \mid n^2 - 1$. It is known that, for odd r, $|\gamma_b(\xi)| = \sqrt{cr}$ is never 0. Here $c = (b, r)$ is the greatest common divisor of b and r.

2.4 Definition of the WRT invariant

Suppose the components of L' are colored by fixed integers $j_1, \dots, j_l$. Let

$$F_{L\sqcup L'}(\xi) := \sum_{n_1,\dots,n_m}^{\xi} \left\{ J_{L\sqcup L'}(n_1, \dots, n_m, j_1, \dots, j_l) \prod_{i=1}^{m} [n_i] \right\}.$$

An important special case is when $L = U^b$, the unknot with framing $b \neq 0$, and $L' = \emptyset$. In that case $F_{U^b}(\xi)$ can be calculated using the Gauss sum and is nonzero, see Sect. 7 below.

Let σ_+ (respectively σ_-) be the number of positive (negative) eigenvalues of the linking matrix of L. Then the quantum $SO(3)$ invariant of the pair (M, L') is defined by (see e.g. [12, 28])

$$\tau_{M,L'}(\xi) = \frac{F_{L\sqcup L'}(\xi)}{(F_{U^{+1}}(\xi))^{\sigma_+}(F_{U^{-1}}(\xi))^{\sigma_-}}. \tag{5}$$

The invariant $\tau_{M,L'}(\xi)$ is multiplicative with respect to the connected sum.

For example, the $SO(3)$ invariant of the lens space $L(b, 1)$, obtained by surgery along U^b, is

$$\tau_{L(b,1)}(\xi) = \frac{F_{U^b}(\xi)}{F_{U^{\mathrm{sn}(b)}}(\xi)}, \tag{6}$$

where $\mathrm{sn}(b)$ is the sign of the integer b.

Let us focus on the special case when the linking matrix of L is diagonal, with $b_1, b_2, \dots, b_m$ on the diagonal. Assume each b_i is a power of a prime up to sign. Then $H_1(M, \mathbb{Z}) = \bigoplus_{i=1}^{m} \mathbb{Z}/|b_i|$, and

$$\sigma_+ = \mathrm{card}\,\{i \mid b_i > 0\}, \qquad \sigma_- = \mathrm{card}\,\{i \mid b_i < 0\}.$$

Thus from the definitions (5), (6) and (1) we have

$$\tau'_{M,L'}(\xi) = \left(\prod_{i=1}^{m} \tau'_{L(b_i,1)}(\xi) \right) \frac{F_{L\sqcup L'}(\xi)}{\prod_{i=1}^{m} F_{U^{b_i}}(\xi)}, \tag{7}$$

with

$$\tau'_{L(b_i,1)}(\xi) = \frac{\tau_{L(b_i,1)}(\xi)}{\tau_{L(|b_i|,1)}(\xi)}.$$

2.5 Habiro's cyclotomic expansion of the colored Jones polynomial

Recall that L and L' have m and l components, respectively. Let us color L' by fixed $\mathbf{j} = (j_1, \dots, j_l)$ and vary the colors $\mathbf{n} = (n_1, \dots, n_m)$ of L.

For non-negative integers n, k we define

$$A(n,k) := \frac{\prod_{i=0}^{k}(q^n + q^{-n} - q^i - q^{-i})}{(1-q)(q^{k+1};q)_{k+1}}.$$

For $\mathbf{k} = (k_1, \dots, k_m)$ let

$$A(\mathbf{n},\mathbf{k}) := \prod_{j=1}^{m} A(n_j, k_j).$$

Note that $A(\mathbf{n},\mathbf{k}) = 0$ if $k_j \geq n_j$ for some index j. Also

$$A(n,0) = q^{-1} J_U(n)^2.$$

The colored Jones polynomial $J_{L\sqcup L'}(\mathbf{n},\mathbf{j})$, when $\mathbf{j}$ is fixed, can be repackaged into the invariant $C_{L\sqcup L'}(\mathbf{k},\mathbf{j})$ as stated in the following theorem.

Theorem 3 *Suppose $L \sqcup L'$ is a link in S^3, with L having zero linking matrix. Assume the components of L' have fixed* odd *colors $\mathbf{j} = (j_1, \dots, j_l)$. Then there are invariants*

$$C_{L\sqcup L'}(\mathbf{k},\mathbf{j}) \in \frac{(q^{k+1};q)_{k+1}}{(1-q)}\mathbb{Z}[q^{\pm 1}], \quad \textit{where } k = \max\{k_1, \dots, k_m\} \tag{8}$$

such that for every $\mathbf{n} = (n_1, \dots, n_m)$

$$J_{L\sqcup L'}(\mathbf{n},\mathbf{j}) \prod_{i=1}^{m}[n_i] = \sum_{0\leq k_i \leq n_i - 1} C_{L\sqcup L'}(\mathbf{k},\mathbf{j}) A(\mathbf{n},\mathbf{k}). \tag{9}$$

When $L' = \emptyset$, this is Theorem 8.2 in [7]. This generalization, essentially also due to Habiro, can be proved similarly as in [7]. For completeness we give a proof in the Appendix. Note that the existence of $C_{L\sqcup L'}(\mathbf{k},\mathbf{j})$ as rational functions in q satisfying (9) is easy to establish. The difficulty here is to show the integrality of (8).

Since $A(\mathbf{n},\mathbf{k}) = 0$ unless $\mathbf{k} < \mathbf{n}$, in the sum on the right hand side of (9) one can assume that $\mathbf{k}$ runs over the set of all m-tuples $\mathbf{k}$ with non-negative integer components. We will use this fact later.

3 Strategy of the proof of the main theorem

Here we give the proof of Theorem 1 using technical results that will be proved later.

As before, $L \sqcup L'$ is a framed link in S^3 with disjoint sublinks L and L', with m and l components, respectively. Assume that L' is colored by fixed $\mathbf{j} = (j_1, \dots, j_l)$, with j_i's odd. Surgery along the framed link L transforms (S^3, L') into (M, L'). We will define $I_{M,L'} \in \mathcal{R}_b$, such that

$$\tau'_{M,L'}(\xi) = \mathrm{ev}_\xi(I_{M,L'}) \tag{10}$$

for any root of unity ξ of odd order. This unified invariant is multiplicative with respect to the connected sum.

The following observation is important. By Proposition 2, there is *at most one* element $f(q) \in \mathcal{R}_b$ such that for every root ξ of odd order one has

$$\tau'_{M,L}(\xi) = \mathrm{ev}_\xi(f(q)).$$

That is, if we can find such an element, it is unique, and we put $I_{M,L'} := f(q)$.

3.1 Laplace transform

The following is the main technical result of the paper. A proof will be given in Sect. 8.

Theorem 4 *Suppose $b = \pm 1$ or $b = \pm p^l$ where p is a prime and l is positive. For any non-negative integer k, there exists an element $Q_{b,k} \in \mathcal{R}_b$ such that for every root ξ of odd order one has*

$$\frac{\sum_n^{\xi} q^{b\frac{n^2-1}{4}} A(n,k)}{F_{U^b}(\xi)} = \mathrm{ev}_\xi(Q_{b,k}).$$

In addition, if b is odd, $Q_{b,k} \in \mathcal{S}_b$.

3.2 Definition of the unified invariant: diagonal case

Suppose that the linking number between any two components of L is 0, and the framing on components of L are $b_i = \pm p_i^{k_i}$ for $i = 1, \dots, m$, where each p_i is prime or 1. Let us denote the link L with all framings switched to zero by L_0.

Using (9), taking into account the framings b_i's, we have

$$J_{L\sqcup L'}(\mathbf{n}, \mathbf{j}) \prod_{i=1}^m [n_i] = \sum_{\mathbf{k} \geq 0} C_{L_0 \sqcup L'}(\mathbf{k}, \mathbf{j}) \prod_{i=1}^m q^{b_i \frac{n_i^2-1}{4}} A(n_i, k_i).$$

By the definition of $F_{L\sqcup L'}$, we have

$$F_{L\sqcup L'}(\xi)=\sum_{\mathbf{k}\geq 0}\mathrm{ev}_\xi\left(C_{L_0\sqcup L'}(\mathbf{k},\mathbf{j})\right)\prod_{i=1}^{m}\sum_{n_i}^{\xi} q^{b_i\frac{n_i^2-1}{4}}A(n_i,k_i).$$

From (7) and Theorem 4, we get

$$\tau'_{M,L'}(\xi)=\mathrm{ev}_\xi\left\{\prod_{i=1}^{m} I_{L(b_i,1)}\sum_{\mathbf{k}} C_{L_0\sqcup L'}(\mathbf{k},\mathbf{j})\prod_{i=1}^{m} Q_{b_i,k_i}\right\},$$

where the unified invariant of the lens space $I_{L(b_i,1)}\in\mathcal{R}_b$, with $\mathrm{ev}_\xi(I_{L(b_i,1)})=\tau'_{L(b_i,1)}(\xi)$, exists by Lemma 6 below. Thus if we define

$$I_{(M,L')}:=\prod_{i=1}^{m} I_{L(b_i,1)}\sum_{\mathbf{k}} C_{L_0\sqcup L'}(\mathbf{k},\mathbf{j})\prod_{i=1}^{m} Q_{b_i,k_i},$$

then (10) is satisfied. By Theorem 3, $C_{L_0\sqcup L'}(\mathbf{k},\mathbf{j})$ is divisible by $(q^{k+1};q)_{k+1}/(1-q)$, which is divisible by $(q;q)_k$, where $k=\max k_i$. It follows that $I_{(M,L')}\in\mathcal{R}_b$. In addition, if b is odd, then $I_{(M,L')}\in\mathcal{S}_b$.

3.3 Diagonalization using lens spaces

The general case reduces to the diagonal case by the well-known trick of diagonalization using lens spaces. We say that M is *diagonal* if it can be obtained from S^3 by surgery along a framed link L with diagonal linking matrix, where the diagonal entries are of the form $\pm p^k$ with $p=0,1$ or a prime. The following lemma was proved in [14, Proposition 3.2(a)].

Lemma 5 *For every rational homology sphere M, there are lens spaces $L(b_i,a_i)$ such that the connected sum of M and these lens spaces is diagonal. Moreover, each b_i is a prime power divisor of $|H_1(M,\mathbb{Z})|$.*

To define the unified invariant for a general rational homology sphere M, one first adds to M lens spaces to get a diagonal M', for which the unified invariant $I_{M'}$ had been defined in Sect. 3.2. Then I_M is the quotient of $I_{M'}$ by the unified invariants of the lens spaces. But unlike the simpler case of [14], the unified invariant of lens spaces are *not* invertible in general. To overcome this difficulty we insert knots in lens spaces and split the unified invariant into different components. This will be explained in the remaining part of this section.

3.4 Splitting of the invariant

Suppose p is a prime divisor of b, then it's clear that $\mathcal{R}_p \subset \mathcal{R}_b$.

In Sect. 4 we will see that there is a decomposition

$$\mathcal{R}_b = \mathcal{R}_b^{p,0} \times \mathcal{R}_b^{p,\bar{0}},$$

with canonical projections $\pi_0^p : \mathcal{R}_b \to \mathcal{R}_b^{p,0}$ and $\pi_{\bar{0}}^p : \mathcal{R}_b \to \mathcal{R}_b^{p,\bar{0}}$. If $f \in \mathcal{R}_b^{p,0}$ then $\mathrm{ev}_\xi(f)$ can be defined when the order of ξ is coprime with p; and in this case $\mathrm{ev}_\xi(g) = \mathrm{ev}_\xi(\pi_0^p(g))$ for every $g \in \mathcal{R}_b$.

On the other hand, if $f \in \mathcal{R}_b^{p,\bar{0}}$ then $\mathrm{ev}_\xi(f)$ can be defined when the order of ξ is divisible by p, and one has $\mathrm{ev}_\xi(g) = \mathrm{ev}_\xi(\pi_{\bar{0}}^p(g))$ for every $g \in \mathcal{R}_b$.

It also follows from the definition that $\mathcal{R}_p^{p,\varepsilon} \subset \mathcal{R}_b^{p,\varepsilon}$ for $\varepsilon = 0$ or $\bar{0}$.

For $\mathcal{S}_b$, there exists a completely analogous decomposition. For any odd divisor p of b, an element $x \in \mathcal{R}_b$ (or $\mathcal{S}_b$) determines and is totally determined by the pair $(\pi_0^p(x), \pi_{\bar{0}}^p(x))$. If $p = 2$ divides b, then for any $x \in \mathcal{R}_b$, $x = \pi_0^p(x)$.

Hence, to define I_M it is enough to fix $I_M^0 = \pi_0^p(I_M)$ and $I_M^{\bar{0}} = \pi_{\bar{0}}^p(I_M)$. The first part $I_M^0 = \pi_0^p(I_M)$, when $b = p$, was defined in [14] (up to normalization), where the third author considered the case when the order of roots of unity is coprime with b. We will give a self-contained definition of I_M^0, and show that it is coincident (up to normalization) with the one introduced in [14].

3.5 Lens spaces

Suppose b, a, d are integers with $(b, a) = 1$ and $b \neq 0$. Let $M(b, a; d)$ be the pair of a lens space $L(b, a)$ and a knot $K \subset L(b, a)$, colored by d, as described in Fig. 1.

Among these pairs we want to single out some whose quantum invariants are invertible.

For $\varepsilon \in \{0, \bar{0}\}$, let $M^\varepsilon(b, a) := M(b, a; d(\varepsilon))$, where $d(0) := 1$ and $d(\bar{0})$ is the smallest odd positive integer such that $|a| d(\bar{0}) \equiv 1 \pmod{b}$. Note that if $|a| = 1$, $d(0) = d(\bar{0}) = 1$.

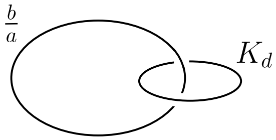


Fig. 1 The lens space $(L(b, a), K_d)$ is obtained by b/a surgery on the first component of the Hopf link, the second component is the knot K colored by d

It is known that if the color of a link component is 1, then the component can be removed from the link without affecting the value of quantum invariants. Hence

$$\tau_{M(b,a;1)} = \tau_{L(b,a)}.$$

Lemma 6 *Suppose $b = \pm p^l$ is a prime power. For $\varepsilon \in \{0, \bar{0}\}$, there exists an invertible invariant $I^{\varepsilon}_{M^{\varepsilon}(b,a)} \in \mathcal{R}^{p,\varepsilon}_p$ such that*

$$\tau'_{M^{\varepsilon}(b,a)}(\xi) = \mathrm{ev}_{\xi}\left(I^{\varepsilon}_{M^{\varepsilon}(b,a)}\right)$$

where $\varepsilon = 0$ if the order of ξ is not divisible by p, and $\varepsilon = \bar{0}$ otherwise. Moreover, if p is odd, then $I^{\varepsilon}_{M^{\varepsilon}(b,a)}$ belongs to and is invertible in $\mathcal{S}^{p,\varepsilon}_p$.

A proof of Lemma 6 will be given in Sect. 7.

3.6 Definition of the unified invariant: general case

Now suppose (M, L') is an arbitrary pair of a rational homology 3-sphere with a link L' in it colored by odd numbers $j_1, \dots, j_l$. Let $L(b_i, a_i)$ for $i = 1, \dots, m$ be the lens spaces of Lemma 5. We use induction on m. If $m = 0$, then M is diagonal and $I_{M,L'}$ has been defined in Sect. 3.2.

Since $(M, L')\#M(b_1, a_1; d)$ becomes diagonal after adding $m - 1$ lens spaces, the unified invariant of $(M, L')\#M(b_1, a_1; d)$ can be defined by induction, for any odd integer d. In particular, one can define $I_{M^{\varepsilon}}$, where $M^{\varepsilon} := (M, L')\#M^{\varepsilon}(b_1, a_1)$. Here $\varepsilon = 0$ or $\varepsilon = \bar{0}$ and b_1 is a power of a prime p dividing b. It follows that the components $\pi^p_{\varepsilon}(I_{M^{\varepsilon}}) \in \mathcal{R}^{p,\varepsilon}_b$ are defined.

By Lemma 6, $I^{\varepsilon}_{M^{\varepsilon}(b_1,a_1)}$ is defined and invertible. Now we put

$$I^{\varepsilon}_{M,L'} := I^{\varepsilon}_{M^{\varepsilon}} \cdot (I^{\varepsilon}_{M^{\varepsilon}(b_1,a_1)})^{-1}.$$

It is easy to see that $I_{M,L'} := (I^0_{M,L'}, I^{\bar{0}}_{M,L'})$ satisfies (10). This completes the construction of $I_{M,L'}$. It remains to prove Lemma 6 and Theorem 4.

4 Cyclotomic completions of polynomial rings

In this section we adapt the results of Habiro on cyclotomic completions of polynomial rings [9] to our rings.

4.1 On cyclotomic polynomial

Recall that $e_n := \exp(2\pi I/n)$ and denote by $\Phi_n(q)$ the cyclotomic polynomial

$$\Phi_n(q) = \prod_{\substack{(j,n)=1\\ 0<j<n}} (q - e_n^j).$$

The degree of $\Phi_n(q) \in \mathbb{Z}[q]$ is given by the Euler function $\varphi(n)$. Suppose p is a prime and n an integer. Then (see e.g. [24])

$$\Phi_n(q^p) = \begin{cases} \Phi_{np}(q) & \text{if } p \mid n, \\ \Phi_{np}(q)\Phi_n(q) & \text{if } p \nmid n. \end{cases} \tag{11}$$

It follows that $\Phi_n(q^p)$ is always divisible by $\Phi_{np}(q)$.

The ideal of $\mathbb{Z}[q]$ generated by $\Phi_n(q)$ and $\Phi_m(q)$ is well-known, see e.g. [14, Lemma 5.4]:

Lemma 7

(a) *If* $\frac{m}{n} \neq p^e$ *for any prime* p *and any integer* $e \neq 0$, *then* $(\Phi_n) + (\Phi_m) = (1)$ *in* $\mathbb{Z}[q]$.
(b) *If* $\frac{m}{n} = p^e$ *for a prime* p *and some integer* $e \neq 0$, *then* $(\Phi_n) + (\Phi_m) = (1)$ *in* $\mathbb{Z}[1/p][q]$.

Note that in a commutative ring R, $(x) + (y) = (1)$ if and only if x is invertible in $R/(y)$. Also $(x) + (y) = (1)$ implies $(x^k) + (y^l) = (1)$ for any integers $k, l \geq 1$.

4.2 Habiro's results

Let us summarize some of Habiro's results on cyclotomic completions of polynomial rings [9]. Let R be a commutative integral domain of characteristic zero and $R[q]$ the polynomial ring over R. For any $S \subset \mathbb{N}$, Habiro defined the S-cyclotomic completion ring $R[q]^S$ as follows:

$$R[q]^S := \varprojlim_{f(q)\in\Phi_S^*} \frac{R[q]}{(f(q))} \tag{12}$$

where Φ_S^* denotes the multiplicative set in $\mathbb{Z}[q]$ generated by $\Phi_S = \{\Phi_n(q) \mid n \in S\}$ and directed with respect to the divisibility relation.

For example, since the sequence $(q;q)_n$, $n \in \mathbb{N}$, is cofinal to $\Phi_{\mathbb{N}}^*$, we have

$$\widehat{\mathbb{Z}[q]} \simeq \mathbb{Z}[q]^{\mathbb{N}}. \tag{13}$$

Note that if S is finite, then $R[q]^S$ is identified with the $(\prod \Phi_S)$-adic completion of $R[q]$. In particular,

$$R[q]^{\{1\}} \simeq R[[q-1]], \qquad R[q]^{\{2\}} \simeq R[[q+1]].$$

Suppose $S' \subset S$, then $\Phi^*_{S'} \subset \Phi^*_S$, hence there is a natural map

$$\rho^R_{S,S'} : R[q]^S \to R[q]^{S'}.$$

Recall important results concerning $R[q]^S$ from [9]. Two positive integers n, n' are called *adjacent* if $n'/n = p^e$ with a nonzero $e \in \mathbb{Z}$ and a prime p, such that the ring R is p-adically separated, i.e. $\bigcap_{n=1}^{\infty}(p^n) = 0$ in R. A set of positive integers is *R-connected* if for any two distinct elements n, n' there is a sequence $n = n_1, n_2, \dots, n_{k-1}, n_k = n'$ in the set, such that any two consecutive numbers of this sequence are adjacent. Theorem 4.1 of [9] says that if S is R-connected, then for any subset $S' \subset S$ the natural map $\rho^R_{S,S'} : R[q]^S \hookrightarrow R[q]^{S'}$ is an embedding.

If ζ is a root of unity of order in S, then for every $f(q) \in R[q]^S$ the evaluation $\mathrm{ev}_\zeta(f(q)) \in R[\zeta]$ can be defined by sending $q \to \zeta$. For a set Ξ of roots of unity whose orders form a subset $\mathcal{T} \subset S$, one defines the evaluation

$$\mathrm{ev}_\Xi : R[q]^S \to \prod_{\zeta \in \Xi} R[\zeta].$$

Theorem 6.1 of [9] shows that if $R \subset \mathbb{Q}$, S is R-connected, and there exists $n \in S$ that is adjacent to infinitely many elements in $\mathcal{T}$, then ev_Ξ is injective.

4.3 Taylor expansion

Fix a natural number n, then we have

$$R[q]^{\{n\}} = \lim_{\overleftarrow{k}} \frac{R[q]}{(\Phi_n^k(q))}.$$

Suppose $\mathbb{Z} \subset R \subset \mathbb{Q}$, then the natural algebra homomorphism

$$h : \frac{R[q]}{(\Phi_n^k(q))} \to \frac{R[e_n][q]}{((q-e_n)^k)}$$

is injective, by Proposition 13 below. Taking the inverse limit, we see that there is a natural injective algebra homomorphism

$$h : R[q]^{\{n\}} \to R[e_n][[q-e_n]].$$

Suppose $n \in S$. Combining h and $\rho_{S,\{n\}} : R[q]^S \to R[q]^{\{n\}}$, we get an algebra map

$$\mathfrak{t}_n : R[q]^S \to R[e_n][[q - e_n]].$$

If $f \in R[q]^S$, then $\mathfrak{t}_n(f)$ is called the Taylor expansion of f at e_n.

4.4 Splitting of $\mathcal{S}_p$ and evaluation

For every integer a, we put $\mathbb{N}_a := \{n \in \mathbb{N} \mid (a, n) = 1\}$.

Suppose p is a prime. Analogously to (13), we have

$$\mathcal{S}_p \simeq \mathbb{Z}[1/p][q]^{\mathbb{N}}.$$

Observe that $\mathbb{N}$ is not $\mathbb{Z}[1/p]$-connected. In fact one has $\mathbb{N} = \bigsqcup_{j=0}^{\infty} p^j \mathbb{N}_p$, where each $p^j \mathbb{N}_p$ is $\mathbb{Z}[1/p]$-connected. Let us define

$$\mathcal{S}_{p,j} := \mathbb{Z}[1/p][q]^{p^j \mathbb{N}_p}.$$

Note that for every $f \in \mathcal{S}_p$, the evaluation $\mathrm{ev}_\xi(f)$ can be defined for every root ξ of unity. For $f \in \mathcal{S}_{p,j}$, the evaluation $\mathrm{ev}_\xi(f)$ can be defined when ξ is a root of unity of order in $p^j \mathbb{N}_p$.

Proposition 8 *For every prime p one has*

$$\mathcal{S}_p \simeq \prod_{j=0}^{\infty} \mathcal{S}_{p,j}. \tag{14}$$

Proof Suppose $n_i \in p^{j_i} \mathbb{N}_p$ for $i = 1, \ldots, m$, with distinct j_i's. Then n_i/n_s, with $i \neq s$, is either not a power of a prime or a non-zero power of p, hence by Lemma 7 (and the remark right after Lemma 7), for any positive integers $k_1, \ldots, k_m$, we have

$$(\Phi_{n_i}^{k_i}) + (\Phi_{n_s}^{k_s}) = (1) \quad \text{in } \mathbb{Z}[1/p][q].$$

By the Chinese remainder theorem, we have

$$\frac{\mathbb{Z}[1/p][q]}{(\prod_{i=1}^m \Phi_{n_i}^{k_i})} \simeq \prod_{i=1}^m \frac{\mathbb{Z}[1/p][q]}{(\Phi_{n_i}^{k_i})}.$$

Taking the inverse limit, we get (14). □

Let $\pi_j : \mathcal{S}_p \to \mathcal{S}_{p,j}$ denote the projection onto the jth component in the above decomposition.

Lemma 9 *Suppose ξ is a root of unity of order $r = p^j r'$, with $(r', p) = 1$. Then for any $x \in \mathcal{S}_p$, one has*

$$\mathrm{ev}_\xi(x) = \mathrm{ev}_\xi(\pi_j(x)).$$

If $i \neq j$ then $\mathrm{ev}_\xi(\pi_i(x)) = 0$.

Proof Note that $\mathrm{ev}_\xi(x)$ is the image of x under the projection $\mathcal{S}_p \to \mathcal{S}_p/(\Phi_r(q)) = \mathbb{Z}[1/p][\xi]$. It remains to notice that $\mathcal{S}_{p,i}/(\Phi_r(q)) = 0$ if $i \neq j$. □

4.5 Splitting of $\mathcal{S}_b$

Suppose p is a prime divisor of b. Let

$$\mathcal{S}_b^{p,0} := \mathbb{Z}[1/b][q]^{\mathbb{N}_p} \quad \text{and} \quad \mathcal{S}_b^{p,\bar{0}} := \mathbb{Z}[1/b][q]^{p\mathbb{N}} \simeq \prod_{j>0} \mathbb{Z}[1/b][q]^{p^j \mathbb{N}_p}.$$

We have similarly

$$\mathcal{S}_b = \mathcal{S}_b^{p,0} \times \mathcal{S}_b^{p,\bar{0}}$$

with canonical projections $\pi_0^p : \mathcal{S}_b \to \mathcal{S}_b^{p,0}$ and $\pi_{\bar{0}}^p : \mathcal{S}_b \to \mathcal{S}_b^{p,\bar{0}}$. Note that if $b = p$, then $\mathcal{S}_p^{p,0} = \mathcal{S}_{p,0}$ and $\mathcal{S}_p^{p,\bar{0}} = \prod_{j>0} \mathcal{S}_{p,j}$. As before we set $\mathcal{S}_{b,0} := \mathbb{Z}[1/b][q]^{\mathbb{N}_b}$ and $\pi_0 : \mathcal{S}_b \to \mathcal{S}_{b,0}$.

Suppose $f \in \mathcal{S}_b$. If ξ is a root of unity of order coprime with p, then $\mathrm{ev}_\xi(f) = \mathrm{ev}_\xi(\pi_0^p(f))$. Similarly, if the order of ξ is divisible by p, then $\mathrm{ev}_\xi(f) = \mathrm{ev}_\xi(\pi_{\bar{0}}^p(f))$.

4.6 Properties of the ring $\mathcal{R}_b$

For any $b \in \mathbb{N}$, we have

$$\mathcal{R}_b \simeq \mathbb{Z}[1/b][q]^{\mathbb{N}_2}$$

since the sequence $(q;q^2)_k$, $k \in \mathbb{N}$, is cofinal to $\Phi^*_{\mathbb{N}_2}$. Here $\mathbb{N}_2$ is the set of all odd numbers.

Let $\{p_i \mid i = 1, \dots, m\}$ be the set of all distinct *odd* prime divisors of b. For $\mathbf{n} = (n_1, \dots, n_m)$, a tuple of numbers $n_i \in \mathbb{N}$, let $\mathbf{p}^\mathbf{n} = \prod_i p_i^{n_i}$. Let $S_\mathbf{n} := \mathbf{p}^\mathbf{n} \mathbb{N}_{2b}$. Then $\mathbb{N}_2 = \bigsqcup_\mathbf{n} S_\mathbf{n}$. Moreover, for $a \in S_\mathbf{n}$, $a' \in S_{\mathbf{n}'}$, we have $(\Phi_a(q), \Phi_{a'}(q)) = (1)$ in $\mathbb{Z}[1/b]$ if $\mathbf{n} \neq \mathbf{n}'$. In addition, each $S_\mathbf{n}$ is $\mathbb{Z}[1/b]$-connected. An argument similar to that for (14) gives

$$\mathcal{R}_b \simeq \prod_\mathbf{n} \mathbb{Z}[1/b][q]^{S_\mathbf{n}}.$$

In particular, $\mathcal{R}_b^{p_i,0} := \mathbb{Z}[1/b][q]^{\mathbb{N}_{2p_i}}$ and $\mathcal{R}_b^{p_i,\bar{0}} := \mathbb{Z}[1/b][q]^{p_i\mathbb{N}_2}$ for any $1 \leq i \leq m$. If $2 \mid b$, then $\mathcal{R}_b^{2,0}$ coincides with $\mathcal{R}_b$.

Let T be an infinite set of powers of an odd prime not dividing b and let P be an infinite set of odd primes not dividing b.

Proposition 10 *With the above notations, one has the following.*

(a) *For any $l \in S_{\mathbf{n}}$, the Taylor map $\mathfrak{t}_l : \mathbb{Z}[1/b][q]^{S_{\mathbf{n}}} \to \mathbb{Z}[1/b][e_l][[q-e_l]]$ is injective.*
(b) *Suppose $f, g \in \mathbb{Z}[1/b][q]^{S_{\mathbf{n}}}$ such that $\mathrm{ev}_\xi(f) = \mathrm{ev}_\xi(g)$ for any root of unity ξ with ord$(\xi) \in \mathbf{p}^{\mathbf{n}}T$, then $f = g$. The same holds true if $\mathbf{p}^{\mathbf{n}}T$ is replaced by $\mathbf{p}^{\mathbf{n}}P$.*
(c) *For odd b, the natural homomorphism $\rho_{\mathbb{N},\mathbb{N}_2} : \mathcal{S}_b \to \mathcal{R}_b$ is injective. If $2 \mid b$, then the natural homomorphism $\mathcal{S}_b^{2,0} \to \mathcal{R}_b$ is an isomorphism.*

Proof (a) Since each $S_{\mathbf{n}}$ is $\mathbb{Z}[1/b]$-connected in Habiro sense, by [9, Theorem 4.1], for any $l \in S_{\mathbf{n}}$

$$\rho_{S,\{l\}} : \mathbb{Z}[1/b][q]^{S_{\mathbf{n}}} \to \mathbb{Z}[1/b][q]^{\{l\}} \tag{15}$$

is injective. Hence $\mathfrak{t}_l = h \circ \rho_{S,\{l\}}$ is injective too.

(b) Since both sets contain infinitely many numbers adjacent to $\mathbf{p}^{\mathbf{n}}$, the claim follows from Theorem 6.1 in [9].

(c) Note that for odd b

$$\mathcal{S}_b \simeq \prod_{\mathbf{n}} \mathbb{Z}[1/b][q]^{S'_{\mathbf{n}}}$$

where $S'_{\mathbf{n}} := \mathbf{p}^{\mathbf{n}}\mathbb{N}_b$. Further observe that $S'_{\mathbf{n}}$ is $\mathbb{Z}[1/b]$-connected if b is odd. Then by [9, Theorem 4.1] the map

$$\mathbb{Z}[1/b][q]^{S'_{\mathbf{n}}} \hookrightarrow \mathbb{Z}[1/b][q]^{S_{\mathbf{n}}}$$

is an embedding. If $2 \mid b$, then $\mathcal{S}_b^{2,0} := \mathbb{Z}[1/b][q]^{\mathbb{N}_2} \simeq \mathcal{R}_b$. □

Assuming Theorem 1, Proposition 10(b) implies Proposition 2.

5 On the Ohtsuki series at roots of unity

The Ohtsuki series was defined for $SO(3)$ invariants by Ohtsuki [25] and extended to all other Lie algebras by the third author [15, 16].

In the works [15, 16, 25], it was proved that the sequence of quantum invariants at e_p, where p runs through the set of primes, obeys some congruence properties that allow to define uniquely the coefficients of the Ohtsuki series. The proof of the existence of such congruence relations is difficult. In [7], Habiro proved that Ohtsuki series coincide with the Taylor expansion of the unified invariant at $q=1$ in the case of integral homology spheres; this result was generalized to rational homology spheres by the third author [14].

Here, we prove that the sequence of $SO(3)$ invariants at the prth roots $e_r e_p$, where r is a fixed odd number and p runs through the set of primes, obeys some congruence properties that allow to define uniquely the coefficients of the "Ohtsuki series" at e_r, which is coincident with the Taylor expansion at e_r.

5.1 Extension of $\mathbb{Z}[1/b][e_r]$

Fix an odd positive integer r. Assume p is a prime bigger than b and r. The cyclotomic rings $\mathbb{Z}[1/b][e_{pr}]$ and $\mathbb{Z}[1/b][e_r]$ are extensions of $\mathbb{Z}[1/b]$ of degree $\varphi(rp)=\varphi(r)\varphi(p)$ and $\varphi(r)$, respectively. Hence $\mathbb{Z}[1/b][e_{pr}]$ is an extension of $\mathbb{Z}[1/b][e_r]$ of degree $\varphi(p)=p-1$. Actually, it is easy to see that for

$$f_p(q) := \frac{q^p - e_r^p}{q-e_r},$$

the map

$$\frac{\mathbb{Z}[1/b, e_r][q]}{(f_p(q))} \to \mathbb{Z}[1/b][e_{pr}], \quad q \mapsto e_p e_r,$$

is an isomorphism. We put $x = q - e_r$ and get

$$\mathbb{Z}[1/b][e_{pr}] \simeq \frac{\mathbb{Z}[1/b, e_r][x]}{(f_p(x+e_r))}. \tag{16}$$

Note that

$$f_p(x+e_r) = \sum_{n=0}^{p-1} \binom{p}{n+1} x^n e_r^{p-n-1}$$

is a monic polynomial in x of degree $p-1$, and the coefficient of x^n in $f_p(x+e_r)$ is divisible by p if $n \le p-2$.

5.2 Arithmetic expansion of τ'_M

Suppose M is a rational homology 3-sphere with $|H_1(M,\mathbb{Z})| = b$. By Theorem 1, for any root of unity ξ of order pr

$$\tau'_M(\xi) \in \mathbb{Z}[1/b][e_{pr}] \simeq \frac{\mathbb{Z}[1/b, e_r][x]}{(f_p(x+e_r))}.$$

Hence we can write

$$\tau'_M(e_r e_p) = \sum_{n=0}^{p-2} a_{p,n} x^n \tag{17}$$

where $a_{p,n} \in \mathbb{Z}[1/b, e_r]$. The following proposition shows that the coefficients $a_{p,n}$ stabilize as $p \to \infty$.

Proposition 11 *Suppose M is a rational homology 3-sphere with $|H_1(M,\mathbb{Z})| = b$, and r is an odd positive integer. For every non-negative integer n, there exists a unique invariant $a_n = a_n(M) \in \mathbb{Z}[1/b, e_r]$ such that for every prime $p > \max(b, r)$, we have*

$$a_n \equiv a_{p,n} \pmod{p} \quad \text{in } \mathbb{Z}[1/b, e_r] \text{ for } 0 \le n \le p-2. \tag{18}$$

Moreover, the formal series $\sum_n a_n (q - e_r)^n$ is equal to the Taylor expansion of the unified invariant I_M at e_r.

Proof The uniqueness of a_n follows from the easy fact that if $a \in \mathbb{Z}[1/b, e_r]$ is divisible by infinitely many rational primes p, then $a = 0$.

Assume Theorem 1 holds. We define a_n to be the coefficient of $(q - e_r)^n$ in the Taylor series of I_M at e_r, and will show that (18) holds true.

Recall that $x = q - e_r$. The diagram

$$\begin{array}{ccccc}
\mathbb{Z}[\frac{1}{b}][q]^{\mathbb{N}_2} & \longrightarrow & \mathbb{Z}[\frac{1}{b}, e_r][q]^{r\mathbb{N}_2} & \longrightarrow & \mathbb{Z}[\frac{1}{b}, e_r][[x]] \\
\big\downarrow{\scriptstyle q \to e_r e_p} & & \big\downarrow{\scriptstyle /(f_p(q))} & & \big\downarrow{\scriptstyle /(f_p(x+e_r))} \\
\mathbb{Z}[\frac{1}{b}][e_{rp}] & \xrightarrow{e_r e_p \to q} & \frac{\mathbb{Z}[\frac{1}{b}, e_r][q]}{(f_p(q))} & \longrightarrow & \frac{\mathbb{Z}[\frac{1}{b}, e_r][[x]]}{(f_p(x+e_r))}
\end{array}$$

is commutative. Here the middle and the right vertical maps are the quotient maps by the corresponding ideals. Note that I_M belongs to the upper left corner ring, its Taylor series is the image in the upper right corner ring, while the evaluation (17) is in the lower middle ring. Using the commutativity at the lower right corner ring, we see that

$$\sum_{n=0}^{p-2} a_{p,n} x^n = \sum_{n=0}^{\infty} a_n x^n \pmod{f_p(x+e_r)} \quad \text{in } \mathbb{Z}[1/b, e_r][[x]].$$

Since the coefficients of $f_p(x + e_r)$ up to degree $p - 2$ are divisible by p, we get the congruence (18). □

Remark 12 Proposition 11, when $r = 1$, was the main result of Ohtsuki [25], which leads to the development of the theory of finite type invariant and the LMO invariant.

When $(r,b)=1$, then Taylor series at e_r determines and is determined by the Ohtsuki series. But when, say, r is a divisor of b, a priori the two Taylor series, one at e_r and the other at 1, are independent. We suspect that the Taylor series at e_r, with $r \mid b$, corresponds to a new type of LMO invariant.

6 Frobenius maps

The proof of Theorem 4, and hence of the main theorem, uses the Laplace transform method. The aim of this section is to show that the image of the Laplace transform, defined in Sect. 8, belongs to $\mathcal{R}_b$, i.e. that certain roots of q exist in $\mathcal{R}_b$.

6.1 On the module $\mathbb{Z}[q]/(\Phi_n^k(q))$

Since cyclotomic completions are built from modules like $\mathbb{Z}[q]/(\Phi_n^k(q))$, we first consider these modules. Fix $n,k\geq 1$. Let

$$E := \frac{\mathbb{Z}[q]}{(\Phi_n^k(q))}, \quad \text{and} \quad G := \frac{\mathbb{Z}[e_n][x]}{(x^k)}.$$

The following is probably well-known.

Proposition 13

(a) *Both E and G are free $\mathbb{Z}$-modules of the same rank $k\varphi(n)$.*
(b) *The algebra map $h:\mathbb{Z}[q]\to\mathbb{Z}[e_n][x]$ defined by*

$$h(q)=e_n+x$$

descends to a well-defined algebra homomorphism, also denoted by h, from E to G. Moreover, the algebra homomorphism $h: E\to G$ is injective.

Proof (a) Since $\Phi_n^k(q)$ is a monic polynomial in q of degree $k\varphi(n)$, it is clear that

$$E=\mathbb{Z}[q]/(\Phi_n^k(q))$$

is a free $\mathbb{Z}$-module of rank $k\varphi(n)$. Since $G=\mathbb{Z}[e_n]\otimes_{\mathbb{Z}}\mathbb{Z}[x]/(x^k)$, we see that G is free over $\mathbb{Z}$ of rank $k\varphi(n)$.

(b) To prove that h descends to a map $E\to G$, one needs to verify that $h(\Phi_n^k(q))=0$. Note that

$$h(\Phi_n^k(q))=\Phi_n^k(x+e_n)=\prod_{(j,n)=1}(x+e_n-e_n^j)^k.$$

When $j=1$, the factor is x^k, which is 0 in $\mathbb{Z}[e_n][x]/(x^k)$. Hence $h(\Phi_n^k(q)) = 0$.

Now we prove that h is injective. Let $f(q)\in\mathbb{Z}[q]$. Suppose $h(f(q))=0$, or $f(x+e_n)=0$ in $\mathbb{Z}[e_n][x]/(x^k)$. It follows that $f(x+e_n)$ is divisible by x^k; or that $f(x)$ is divisible by $(x-e_n)^k$. Since f is a polynomial with coefficients in $\mathbb{Z}$, it follows that $f(x)$ is divisible by all Galois conjugates $(x-e_n^j)^k$ with $(j,n)=1$. Then f is divisible by $\Phi_n^k(q)$. In other words, $f=0$ in $E=\mathbb{Z}[q]/(\Phi_n^k(q))$. □

6.2 A Frobenius homomorphism

We use E and G of the previous subsection. Suppose b is a positive integer coprime with n. If ξ is a primitive nth root of 1, i.e. $\Phi_n(\xi)=0$, then ξ^b is also a primitive nth root of 1, i.e. $\Phi_n(\xi^b)=0$. It follows that $\Phi_n(q^b)$ is divisible by $\Phi_n(q)$.

Therefore the algebra map $F_b:\mathbb{Z}[q]\to\mathbb{Z}[q]$, defined by $F_b(q)=q^b$, descends to a well-defined algebra map, also denoted by F_b, from E to E. We want to understand the image $F_b(E)$.

Proposition 14 *The image $F_b(E)$ is a free $\mathbb{Z}$-submodule of E of maximal rank, i.e.* $\mathrm{rk}(F_b(E))=\mathrm{rk}(E)$. *Moreover, the index of $F_b(E)$ in E is $b^{k(k-1)\varphi(n)/2}$.*

Proof Using Proposition 13 we identify E with its image $h(E)$ in G.

Let $\tilde{F}_b:G\to G$ be the $\mathbb{Z}$-algebra homomorphism defined by $\tilde{F}_b(e_n)=e_n^b$, $\tilde{F}_b(x)=(x+e_n)^b-e_n^b$.

Note that $\tilde{F}_b(x)=be_n^{b-1}x+O(x^2)$, hence $\tilde{F}_b(x^k)=0$. It is easy to see that $\tilde{F}_b$ is a well-defined algebra homomorphism, and that $\tilde{F}_b$ restricted to E is exactly F_b. Since E is a lattice of maximal rank in $G\otimes\mathbb{Q}$, it follows that the index of F_b is exactly the determinant of $\tilde{F}_b$, acting on $G\otimes\mathbb{Q}$.

A basis of G is $e_n^jx^l$, with $(j,n)=1, 0<j<n$ and $j=0$, and $0\le l<k$. Note that

$$\tilde{F}_b(e_n^jx^l)=b^le_n^{jb}e_n^{(b-1)l}x^l+O(x^{l+1}).$$

Since $(b,n)=1$, the set e_n^{jb}, with $(j,n)=1$ is the same as the set e_n^j, with $(j,n)=1$. Let $f_1:G\to G$ be the $\mathbb{Z}$-linear map defined by $f_1(e_n^{jb}x^l)=e_n^jx^l$. Since f_1 permutes the basis elements, its determinant is ±1. Let $f_2:G\to G$ be the $\mathbb{Z}$-linear map defined by $f_2(e_n^jx^l)=e_n^j(e_n^{1-b}x)^l$. The determinant of f_2 is again ±1. This is because, for any fixed l, f_2 restricts to the automorphism of $\mathbb{Z}[e_n]$ sending a to e_n^sa, each of these maps has a well-defined inverse: $a\mapsto e_n^{-s}a$. Now

$$f_1f_2\tilde{F}_b(e_n^jx^l)=b^le_n^jx^l+O(x^{l+1})$$

can be described by an upper triangular matrix with b^l's on the diagonal; its determinant is equal to $b^{k(k-1)\varphi(n)/2}$. □

From the proposition we see that if b is invertible, then the index is equal to 1, hence we have

Proposition 15 *For any n coprime with b and $k \in \mathbb{N}$, the Frobenius homomorphism $F_b : \mathbb{Z}[1/b][q]/(\Phi_n^k(q)) \to \mathbb{Z}[1/b][q]/(\Phi_n^k(q))$, defined by $F_b(q) = q^b$, is an isomorphism.*

6.3 Frobenius endomorphism of $\mathcal{S}_{b,0}$

For finitely many $n_i \in \mathbb{N}_b$ and $k_i \in \mathbb{N}$, the Frobenius endomorphism

$$F_b : \frac{\mathbb{Z}[1/b][q]}{\left(\prod_i \Phi_{n_i}^{k_i}(q)\right)} \to \frac{\mathbb{Z}[1/b][q]}{\left(\prod_i \Phi_{n_i}^{k_i}(q)\right)}$$

sending q to q^b, is again well-defined. Taking the inverse limit, we get an algebra endomorphism

$$F_b : \mathbb{Z}[1/b][q]^{\mathbb{N}_b} \to \mathbb{Z}[1/b][q]^{\mathbb{N}_b}.$$

Theorem 16 *For any subset $T \subset \mathbb{N}_b$, the Frobenius endomorphism $F_b : \mathbb{Z}[1/b][q]^T \to \mathbb{Z}[1/b][q]^T$, sending q to q^b, is an isomorphism.*

Proof For finitely many $n_i \in \mathbb{N}_b$ and $k_i \in \mathbb{N}$, consider the natural algebra homomorphism

$$J : \frac{\mathbb{Z}[1/b][q]}{\left(\prod_i \Phi_{n_i}^{k_i}(q)\right)} \to \prod_i \frac{\mathbb{Z}[1/b][q]}{\left(\Phi_{n_i}^{k_i}(q)\right)}.$$

This map is injective, because in the unique factorization domain $\mathbb{Z}[1/b][q]$, one has

$$(\Phi_{n_1}(q)^{k_1} \dots \Phi_{n_s}(q)^{k_s}) = \bigcap_{j=1}^{s} \Phi_{n_j}(q)^{k_j}.$$

Since the Frobenius homomorphism commutes with J and is an isomorphism on the target of J by Proposition 15, it is an isomorphism on the domain of J. Taking the inverse limit, we get the claim. □

6.4 Existence of bth root of q in $\mathcal{S}_{b,0}$

Lemma 17 *Suppose n and b are coprime positive integers and $y \in \mathbb{Q}[e_n]$ such that $y^b = 1$. Then $y = \pm 1$. If b is odd then $y = 1$.*

Proof Let $d \mid b$ be the order of y, i.e. y is a primitive dth root of 1. Then $\mathbb{Q}[e_n]$ contains y, and hence e_d. Since $(n,d)=1$, one has $\mathbb{Q}[e_n] \cap \mathbb{Q}[e_d] = \mathbb{Q}$ (see e.g. [13, Corollary of IV.3.2]). Hence if $e_d \in \mathbb{Q}[e_n]$, then $e_d \in \mathbb{Q}$, it follows that $d=1$ or 2. Thus $y=1$ or $y=-1$. If b is odd, then y cannot be -1. □

Lemma 18 *Let b be a positive integer, $T \subset \mathbb{N}_b$, and $y \in \mathbb{Q}[q]^T$ satisfying $y^b = 1$. Then $y = \pm 1$. If b is odd then $y = 1$.*

Proof It suffices to show that for any $n_1, n_2, \dots, n_m \in T$, the ring $\mathbb{Q}[q]/(\Phi_{n_1}^{k_1} \dots \Phi_{n_m}^{k_m})$ does not contains a bth root of 1 except possibly for ± 1. Using the Chinese remainder theorem, it suffices to consider the case where $m = 1$.

The ring $\mathbb{Q}[q]/(\Phi_n^k(q))$ is isomorphic to $\mathbb{Q}[e_n][x]/(x^k)$, by Proposition 13. If

$$y = \sum_{j=0}^{k-1} a_j x^j, \quad a_j \in \mathbb{Q}[e_n]$$

satisfies $y^b = 1$, then it follows that $a_0^b = 1$. By Lemma 17 we have $a_0 = \pm 1$. One can easily see that $a_1 = \cdots = a_{k-1} = 0$. Thus $y = \pm 1$. □

In contrast with Lemma 18, we have

Proposition 19 *For any odd positive b, and any subset $T \subset \mathbb{N}_b$, the ring $\mathbb{Z}[1/b][q]^T$ contains a unique bth root of q, which is invertible in $\mathbb{Z}[1/b][q]^T$.*

For any even positive b, and any subset $T \subset \mathbb{N}_b$, the ring $\mathbb{Z}[1/b][q]^T$ contains two bth roots of q, which are invertible in $\mathbb{Z}[1/b][q]^T$; one is the negative of the other.

Proof Let us first consider the case $T = \mathbb{N}_b$. Since F_b is an isomorphism by Theorem 16, we can define a bth root of q by

$$q^{1/b} := F_b^{-1}(q) \in \mathcal{S}_{b,0}.$$

If y_1 and y_2 are two bth root of the same element, then their ratio y_1/y_2 is a bth root of 1. From Lemma 18 it follows that if b is odd, there is only one bth root of q in $\mathbb{Z}[1/b][q]^{\mathbb{N}_b}$, and if b is even, there are 2 such roots, one is the minus of the other. We will denote them $\pm q^{1/b}$.

Further it is known that q is invertible in $\mathbb{Z}[q]^{\mathbb{N}}$ (see [9]). Actually, there is an explicit expression $q^{-1} = \sum_n q^n (q;q)_n$. Hence $q^{-1} \in \mathbb{Z}[1/b][q]^{\mathbb{N}_b}$, since the natural homomorphism from $\mathbb{Z}[q]^{\mathbb{N}}$ to $\mathbb{Z}[1/b][q]^{\mathbb{N}_b}$ maps q to q. In a commutative ring, if $x \mid y$ and y is invertible, then so is x. Hence any root of q is invertible.

In the general case of $T \subset \mathbb{N}_b$, we use the natural map $\mathbb{Z}[1/b][q]^{\mathbb{N}_b} \hookrightarrow \mathbb{Z}[1/b][q]^T$. □

Relation with [14]

By Proposition 19, $\mathcal{S}_{b,0}$ is isomorphic to the ring $\Lambda_b^{\mathbb{N}_b} := \mathbb{Z}[1/b][q^{1/b}]^{\mathbb{N}_b}$ used in [14]. Furthermore, our invariant $\pi_0 I_M$ and the one defined in [14] belong to $\mathcal{S}_{b,0}$. This follows from Theorem 1 for b odd, and from Proposition 10(c) for b even. Finally, the invariant defined in [14] for M divided by the invariant of $\#_i L(b_i^{k_i}, 1)$ (which is invertible in $\mathcal{S}_{b,0}$ [14, Sect. 4.1]) coincides with $\pi_0 I_M$ up to factor $q^{\frac{1-b}{4}}$ by Theorem 1, [14, Theorem 3] and Proposition 10(b).

6.5 Another Frobenius homomorphism

We define another Frobenius type algebra homomorphism. The difference of the two types of Frobenius homomorphisms is in the target spaces of these homomorphisms.

Suppose m is a positive integer. Define the algebra homomorphism

$$G_m : R[q]^T \to R[q]^{mT} \quad \text{by } G_m(q) = q^m.$$

Since $\Phi_{mr}(q)$ always divides $\Phi_r(q^m)$, G_m is well-defined.

6.6 Realization of $q^{a^2/b}$ in $\mathcal{S}_p$

In this subsection we will construct elements $z_{a,b} \in \mathcal{S}_p$ which will be used in Sect. 8 and which realize the value of $q^{a^2/b}$, evaluated at $q = \xi$ in a certain way.

Throughout this subsection, let p be a prime or 1, $b = \pm p^l$ for an $l \in \mathbb{N}$, and a an integer. Let $B_{p,j} = G_{p^j}(\mathcal{S}_{p,0})$. Note that $B_{p,j} \subset \mathcal{S}_{p,j}$. If b is odd, by Proposition 19 there is a unique bth root of q in $\mathcal{S}_{p,0}$; we denote it by $x_{b;0}$. If b is even, by Proposition 19 there are exactly two bth root of q, namely $\pm q^{1/b}$. We put $x_{b;0} = q^{1/b}$. We define an element $z_{b,a} \in \mathcal{S}_p$ as follows.

If $b \mid a$, let $z_{b,a} := q^{a^2/b} \in \mathcal{S}_p$.

If $b = \pm p^l \nmid a$, then $z_{b,a} \in \mathcal{S}_p$ is defined by specifying its projections $\pi_j(z_{b,a}) := z_{b,a;j} \in \mathcal{S}_{p,j}$ as follows. Suppose $a = p^s e$, with $(e, p) = 1$. Then $s < l$. For $j > s$ let $z_{b,a;j} := 0$. For $0 \le j \le s$ let

$$z_{b,a;j} := [G_{p^j}(x_{b;0})]^{a^2/p^j} = [G_{p^j}(x_{b;0})]^{e^2 p^{2s-j}} \in B_{p,j} \subset \mathcal{S}_{p,j}.$$

Similarly, for $b = \pm p^l$ we define an element $x_b \in \mathcal{S}_p$ as follows. We put $\pi_0(x_b) := x_{b;0}$. For $j < l$, $\pi_j(x_b) := [G_{p^j}(x_{b;0})]^{p^j}$. If $j \ge l$, $\pi_j(x_b) := q^b$. Notice that for $c = (b, p^j)$ we have

$$\pi_j(x_b) = z_{b,c;j}.$$

Proposition 20 *Suppose ξ is a root of unity of order $r = cr'$, where $c = (r, b)$. Then*

$$\mathrm{ev}_\xi(z_{b,a}) = \begin{cases} 0 & \text{if } c \nmid a, \\ (\xi^c)^{a_1^2 b'_*} & \text{if } a = ca_1, \end{cases}$$

where b'_* is the unique element in $\mathbb{Z}/r'\mathbb{Z}$ such that $b'_*(b/c) \equiv 1 \pmod{r'}$. Moreover,

$$\mathrm{ev}_\xi(x_b) = (\xi^c)^{b'_*}.$$

Proof Let us compute $\mathrm{ev}_\xi(z_{b,a})$. The case of $\mathrm{ev}_\xi(x_b)$ is completely analogous.

If $b \mid a$, then $c \mid a$, and the proof is obvious.

Suppose $b \nmid a$. Let $a = p^s e$ and $c = p^i$. Then $s < l$. Recall that $z_{b,a} = \prod_{j=0}^{\infty} z_{b,a;j}$. By Lemma 9,

$$\mathrm{ev}_\xi(z_{b,a}) = \mathrm{ev}_\xi(z_{b,a;i}).$$

If $c \nmid a$, then $i > s$. By definition, $z_{b,a;i} = 0$, hence the statement holds true.

It remains the case $c \mid a$, or $i \le s$. Note that $\zeta = \xi^c$ is a primitive root of order r' and $(p, r') = 1$. Since $z_{b,a;i} \in B_{p,i}$,

$$\mathrm{ev}_\xi(z_{b,a;i}) \in \mathbb{Z}[1/p][\zeta].$$

From the definition of $z_{b,a;i}$ it follows that $(z_{b,a;i})^{b/c} = (q^c)^{a^2/c^2}$, hence after evaluation we have

$$[\mathrm{ev}_\xi(z_{b,a;i})]^{b/c} = (\zeta)^{a_1^2}.$$

Note also that

$$[(\xi^c)^{a_1^2 b'_*}]^{b/c} = (\zeta)^{a_1^2}.$$

Using Lemma 17 we conclude $\mathrm{ev}_\xi(z_{b,a;i}) = (\xi^c)^{a_1^2 b'_*}$ if b is odd, and $\mathrm{ev}_\xi(z_{b,a;i}) = (\xi^c)^{a_1^2 b'_*}$ or $\mathrm{ev}_\xi(z_{b,a;i}) = -(\xi^c)^{a_1^2 b'_*}$ if b is even. Since $\mathrm{ev}_1(q^{1/b}) = 1$ and therefore $\mathrm{ev}_\xi(q^{1/b}) = \xi^{b*}$ (and not $-\xi^{b*}$) we get the claim. □

7 Invariant of lens spaces

The purpose of this section is to prove Lemma 6. Throughout this section we will use the following notations.

Let a and b be coprime integers. Choose $\hat{a}$ and $\hat{b}$ such that $b\hat{b} + a\hat{a} = 1$ with $0 < \mathrm{sn}(a)\hat{a} < |b|$. Notice that for $a = 1$ we have $\hat{1} = 1$ and $\hat{b} = 0$.

Let r be a fixed odd integer (the order of ξ). For $l \in \mathbb{Z}$ coprime to r, let l_* denote the inverse of l modulo r. If $(b, r) = c$, let b'_* denote the inverse of $b' := \frac{b}{c}$ modulo $r' := \frac{r}{c}$. Notice that for $c = 1$, we have $b_* = b'_*$.

Further, we denote by $(\frac{x}{y})$ the Jacobi symbol and by $s(a, b)$ the Dedekind sum (see e.g. [11]).

7.1 Invariants of lens spaces

Let us compute the $SO(3)$ invariant of the lens space $M(b, a; d)$. Recall that $M(b, a; d)$ is the lens space $L(b, a)$ together with a knot K inside colored by d (see Fig. 1).

Proposition 21 *Suppose $c = (b, r)$ divides $d - \operatorname{sn}(a)\hat{a}$. Then*

$$\tau'_{M(b,a;d)}(\xi) = (-1)^{\frac{c+1}{2}\frac{\operatorname{sn}(ab)-1}{2}} \left(\frac{|a|}{c}\right) \left(\frac{1-\xi^{-\operatorname{sn}(a)db'_*}}{1-\xi^{-\operatorname{sn}(b)b'_*}}\right)^{\chi(c)}$$
$$\times\, \xi^{4_*u - 4_*b'_*\frac{a(\hat{a}-\operatorname{sn}(a)d)^2}{c}}$$

where

$$u = 12s(1, b) - 12\operatorname{sn}(b)s(a, b)$$
$$+ \frac{1}{b}\big(a(1-d^2) + 2(\operatorname{sn}(a)d - \operatorname{sn}(b)) + a(\hat{a} - \operatorname{sn}(a)d)^2\big) \in \mathbb{Z}$$

and $\chi(c) = 1$ if $c = 1$ and is zero otherwise. If $c \nmid (\hat{a} \pm d)$, $\tau_{M(b,a;d)}(\xi) = 0$.

In particular, it follows that $\tau_{L(b,a)}(\xi) = 0$ if $c \nmid \hat{a} \pm 1$.

Proof We consider first the case where $b, a > 0$. Since two lens spaces $L(b, a_1)$ and $L(b, a_2)$ are homeomorphic if $a_1 \equiv a_2 \pmod{b}$, we can assume $a < b$. Let b/a be given by a continued fraction

$$\frac{b}{a} = m_n - \cfrac{1}{m_{n-1} - \cfrac{1}{m_{n-2} - \cdots \cfrac{1}{m_2 - \cfrac{1}{m_1}}}}.$$

Using the Lagrange identity

$$a - \frac{1}{b} = (a-1) + \cfrac{1}{1 + \cfrac{1}{(b-1)}}$$

we can assume $m_i \geq 2$ for all i.

The $\tau_{M(b,a;d)}(\xi)$ can be computed in the same way as the invariant $\xi_r(L(b,a),A)$ in [19], after replacing A^2 (respectively A) by ξ^{2_*} (respectively ξ^{4_*}). Representing the b/a-framed unknot in Fig. 1 by a Hopf chain (as e.g. in Lemma 3.1 of [3]), we have

$$\begin{aligned}F_{L\sqcup K}(\xi,d) &= \sum_{j_1,\dots,j_n}^{\xi} \prod_{i=1}^{n} q^{m_i\frac{j_i^2-1}{4}} \prod_{i=1}^{n-1}[j_i j_{i+1}]\cdot[j_n d][j_1]\\ &= \frac{S_n(d)}{(\xi^{2_*}-\xi^{-2_*})^{n+1}}\cdot \xi^{-4_*\sum_{i=1}^n m_i}\end{aligned}$$

where

$$\begin{aligned}S_n(d) = \sum_{\substack{j_i=1\\ \text{odd}}}^{2r} \xi^{4_*\sum m_i j_i^2}(\xi^{2_* j_1}-\xi^{-2_* j_1})(\xi^{2_* j_1 j_2}-\xi^{-2_* j_1 j_2})\dots\\ (\xi^{2_* j_{n-1}j_n}-\xi^{-2_* j_{n-1}j_n})(\xi^{2_* j_n d}-\xi^{-2_* j_n d}).\end{aligned}$$

Using Lemmas 4.11, 4.12 and 4.20 of [19][1] (and replacing e_r by ξ^{4_*}, c_n by c, $N_{n,1}=p$ by b, $N_{n-1,1}=q$ by a, $N_{n,2}=q^*$ by $\hat{a}$ and $-N_{n-1,2}=p^*$ by $\hat{b}$), we get

$$\begin{aligned}S_n(d) &= (-2)^n(\sqrt{r}\epsilon(r))^n\sqrt{c}\epsilon(c)\left(\frac{\frac{b}{c}}{\frac{r}{c}}\right)\left(\frac{a}{c}\right)\\ &\quad\times(-1)^{\frac{r-1}{2}\frac{c-1}{2}}\cdot\sum_{\pm}\chi^{\pm}(d)\xi^{-ca4_*b'_*\left(\frac{d\mp\hat{a}}{c}\right)^2\pm 2_*\hat{b}(d\mp\hat{a})+4_*\hat{a}\hat{b}}\end{aligned}$$

where $\chi^{\pm}(d)=\pm 1$ if $c\mid d\mp\hat{a}$ and is zero otherwise. Further $\epsilon(x)=1$ if $x\equiv 1 \pmod 4$ and $\epsilon(x)=I$ if $x\equiv 3 \pmod 4$. This implies the second claim of the lemma.

Note that when $c=1$, both $\chi^{\pm}(d)$ are nonzero. If $c>1$ and $c\mid(d-\hat{a})$, $\chi^+(d)=1$, but $\chi^-(d)=0$. Indeed, for c dividing $d-\hat{a}$, $c\mid(d+\hat{a})$ if and only if $c\mid\hat{a}$ which is impossible, because $c\mid b$ but $(b,\hat{a})=1$.

Inserting the last formula into the Definition (5) we get

$$\tau_{M(b,a;d)}(\xi) = \frac{S_n(d)}{\xi^{2_*}-\xi^{-2_*}}\left(-2\xi^{-3\cdot 4_*}\sum_{j=1}^{r}\xi^{4_* j^2}\right)^{-n}\xi^{-4_*\sum_{i=1}^n m_i}$$

[1] There are misprints in Lemma 4.21: $q^*\pm n$ should be replaced by $q^*\mp n$ for $n=1,2$.

where we used that $\sigma_+ = n$ and $\sigma_- = 0$ (compare [11, p. 243]). From $\sum_{j=1}^{r} \xi^{4_* j^2} = \epsilon(r)\sqrt{r}$, we obtain

$$\tau_{M(b,a;d)}(\xi) = (-1)^{\frac{(c-1)(r-1)}{4}} \epsilon(c) \left(\frac{b'}{r'}\right)\left(\frac{a}{c}\right) \times \sqrt{c}\frac{(1-\xi^{-db'_*})^{\chi(c)}}{\xi^{2_*}-\xi^{-2_*}}\, \xi^{4_*(3n-\sum_i m_i)-4_*\hat{b}(\hat{a}-2d)-4_* b'_* \frac{a(d-\hat{a})^2}{c}}.$$

Applying the following formulas for the Dedekind sum (compare [11, Theorem 1.12])

$$3n - \sum_i m_i = -12s(a,b) + \frac{a+\hat{a}}{b}, \qquad -3+b = 12s(1,b) - \frac{2}{b} \tag{19}$$

and dividing the formula for $\tau_{M(b,a;d)}(\xi)$ by the formula for $\tau_{L(b,1)}(\xi)$ we get

$$\tau'_{M(b,a;d)}(\xi) = \left(\frac{a}{c}\right)\left(\frac{1-\xi^{-db'_*}}{1-\xi^{-b'_*}}\right)^{\chi(c)} \xi^{4_* u - 4_* b'_* \frac{a(d-\hat{a})^2}{c}}$$

where

$$u = -12s(a,b) + 12s(1,b) + \frac{1}{b}\Big(a+\hat{a}-2-\hat{b}b(\hat{a}-2d)\Big).$$

Notice, that $u \in \mathbb{Z}$. Further observe, that by using $a\hat{a}+b\hat{b}=1$, we get

$$a+\hat{a}-2-\hat{b}b(\hat{a}-2d) = 2(d-1) + a(1-d^2) + a(\hat{a}-d)^2.$$

This implies the result for $0<a<b$.

To compute $\tau_{M(-b,a;d)}(\xi)$, observe that $\tau_{M(b,-a;d)} = \tau_{M(-b,a;d)}$ is equal to the complex conjugate of $\tau_{M(b,a;d)}$. The ratio

$$\tau'_{M(-b,a;d)}(\xi) = \frac{\overline{\tau_{M(b,a;d)}(\xi)}}{\tau_{L(b,1)}(\xi)}$$

can be computed analogously. Using $\overline{\epsilon(c)} = (-1)^{\frac{c-1}{2}}\epsilon(c)$, we have for $a, b > 0$

$$\tau'_{M(-b,a,d)}(\xi) = (-1)^{\frac{c+1}{2}}\left(\frac{a}{c}\right)\left(\frac{1-\xi^{db'_*}}{1-\xi^{-b'_*}}\right)^{\chi(c)} \xi^{4_*\tilde{u}+4_* b'_* \frac{a(d-\hat{a})^2}{c}}$$

where

$$\tilde{u} = 12s(a,b) + 12s(1,b) + \frac{1}{b}\big(-a - \hat{a} - 2 + \hat{b}b(\hat{a} - 2d)\big).$$

Using $s(a,b) = s(a,-b) = -s(-a,b)$, we get the result. □

Example For $b > 0$, we have

$$\tau'_{L(-b,1)}(\xi) = (-1)^{\frac{c+1}{2} - \chi(c)} \xi^{2_*(b-3)+b_*\chi(c)}.$$

7.2 Proof of Lemma 6

Assume $b = \pm p^l$ and p is prime. We have to define the unified invariant of $M^\varepsilon(b,a) := M(b,a;d(\varepsilon))$, where $d(0) = 1$ and $d(\bar{0})$ is the smallest odd positive integer such that $\operatorname{sn}(a)ad(\bar{0}) \equiv 1 \pmod{b}$. First observe that such $d(\bar{0})$ always exists. Indeed, if p is odd, we can achieve this by adding b, otherwise the inverse of any odd number modulo 2^l is odd.

Recall that we denote the unique positive bth root of q in $S_{p,0}$ by $q^{\frac{1}{b}}$. We define the unified invariant $I_{M^\varepsilon(b,a)} \in \mathcal{R}_b$ as follows. If $p \neq 2$, then $I_{M^\varepsilon(b,a)} \in \mathcal{S}_p$ is defined by specifying its projections

$$\pi_j I_{M^\varepsilon(b,a)} := \begin{cases} q^{3s(1,b)-3\operatorname{sn}(b)s(a,b)} & \text{if } j = 0,\ \varepsilon = 0, \\ (-1)^{\frac{p^j+1}{2}\frac{\operatorname{sn}(ab)-1}{2}} \left(\frac{|a|}{p}\right)^j q^{\frac{u'}{4}} & \text{if } 0 < j < l,\ \varepsilon = \bar{0}, \\ (-1)^{\frac{p^l+1}{2}\frac{\operatorname{sn}(ab)-1}{2}} \left(\frac{|a|}{p}\right)^l q^{\frac{u'}{4}} & \text{if } j \geq l,\ \varepsilon = \bar{0}, \end{cases}$$

where $u' := u - \frac{a(\hat{a}-\operatorname{sn}(a)d(\bar{0}))^2}{b}$ and u is defined in Proposition 21. If $p = 2$, then only $\pi_0 I_{M(b,a)} \in \mathcal{S}_{2,0} = \mathcal{R}_2$ is non-zero and it is defined to be $q^{3s(1,b)-3\operatorname{sn}(b)s(a,b)}$.

The $I_{M^\varepsilon(b,a)}$ is well-defined due to Lemma 22 below, i.e. all powers of q in $I_{M^\varepsilon(b,a)}$ are integers for $j > 0$ or lie in $\frac{1}{b}\mathbb{Z}$ for $j = 0$. Further, for b odd (respectively even) $I_{M^\varepsilon(b,a)}$ is invertible in $\mathcal{S}_p^{p,\varepsilon}$ (respectively $\mathcal{R}_p^{p,\varepsilon}$) since q and $q^{\frac{1}{b}}$ are invertible in these rings.

In particular, for odd $b = p^l$, we have $I_{L(b,1)} = 1$, and

$$\pi_j I_{L(-b,1)} = \begin{cases} q^{\frac{b-3}{2}+\frac{1}{b}} & \text{if } j = 0, \\ (-1)^{\frac{p^j+1}{2}} q^{\frac{b-3}{2}} & \text{if } 0 < j < l,\ p \text{ odd}, \\ (-1)^{\frac{p^l+1}{2}} q^{\frac{b-3}{2}} & \text{if } j \geq l,\ p \text{ odd}. \end{cases}$$

It is left to show, that for any ξ of order r coprime with p, we have

$$\operatorname{ev}_\xi(I_{M^0(b,a)}) = \tau'_{M^0(b,a)}(\xi)$$

and if $r = p^j k$ with $j > 0$, then

$$\mathrm{ev}_\xi(I_{M^{\bar{0}}(b,a)}) = \tau'_{M^{\bar{0}}(b,a)}(\xi)\,.$$

For $\varepsilon = 0$, this follows directly from Propositions 20 and 21 with $c = d = 1$. For $\varepsilon = \bar{0}$, we have $c = (p^j, b) > 1$ and we get the claim by using Proposition 21 and

$$\xi^{\frac{a(\hat{a}-\mathrm{sn}(a)d(\bar{0}))^2}{b}} = \xi^{c\,\frac{a(\hat{a}-\mathrm{sn}(a)d(\bar{0}))^2}{bc}} = \xi^{bb'_*\,\frac{a(\hat{a}-\mathrm{sn}(a)d(\bar{0}))^2}{bc}} = \xi^{b'_*\,\frac{a(\hat{a}-\mathrm{sn}(a)d(\bar{0}))^2}{c}}, \quad (20)$$

where for the second equality we use $c \equiv bb'_* \pmod r$. Notice that due to part (2) of Lemma 22 below, b and c divide $\hat{a} - \mathrm{sn}(a)d(\bar{0})$ and therefore all powers of ξ in (20) are integers. □

The following Lemma is used in the proof of Lemma 6.

Lemma 22 *We have*

(a) $3s(1, b) - 3\,\mathrm{sn}(b)\, s(a, b) \in \frac{1}{b}\mathbb{Z}$,
(b) $b \mid \hat{a} - \mathrm{sn}(a)d(\bar{0})$ *and therefore* $u' \in \mathbb{Z}$, *and*
(c) $4 \mid u'$ *for* $d = d(\bar{0})$.

Proof The first claim follows from the formulas (19) for the Dedekind sum. The second claim follows from the fact that $(a, b) = 1$ and

$$a(\hat{a} - \mathrm{sn}(a)d) = 1 - \mathrm{sn}(a)ad - b\hat{b} \equiv 0 \pmod b,$$

since d is chosen such that $\mathrm{sn}(a)ad \equiv 1 \pmod b$. For the third claim, notice that for odd d we have

$$4 \mid (1 - d^2) \quad \text{and} \quad 4 \mid 2(\mathrm{sn}(a)d - \mathrm{sn}(b)). \qquad \square$$

8 Laplace transform

This section is devoted to the proof of Theorem 4 by using Andrew's identity. Throughout this section, let p be a prime or $p = 1$, and $b = \pm p^l$ for an $l \in \mathbb{N}$.

8.1 Definition

The Laplace transform is a $\mathbb{Z}[q^{\pm 1}]$-linear map defined by

$$\mathcal{L}_b : \mathbb{Z}[z^{\pm 1}, q^{\pm 1}] \to \mathcal{S}_p,$$
$$z^a \mapsto z_{b,a}.$$

In particular, we put $\mathcal{L}_{b;j} := \pi_j \circ \mathcal{L}_b$ and have $\mathcal{L}_{b;j}(z^a) = z_{b,a;j} \in \mathcal{S}_{p,j}$.

Further, for any $f \in \mathbb{Z}[z^{\pm 1}, q^{\pm 1}]$ and $n \in \mathbb{Z}$, we define

$$\hat{f} := f|_{z=q^n} \in \mathbb{Z}[q^{\pm n}, q^{\pm 1}].$$

Lemma 23 *Suppose* $f \in \mathbb{Z}[z^{\pm 1}, q^{\pm 1}]$. *Then for a root of unity* ξ *of odd order,*

$$\sum_n^{\xi} q^{b\frac{n^2-1}{4}} \hat{f} = \gamma_b(\xi)\, \mathrm{ev}_\xi(\mathcal{L}_{-b}(f)).$$

Proof It is sufficient to consider the case $f = z^a$. Then, by the same arguments as in the proof of [3, Lemma 1.3], we have

$$\sum_n^{\xi} q^{b\frac{n^2-1}{4}} q^{na} = \begin{cases} 0 & \text{if } c \nmid a, \\ (\xi^c)^{-a_1^2 b'_*} \gamma_b(\xi) & \text{if } a = ca_1. \end{cases} \tag{21}$$

The result follows now from Proposition 20. □

8.2 Proof of Theorem 4

Recall that

$$A(n,k) = \frac{\prod_{i=0}^k (q^n + q^{-n} - q^i - q^{-i})}{(1-q)(q^{k+1};q)_{k+1}}.$$

We have to show that there exists an element $Q_{b,k} \in \mathcal{R}_b$ such that for every root of unity ξ of odd order r one has

$$\frac{\sum_n^{\xi} q^{b\frac{n^2-1}{4}} A(n,k)}{F_{U^b}(\xi)} = \mathrm{ev}_\xi(Q_{b,k}).$$

Applying Lemma 23 to $F_{U^b}(\xi) = \sum_n^{\xi} q^{b\frac{n^2-1}{4}} [n]^2$, we get for $c = (b, r)$

$$F_{U^b}(\xi) = 2\gamma_b(\xi) \mathrm{ev}_\xi \left(\frac{(1 - x_{-b})^{\chi(c)}}{(1-q^{-1})(1-q)} \right), \tag{22}$$

where as usual, $\chi(c) = 1$ if $c = 1$ and is zero otherwise. We will prove that for an odd prime p and any number $j \geq 0$ there exists an element $Q_k(q, x_b, j) \in \mathcal{S}_{p,j}$ such that

$$\frac{1}{(q^{k+1};q)_{k+1}} \mathcal{L}_{b;j} \left(\prod_{i=0}^k (z + z^{-1} - q^i - q^{-i}) \right) = 2\, Q_k(q^{\mathrm{sn}(b)}, x_b, j). \tag{23}$$

If $p=2$ we will prove the claim for $j=0$ only, since $\mathcal{S}_{2,0}\simeq\mathcal{R}_2$. The case $p=\pm 1$ was done e.g. in [2]. Theorem 4 follows then from Lemma 23 and (22) where $Q_{b,k}$ is defined by its projections

$$\pi_j Q_{b,k} := \frac{1-q^{-1}}{(1-x_{-b})^{\chi(p^j)}}\, Q_k(q^{-\operatorname{sn}(b)}, x_{-b}, j).$$

We split the proof of (23) into two parts. In the first part we will show that there exists an element $Q_k(q,x_b,j)$ such that equality (23) holds. In the second part we show that $Q_k(q,x_b,j)$ lies in $\mathcal{S}_{p,j}$.

Part 1, b odd case

Assume $b=\pm p^l$ with $p\neq 2$. We split the proof into several lemmas.

Lemma 24 *For $x_{b;j}:=\pi_j(x_b)$ and $c=(b,p^j)$,*

$$\mathcal{L}_{b;j}\left(\prod_{i=0}^{k}(z+z^{-1}-q^i-q^{-i})\right)=2(-1)^{k+1}\begin{bmatrix}2k+1\\ k\end{bmatrix}S_{b;j}(k,q),$$

where

$$S_{b;j}(k,q):=1+\sum_{n=1}^{\infty}\frac{q^{(k+1)cn}(q^{-k-1};q)_{cn}}{(q^{k+2};q)_{cn}}(1+q^{cn})x_{b;j}^{n^2}. \tag{24}$$

Observe that for $n>\frac{k+1}{c}$ the term $(q^{-k-1};q)_{cn}$ is zero and therefore the sum in (24) is finite.

Proof Since $\mathcal{L}_b$ is invariant under $z\to z^{-1}$ one has

$$\mathcal{L}_b\left(\prod_{i=0}^{k}(z+z^{-1}-q^i-q^{-i})\right)=-2\mathcal{L}_b(z^{-k}(zq^{-k};q)_{2k+1}),$$

and the q-binomial theorem (e.g. see [5], II.3) gives

$$z^{-k}(zq^{-k};q)_{2k+1}=(-1)^k\sum_{i=-k}^{k+1}(-1)^i\begin{bmatrix}2k+1\\ k+i\end{bmatrix}z^i. \tag{25}$$

Notice that $\mathcal{L}_{b;j}(z^a)\neq 0$ if and only if $c\mid a$. Applying $\mathcal{L}_{b;j}$ to the RHS of (25), only the terms with $c\mid i$ survive and therefore

$$\mathcal{L}_{b;j}\big(z^{-k}(zq^{-k};q)_{2k+1}\big)=(-1)^k\sum_{n=-\lfloor k/c\rfloor}^{\lfloor (k+1)/c\rfloor}(-1)^{cn}\begin{bmatrix}2k+1\\ k+cn\end{bmatrix}z_{b,cn;j}.$$

Separating the case $n=0$ and combining positive and negative n this is equal to

$$(-1)^k \begin{bmatrix} 2k+1 \\ k \end{bmatrix} + (-1)^k \sum_{n=1}^{\lfloor (k+1)/c \rfloor} (-1)^{cn} \left(\begin{bmatrix} 2k+1 \\ k+cn \end{bmatrix} + \begin{bmatrix} 2k+1 \\ k-cn \end{bmatrix} \right) z_{b,cn;j},$$

where we use the convention that $\left[\begin{smallmatrix} x \\ -1 \end{smallmatrix}\right]$ is put to be zero for positive x. Further,

$$\begin{bmatrix} 2k+1 \\ k+cn \end{bmatrix} + \begin{bmatrix} 2k+1 \\ k-cn \end{bmatrix} = \frac{\{k+1\}}{\{2k+2\}} \begin{bmatrix} 2k+2 \\ k+cn+1 \end{bmatrix} (q^{cn/2} + q^{-cn/2})$$

and

$$\frac{\{k+1\}}{\{2k+2\}} \begin{bmatrix} 2k+2 \\ k+cn+1 \end{bmatrix} \begin{bmatrix} 2k+1 \\ k \end{bmatrix}^{-1} = (-1)^{cn} q^{(k+1)cn + \frac{cn}{2}} \frac{(q^{-k-1};q)_{cn}}{(q^{k+2};q)_{cn}}.$$

Using $z_{b,cn;j} = (z_{b,c;j})^{n^2} = x_{b;j}^{n^2}$ we get the result. □

To define $Q_k(q, x_b, j)$ we will need Andrew's identity (3.43) of [1]:

$$\sum_{n \geq 0} (-1)^n \alpha_n t^{-\frac{n(n-1)}{2} + sn + Nn} \frac{(t^{-N})_n}{(t^{N+1})_n} \prod_{i=1}^{s} \frac{(b_i)_n (c_i)_n}{b_i^n c_i^n (\frac{t}{b_i})_n (\frac{t}{c_i})_n}$$
$$= \frac{(t)_N (\frac{q}{b_s c_s})_N}{(\frac{t}{b_s})_N (\frac{t}{c_s})_N} \sum_{n_s \geq \cdots \geq n_2 \geq n_1 \geq 0} \beta_{n_1} \frac{t^{n_s} (t^{-N})_{n_s} (b_s)_{n_s} (c_s)_{n_s}}{(t^{-N} b_s c_s)_{n_s}}$$
$$\times \prod_{i=1}^{s-1} \frac{t^{n_i}}{b_i^{n_i} c_i^{n_i}} \frac{(b_i)_{n_i} (c_i)_{n_i}}{(\frac{t}{b_i})_{n_{i+1}} (\frac{t}{c_i})_{n_{i+1}}} \frac{(\frac{t}{b_i c_i})_{n_{i+1} - n_i}}{(t)_{n_{i+1} - n_i}}.$$

Here and in what follows we use the notation $(a)_n := (a;t)_n$. The special Bailey pair (α_n, β_n) is chosen as follows

$$\alpha_0 = 1, \qquad \alpha_n = (-1)^n t^{\frac{n(n-1)}{2}} (1 + t^n),$$
$$\beta_0 = 1, \qquad \beta_n = 0 \quad \text{for } n \geq 1.$$

Lemma 25 *$S_{b;j}(k,q)$ is equal to the LHS of Andrew's identity with the parameters fixed below.*

Proof Since

$$S_{b;j}(k,q) = S_{-b;j}(k, q^{-1}),$$

it is enough to look at the case when $b > 0$. Define $b' := \frac{b}{c}$ and let ω be a b'th primitive root of unity. For simplicity, put $N := k + 1$ and $t := x_{b;j}$. Using the following identities

$$(q^y; q)_{cn} = \prod_{l=0}^{c-1} (q^{y+l}; q^c)_n,$$

$$(q^{yc}; q^c)_n = \prod_{i=0}^{b'-1} (\omega^i t^y; t)_n,$$

where the later is true due to $t^{b'} = x_{b;j}^{b'} = q^c$ for all j, and choosing a cth root of t denoted by $t^{\frac{1}{c}}$ we can see that

$$S_{b;j}(k,q) = 1 + \sum_{n=1}^{\infty} \prod_{i=0}^{b'-1} \prod_{l=0}^{c-1} \frac{(\omega^i t^{\frac{-N+l}{c}})_n}{(\omega^i t^{\frac{N+1+l}{c}})_n} (1 + t^{b'n}) t^{n^2+b'Nn}.$$

Now we choose the parameters for Andrew's identity as follows. We put $a := \frac{c-1}{2}$, $d := \frac{b'-1}{2}$ and $m := \lfloor \frac{N}{c} \rfloor$. For $l \in \{1, \dots, c-1\}$ there exist unique $u_l, v_l \in \{0, \dots, c-1\}$ such that $u_l \equiv N + l \pmod{c}$ and $v_l \equiv N - l \pmod{c}$. Note that $v_l = u_{c-l}$. We define $U_l := \frac{-N+u_l}{c}$ and $V_l := \frac{-N+v_l}{c}$. Then $U_l, V_l \in \frac{1}{c}\mathbb{Z}$ but $U_l + V_l \in \mathbb{Z}$. We define

$$\begin{aligned}
b_l &:= t^{U_l}, & c_l &:= t^{V_l} && \text{for } l = 1, \dots, a,\\
b_{a+i} &:= \omega^i t^{-m}, & c_{a+i} &:= \omega^{-i} t^{-m} && \text{for } i = 1, \dots, d,\\
b_{a+ld+i} &:= \omega^i t^{U_l}, & c_{a+ld+i} &:= \omega^{-i} t^{V_l} && \text{for } i = 1, \dots, d \text{ and}\\
& & & && \quad l = 1, \dots, c-1,\\
b_{g+i} &:= -\omega^i t, & c_{g+i} &:= -\omega^{-i} t && \text{for } i = 1, \dots, d,\\
b_{s-1} &:= t^{-m}, & c_{s-1} &:= t^{N+1}, & &\\
b_s &\to \infty, & c_s &\to \infty, & &
\end{aligned}$$

where $g = a + cd$ and $s = (c+1)\frac{b'}{2} + 1$.

We now calculate the LHS of Andrew's identity. Using the notation

$$(\omega^{\pm 1} t^x)_n = (\omega t^x)_n (\omega^{-1} t^x)_n$$

and the identities

$$\lim_{c \to \infty} \frac{(c)_n}{c^n} = (-1)^n t^{\frac{n(n-1)}{2}} \quad \text{and} \quad \lim_{c \to \infty} \left(\frac{t}{c} \right)_n = 1$$

we get

$$
\begin{aligned}
LHS = 1 + \sum_{n\geq 1} t^{n(n-1+s+N-y)}(1+t^n)\frac{(t^{-N})_n}{(t^{N+1})_n} \\
\times \prod_{l=1}^{a} \frac{(t^{U_l})_n (t^{V_l})_n}{(t^{1-U_l})_n (t^{1-V_l})_n} \cdot \prod_{i=1}^{d} \frac{(\omega^{\pm i} t^{-m})_n}{(\omega^{\pm i} t^{1+m})_n} \\
\times \prod_{i=1}^{d} \prod_{l=1}^{c-1} \frac{(\omega^{i} t^{U_l})_n (\omega^{-i} t^{V_l})_n}{(\omega^{-i} t^{1-U_l})_n (\omega^{i} t^{1-V_l})_n} \cdot \prod_{i=1}^{d} \frac{(-\omega^{\pm i} t)_n}{(-\omega^{\pm i})_n} \cdot \frac{(t^{-m})_n (t^{N+1})_n}{(t^{1+m})_n (t^{-N})_n}
\end{aligned}
$$

where

$$
y := \sum_{l=1}^{a} (U_l + V_l) + \sum_{i=1}^{d} \sum_{l=1}^{c-1} (U_l + V_l) - m(2d+1) + 2d + 1 + N.
$$

Since $\sum_{l=1}^{c-1}(U_l + V_l) = 2\sum_{l=1}^{a}(U_l + V_l) = 2(-N + m + \frac{c-1}{2})$ and $2d + 1 = b'$, we have

$$
n - 1 + s + N - y = n + Nb'.
$$

Further,

$$
\prod_{i=1}^{d} \frac{(-\omega^{\pm i} t)_n}{(-\omega^{\pm i})_n} = \prod_{i=1}^{b'-1} \frac{1+\omega^i t^n}{1+\omega^i} = \frac{1+t^{b'n}}{1+t^n}
$$

and

$$
\begin{aligned}
\prod_{l=1}^{a} \frac{(t^{U_l})_n (t^{V_l})_n}{(t^{1-U_l})_n (t^{1-V_l})_n} \cdot \prod_{i=1}^{d} \frac{(\omega^{\pm i} t^{-m})_n}{(\omega^{\pm i} t^{1+m})_n} \\
\times \prod_{i=1}^{d} \prod_{l=1}^{c-1} \frac{(\omega^{i} t^{U_l})_n (\omega^{-i} t^{V_l})_n}{(\omega^{-i} t^{1-U_l})_n (\omega^{i} t^{1-V_l})_n} \cdot \frac{(t^{-m})_n}{(t^{1+m})_n} \\
= \prod_{i=0}^{b'-1} \prod_{l=0}^{c-1} \frac{(\omega^i t^{\frac{-N+l}{c}})_n}{(\omega^i t^{\frac{N+1+l}{c}})_n}.
\end{aligned}
$$

Taking all the results together, we see that the LHS is equal to $S_{b;j}(k,q)$. $\square$

Let us now calculate the RHS of Andrew's identity with parameters chosen as above. For simplicity, we put $\delta_j := n_{j+1} - n_j$. Then the RHS is given by

$$
\begin{aligned}
RHS = (t)_N & \sum_{n_s \geq \cdots \geq n_2 \geq n_1 = 0} \frac{t^x \cdot (t^{-N})_{n_s} (b_s)_{n_s} (c_s)_{n_s}}{\prod_{i=1}^{s-1} (t)_{\delta_i} (t^{-N} b_s c_s)_{n_s}} \\
& \times \frac{(t^{-m})_{n_{s-1}} (t^{N+1})_{n_{s-1}} (t^{m-N})_{\delta_{s-1}}}{(t^{m+1})_{n_s} (t^{-N})_{n_s}} \\
& \times \prod_{l=1}^{a} \frac{(t^{U_l})_{n_l} (t^{V_l})_{n_l} (t^{1-U_l-V_l})_{\delta_l}}{(t^{1-U_l})_{n_{l+1}} (t^{1-V_l})_{n_{l+1}}} \\
& \times \prod_{i=1}^{d} \frac{(\omega^{\pm i} t^{-m})_{n_{a+i}} (t^{2m+1})_{\delta_{a+i}}}{(\omega^{\pm i} t^{m+1})_{n_{a+i+1}}} \frac{(-\omega^{\pm i} t)_{n_{g+i}} (t^{-1})_{\delta_{g+i}}}{(-\omega^{\pm i})_{n_{g+i+1}}} \\
& \times \prod_{i=1}^{d} \prod_{l=1}^{c-1} \frac{(\omega^{i} t^{U_l})_{n_{a+ld+i}} (\omega^{-i} t^{V_l})_{n_{a+ld+i}} (t^{1-U_l-V_l})_{\delta_{a+ld+i}}}{(\omega^{-i} t^{1-U_l})_{n_{a+ld+i+1}} (\omega^{i} t^{1-V_l})_{n_{a+ld+i+1}}}
\end{aligned}
$$

where

$$
\begin{aligned}
x = & \sum_{l=1}^{a} (1 - U_l - V_l) n_l + \sum_{i=1}^{d} (2m+1) n_{a+i} \\
& + \sum_{i=1}^{d} \sum_{l=1}^{c-1} (1 - U_l - V_l) n_{a+ld+i} - \sum_{i=1}^{d} n_{g+i} + (m - N) n_{s-1} + n_s .
\end{aligned}
$$

For $c = 1$ or $d = 0$, we use the convention that empty products are set to be 1 and empty sums are set to be zero.

Let us now have a closer look at the RHS. Notice, that

$$
\lim_{b_s, c_s \to \infty} \frac{(b_s)_{n_s} (c_s)_{n_s}}{(t^{-N} b_s c_s)_{n_s}} = (-1)^{n_s} t^{\frac{n_s(n_s-1)}{2}} t^{N n_s} .
$$

The term $(t^{-1})_{\delta_{g+i}}$ is zero unless $\delta_{g+i} \in \{0, 1\}$. Therefore, we get

$$
\prod_{i=1}^{d} \frac{(-\omega^{\pm i} t)_{n_{g+i}}}{(-\omega^{\pm i})_{n_{g+i+1}}} = \prod_{i=1}^{d} (1 + \omega^{\pm i} t^{n_{g+i}})^{1 - \delta_{g+i}} .
$$

Due to the term $(t^{-m})_{n_s}$, we have $n_s \leq m$ and therefore $n_i \leq m$ for all $i \leq s - 1$. Multiplying the numerator and denominator of each term of the RHS by

$$\prod_{l=1}^{a}(t^{1-U_l+n_{l+1}})_{m-n_{l+1}}(t^{1-V_l+n_{l+1}})_{m-n_{l+1}}\prod_{i=1}^{d}(\omega^{\pm i}t^{m+1+n_{a+i+1}})_{m-n_{a+i+1}}$$
$$\times\prod_{i=1}^{d}\prod_{l=1}^{c-1}(\omega^{-i}t^{1-U_l+n_{a+ld+i+1}})_{m-n_{a+ld+i+1}}(\omega^{i}t^{1-V_l+n_{a+ld+i+1}})_{m-n_{a+ld+i+1}}$$

gives in the denominator $\prod_{i=0}^{b'-1}\prod_{l=1}^{c-1}(\omega^i t^{1-U_l})_m\cdot\prod_{i=1}^{b'-1}(\omega^i t^{m+1})_m$. This is equal to

$$\prod_{l=1}^{c-1}(t^{b'(1-U_l)};t^{b'})_m\cdot\frac{(t^{b'(m+1)};t^{b'})_m}{(t^{m+1};t)_m}=\frac{(q^{N+1};q)_{cm}}{(t^{m+1};t)_m}.$$

Further,

$$(t)_N(t^{N+1})_{n_{s-1}}=(t)_{N+n_{s-1}}=(t)_m(t^{m+1})_{N-m+n_{s-1}}.$$

The term $(t^{-N+m})_{\delta_{s-1}}$ is zero unless $\delta_{s-1}\le N-m$ and therefore

$$\frac{(t^{m+1})_{N-m+n_{s-1}}}{(t^{m+1})_{n_s}}=(t^{m+1+n_s})_{N-m-\delta_{s-1}}.$$

Taking the above calculations into account, we get

$$RHS=\frac{(t;t)_{2m}}{(q^{N+1};q)_{cm}}\cdot T_k(q,t) \tag{26}$$

where

$$T_k(q,t):=\sum_{n_s\ge\cdots\ge n_2\ge n_1=0}(-1)^{n_s}t^{x'}\cdot(t^{-m})_{n_{s-1}}\cdot(t^{m+1+n_s})_{N-m-\delta_{s-1}}$$
$$\times\frac{(t^{-N+m})_{\delta_{s-1}}}{\prod_{i=1}^{s-1}(t)_{\delta_i}}\cdot\prod_{l=1}^{a}(t^{1-U_l-V_l})_{\delta_l}\cdot\prod_{i=1}^{d}(t^{2m+1})_{\delta_{a+i}}(t^{-1})_{\delta_{g+i}}$$
$$\times\prod_{i=1}^{d}\prod_{l=1}^{c-1}(t^{1-U_l-V_l})_{\delta_{a+ld+i}}$$
$$\times\prod_{l=1}^{a}(t^{U_l})_{n_l}(t^{V_l})_{n_l}(t^{1-U_l+n_{l+1}})_{m-n_{l+1}}(t^{1-V_l+n_{l+1}})_{m-n_{l+1}}$$

$$\times \prod_{i=1}^{d}(1+\omega^{\pm i}t^{n_{g+i}})^{1-\delta_{g+i}}$$

$$\times \prod_{i=1}^{d}(\omega^{\pm i}t^{-m})_{n_{a+i}}(\omega^{\pm i}t^{m+1+n_{a+i+1}})_{m-n_{a+i+1}}$$

$$\times \prod_{i=1}^{d}\prod_{l=1}^{c-1}(\omega^{i}t^{U_l})_{n_{a+ld+i}}(\omega^{-i}t^{V_l})_{n_{a+ld+i}}$$

$$\times \prod_{i=1}^{d}\prod_{l=1}^{c-1}(\omega^{-i}t^{1-U_l+n_{a+ld+i+1}})_{m-n_{a+ld+i+1}}$$

$$\times (\omega^{i}t^{1-V_l+n_{a+ld+i+1}})_{m-n_{a+ld+i+1}}$$

and $x' := x + \frac{n_s(n_s-1)}{2} + Nn_s$.

We now define the element $Q_k(q, x_b, j)$ by

$$Q_k(q,x_b,j) := \left((-1)^{k+1}q^{-\frac{k(k+1)}{2}}\right)^{\frac{1+\mathrm{sn}(b)}{2}}\left(q^{(k+1)^2}\right)^{\frac{1-\mathrm{sn}(b)}{2}} \times \frac{(x_{b;j};x_{b;j})_{2m}}{(q;q)_{N+cm}}T_k(q,x_{b;j}).$$

By Lemmas 24 and 25, (26) and the following Lemma 26, we see that this element satisfies (23).

Lemma 26 *The following formula holds.*

$$(-1)^{k+1}\begin{bmatrix}2k+1\\ k\end{bmatrix}(q^{k+1};q)_{k+1}^{-1} = (-1)^{k+1}\frac{q^{-k(k+1)/2}}{(q;q)_{k+1}} = \frac{q^{-(k+1)^2}}{(q^{-1};q^{-1})_{k+1}}.$$

Proof This is an easy calculation using

$$(q^{k+1};q)_{k+1} = (-1)^{k+1}q^{(3k^2+5k+2)/4}\frac{\{2k+1\}!}{\{k\}!}. \qquad \square$$

Part 1, b even case.

Let $b=\pm 2^l$. We have to prove equality (23) only for $j=0$, i.e. we have to show

$$\frac{1}{(q^{k+1};q)_{k+1}}\mathcal{L}_{b;0}\left(\prod_{i=0}^{k}(z+z^{-1}-q^i-q^{-i})\right) = 2\,Q_k(q^{\mathrm{sn}(b)},x_b,0).$$

The calculation works similar to the odd case. Note that we have $c = 1$ here. This case was already done in [3] and [14]. Since their approaches are slightly different and for the sake of completeness, we will give the parameters for Andrew's identity and the formula for $Q_k(q, x_b, 0)$ nevertheless.

We put $t := x_{b;0}$, $d := \frac{b}{2} - 1$, ω a bth root of unity and choose a primitive square root ν of ω. Define the parameters of Andrew's identity by

$$\begin{aligned}
b_i &:= \omega^i t^{-N}, & c_i &:= \omega^{-i} t^{-N} & &\text{for } i = 1, \ldots, d,\\
b_{d+i} &:= -\nu^{2i-1} t, & c_{d+i} &:= -\nu^{-(2i-1)} t & &\text{for } i = 1, \ldots, d+1,\\
b_b &:= -t^{-N}, & c_b &:= -t^0 = -1, & &\\
b_{s-1} &:= t^{-N}, & c_{s-1} &:= t^{N+1}, & &\\
b_s &\to \infty, & c_s &\to \infty, & &
\end{aligned}$$

where $s = b + 2$. Now we can define the element

$$\begin{aligned}
Q_k(q, x_b, 0) := &\left((-1)^{k+1} q^{-\frac{k(k+1)}{2}}\right)^{\frac{1+\mathrm{sn}(b)}{2}} \left(q^{(k+1)^2}\right)^{\frac{1-\mathrm{sn}(b)}{2}}\\
&\times \frac{(x_{b;0}; x_{b;0})_{2N}}{(q; q)_{2N}} \frac{1}{(-x_{b;0}; x_{b;0})_N} T_k(q, x_{b;0})
\end{aligned}$$

where

$$\begin{aligned}
T_k(q, t) := &\sum_{n_{s-1} \geq \cdots \geq n_1 = 0} (-1)^{n_{s-1}} t^{x''}\\
&\times \frac{\prod_{i=1}^{d} (t^{2N+1})_{\delta_i} \cdot \prod_{i=1}^{d+1} (t^{-1})_{\delta_{d+i}} \cdot (t^{N+1})_{\delta_b}}{\prod_{i=1}^{s-2} (t)_{\delta_i}}\\
&\times (t^{-N})_{n_{s-1}} \cdot (-t^{N+1+n_{s-1}})_{N-n_{s-1}} \cdot (-t^{-N})_{n_b}\\
&\times (-t)_{n_b - 1} \cdot (-t^{n_{s-1}+1})_{N-n_{s-1}}\\
&\times \prod_{i=1}^{d} (\omega^{\pm i} t^{-N})_{n_i} (\omega^{\pm i} t^{N+1+n_{i+1}})_{N-n_{i+1}}\\
&\times \prod_{i=1}^{d+1} (1 + \nu^{\pm(2i-1)} t^{n_{d+i}})^{1-\delta_{d+i}}
\end{aligned}$$

and $x'' := \sum_{i=1}^{d} (2N+1) n_i - \sum_{i=1}^{d+1} n_{d+i} + \frac{n_{s-1}(n_{s-1}-1)}{2} + (N+1)(n_b + n_{s-1})$. We use the notation $(a; b)_{-1} = \frac{1}{1 - ab^{-1}}$.

Part 2

We have to show that $Q_k(q, x_b, j) \in \mathcal{S}_{p,j}$, where $j \in \mathbb{N} \cup \{0\}$ if p is odd, and $j = 0$ for $p = 2$. The following two lemmas do the proof.

Lemma 27 *For* $t = x_{b;j}$,

$$T_k(q, t) \in \mathbb{Z}[q^{\pm 1}, t^{\pm 1}].$$

Proof Let us first look at the case b odd and positive. Since for $a \neq 0$, $(t^a)_n$ is always divisible by $(t)_n$, it is easy to see that the denominator of each term of $T_k(q, t)$ divides its numerator. Therefore we proved that $T_k(q, t) \in \mathbb{Z}[t^{\pm 1/c}, \omega]$. Since

$$S_{b;j}(k, q) = \frac{(t; t)_{2m}}{(q^{N+1}; q)_{cm}} \cdot T_k(q, t), \tag{27}$$

there are $f_0, g_0 \in \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ such that $T_k(q, t) = \frac{f_0}{g_0}$. This implies that $T_k(q, t) \in \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ since f_0 and g_0 do not depend on ω and the cth root of t.

The proofs for the even and the negative case work similar. □

Lemma 28 *For* $t = x_{b;j}$,

$$\frac{(t; t)_{2m}}{(q; q)_{N+cm}} \frac{1}{((-t; t)_N)^{\lambda}} \in \mathcal{S}_{p,j}$$

where $\lambda = 1$ *and* $j = 0$ *if* $p = 2$, *and* $\lambda = 0$ *and* $j \in \mathbb{N} \cup \{0\}$ *otherwise.*

Proof Notice that

$$(q; q)_{N+cm} = \widetilde{(q; q)}_{N+cm} (q^c; q^c)_{2m},$$

where we use the notation

$$\widetilde{(q^a; q)}_n := \prod_{\substack{j=0 \\ c \nmid (a+j)}}^{n-1} (1 - q^{a+j}).$$

We have to show that

$$\frac{(q^c; q^c)_{2m}}{(t; t)_{2m}} \cdot \widetilde{(q; q)}_{N+cm} \cdot ((-t; t)_N)^{\lambda}$$

is invertible in $\mathbb{Z}[1/p][q]$ modulo any ideal $(f) = (\prod_n \Phi_n^{k_n}(q))$ where n runs through a subset of $p^j\mathbb{N}_p$. Recall that in a commutative ring A, an element a is invertible in $A/(d)$ if and only if $(a) + (d) = (1)$. If $(a) + (d) = (1)$ and $(a) + (e) = (1)$, multiplying together we get $(a) + (de) = (1)$. Hence, it is enough to consider $f = \Phi_{p^j n}(q)$ with $(n, p) = 1$. For any $X \in \mathbb{N}$, we have

$$\widetilde{(q;q)_X} = \prod_{\substack{i=1 \\ c\nmid i}}^{X} \prod_{d|i} \Phi_d(q), \tag{28}$$

$$(-t;t)_X = \frac{(t^2;t^2)_X}{(t;t)_X} = \prod_{i=1}^{X} \prod_{d|i} \Phi_{2d}(t), \tag{29}$$

$$\frac{(q^c;q^c)_X}{(t;t)_X} = \frac{(t^{b'};t^{b'})_X}{(t;t)_X} = \frac{\prod_{i=1}^{X} \prod_{d|ib'} \Phi_d(t)}{\prod_{i=1}^{X} \prod_{d|i} \Phi_d(t)} \tag{30}$$

for $b' = b/c$. Recall that $(\Phi_r(q), \Phi_a(q)) = (1)$ in $\mathbb{Z}[1/p][q]$ if either r/a is not a power of a prime or a power of p. For $r = p^j n$ odd and a such that $c \nmid a$, one of the conditions is always satisfied. Hence (28) is invertible in $\mathcal{S}_{p,j}$. If $b = c$ or $b' = 1$, (29) and (30) do not contribute. For $c < b$, notice that q is a cnth primitive root of unity in $\mathbb{Z}[1/p][q]/(\Phi_{cn}(q)) = \mathbb{Z}[1/p][e_{cn}]$. Therefore $t^{b'} = q^c$ is an nth primitive root of unity. Since $(n, b') = 1$, t must be a primitive nth root of unity in $\mathbb{Z}[1/p][e_{cn}]$, too, and hence $\Phi_n(t) = 0$ in that ring. Since for j with $(j, p) > 1$, $(\Phi_j(t), \Phi_n(t)) = (1)$ in $\mathbb{Z}[1/p][t]$, we have $\Phi_j(t)$ is invertible in $\mathbb{Z}[1/p][e_{cn}]$, and therefore (29) and (30) are invertible, too. $\square$

Acknowledgements The authors would like to thank Kazuo Habiro and Christian Krattenthaler for stimulating conversations. We are also grateful to the referee for helpful comments.

Appendix A: Proof of Theorem 3

The appendix is devoted to the proof of Theorem 3, a generalization of the deep integrality result of Habiro, namely Theorem 8.2 of [7]. The existence of this generalization and some ideas of the proof were kindly communicated to us by Habiro.

Reduction to a result on values of the colored Jones polynomial

We will use the notations of [7]. We put $q = e^h$, and $v = e^{h/2}$, where h is a free parameter. The quantum algebra $U_h = U_h(sl_2)$, generated by E, F

and H, subject to some relations, is the quantum deformation of the universal enveloping algebra $U(sl_2)$.

Let V_n be the unique $(n+1)$-dimensional irreducible U_h-module. In [7], Habiro defined a new basis $\tilde{P}'_k$, $k=0,1,2,\dots$, for the Grothendieck ring of finite-dimensional $U_h(sl_2)$-modules with

$$\tilde{P}'_k := \frac{v^{\frac{1}{2}k(1-k)}}{\{k\}!}\prod_{i=0}^{k-1}(V_1 - v^{2i+1} - v^{-2i-1}).$$

Put $\tilde{P}'_{\mathbf{k}} = \{\tilde{P}'_{k_1},\dots,\tilde{P}'_{k_m}\}$. It follows from Lemma 6.1 of [7] that we will have identity (9) of Theorem 3 if we put

$$C_{L\sqcup L'}(\mathbf{k},\mathbf{j}) = J_{L\sqcup L'}(\tilde{P}'_{\mathbf{k}},\mathbf{j})\prod_i(-1)^{k_i}q^{k_i^2+k_i+1}.$$

Hence to prove Theorem 3 it is enough to show the following.

Theorem A.1 *Suppose $L\sqcup L'$ is a colored framed link in S^3 such that L has zero linking matrix and L' has odd colors. Then for $k=\max\{k_1,\dots,k_m\}$ we have*

$$J_{L\sqcup L'}(\tilde{P}'_{\mathbf{k}},\mathbf{j}) \in \frac{(q^{k+1};q)_{k+1}}{1-q}\mathbb{Z}[q^{\pm1}].$$

In the case $L'=\emptyset$, this statement was proved in [7, Theorem 8.2]. Since our proof is a modification of the original one, we first sketch Habiro's original proof for the reader's convenience.

A.1 Sketch of the proof of Habiro's integrality theorem

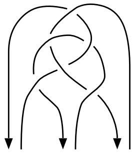
Geometric part

Let us first recall the notion of a bottom tangle, introduced by Habiro in [8].

An n-component bottom tangle $T = T_1\sqcup\cdots\sqcup T_n$ is a framed tangle consisting of n arcs $T_1,\dots,T_n$ in a cube such that all the endpoints of T are on a line at the bottom square of the cube, and for each $i=1,\dots,n$ the component T_i runs from the $2i$th endpoint on the bottom to the $(2i-1)$st endpoint on the bottom, where the endpoints are counted from the left. An example, the Borromean bottom tangle B, is given in Fig. 2.

In [8], Habiro defined a braided subcategory B of the category of framed, oriented tangles which acts on the bottom tangles by composition (vertical pasting). The objects of B are the symbols $\mathsf{b}^{\otimes n}$, $n\geq 0$, where $\mathsf{b} :=\downarrow\uparrow$. For m, $n\geq 0$, a morphism X of B from $\mathsf{b}^{\otimes m}$ to $\mathsf{b}^{\otimes n}$ is the isotopy class of a framed, oriented tangle X which we can compose with m-component bottom tangles

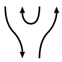
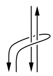
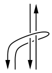
Fig. 2 Borromean bottom tangle B



to get n-component bottom tangles. Let $\mathsf{B}(m,n)$ be the set of morphisms from $\mathsf{b}^{\otimes m}$ to $\mathsf{b}^{\otimes n}$. The composite YX of two morphisms is the gluing of Y to the bottom of X, and the identity morphism $1_{\mathsf{b}^{\otimes m}} = \downarrow\uparrow\cdots\downarrow\uparrow$ is a tangle consisting of $2m$ vertical arcs. The monoidal structure is given by pasting tangles side by side. The braiding for the generating object b with itself is given by $\psi_{\mathsf{b},\mathsf{b}} =$ .

Corollary 9.13 in [8] states the following.

Proposition A.2 (Habiro) *If the linking matrix of a bottom tangle T is zero then T can be presented as $T = WB^{\otimes k}$, where $k \geq 0$ and $W \in \mathsf{B}(3k,n)$ is obtained by horizontal and vertical pasting of finitely many copies of 1_{b}, $\psi_{\mathsf{b},\mathsf{b}}$, $\psi_{\mathsf{b},\mathsf{b}}^{-1}$, and*

$\eta_{\mathsf{b}} =$ , $\mu_{\mathsf{b}} =$ , $\gamma_+ =$ , $\gamma_- =$ .

Algebraic part

Let $K = v^H = e^{\frac{hH}{2}}$. Habiro introduced the integral version $\mathcal{U}_q$, which is the $\mathbb{Z}[q,q^{-1}]$-subalgebra of U_h freely spanned over $\mathbb{Z}[q,q^{-1}]$ by $\tilde{F}^{(i)}K^je^k$ for $i,k \geq 0$, $j \in \mathbb{Z}$, where

$$\tilde{F}^{(n)} = \frac{F^nK^n}{v^{\frac{n(n-1)}{2}}[n]!} \quad \text{and} \quad e = (v - v^{-1})E.$$

There is $\mathbb{Z}/2\mathbb{Z}$-grading, $\mathcal{U}_q = \mathcal{U}_q^0 \oplus \mathcal{U}_q^1$, where $\mathcal{U}_q^0$ (resp. $\mathcal{U}_q^1$) is spanned by $\tilde{F}^{(i)}K^{2j}e^k$ (resp. $\tilde{F}^{(i)}K^{2j+1}e^k$). We call this the ε-grading, and $\mathcal{U}_q^0$ (resp. $\mathcal{U}_q^1$) the even (resp. odd) part.

The two-sided ideal $\mathcal{F}_p$ in $\mathcal{U}_q$ generated by e^p induces a filtration on $(\mathcal{U}_q)^{\otimes n}$, $n \geq 1$, by

$$\mathcal{F}_p((\mathcal{U}_q)^{\otimes n}) = \sum_{i=1}^{n} (\mathcal{U}_q)^{\otimes i-1} \otimes \mathcal{F}_p(\mathcal{U}_q) \otimes (\mathcal{U}_q)^{\otimes n-i} \subset (\mathcal{U}_q)^{\otimes n}.$$

Let $(\tilde{\mathcal{U}}_q)^{\tilde{\otimes} n}$ be the image of the homomorphism

$$\lim_{\overleftarrow{p\geq 0}} \frac{(\mathcal{U}_q)^{\otimes n}}{\mathcal{F}_p((\mathcal{U}_q)^{\otimes n})} \to U_h^{\hat{\otimes} n}$$

where $\hat{\otimes}$ is the h-adically completed tensor product. By using $\mathcal{F}_p(\mathcal{U}_q^\varepsilon) := \mathcal{F}_p(\mathcal{U}_q) \cap \mathcal{U}_q^\varepsilon$ one defines $(\tilde{\mathcal{U}}_q^\varepsilon)^{\tilde{\otimes} n}$ for $\varepsilon \in \{0, 1\}$ in a similar fashion.

By definition (Sect. 4.2 of [7]), the universal sl_2 invariant J_T of an n-component bottom tangle T is an element of $U_h^{\hat{\otimes} n}$. Theorem 4.1 in [7] states that, in fact, for any bottom tangle T with zero linking matrix, J_T is even, i.e.

$$J_T \in (\tilde{\mathcal{U}}_q^0)^{\tilde{\otimes} n}. \tag{A.1}$$

Further, using the fact that J_K of a 0-framed bottom knot K (i.e. a 1-component bottom tangle) belongs to the center of $\tilde{\mathcal{U}}_q^0$, Habiro showed that

$$J_K = \sum_{n\geq 0} (-1)^n q^{n(n+1)} \frac{(1-q)}{(q^{n+1};q)_{n+1}} J_K(\tilde{P}_n')\sigma_n$$

where

$$\sigma_n = \prod_{i=0}^{n} (C^2 - (q^i + 2 + q^{-i}))$$
$$\text{with } C = (v - v^{-1})\tilde{F}^{(1)} K^{-1} e + vK + v^{-1}K^{-1},$$

the quantum Casimir operator. The σ_n provide a basis for the even part of the center. From this, Habiro deduced that $J_K(\tilde{P}_n') \in \frac{(q^{n+1};q)_{n+1}}{(1-q)} \mathbb{Z}[q, q^{-1}]$.

The case of n-component bottom tangles reduces to the 1-component case by partial trace, using certain integrality of traces of even element (Lemma 8.5 of [7]) and the fact that J_T is invariant under the adjoint action.

Algebro-geometric part

The proof of (A.1) uses Proposition A.2, which allows to build any bottom tangle T with zero linking matrix from simple parts, i.e. $T = W(B^{\otimes k})$.

On the other hand, the construction of the universal invariant J_T extends to the braided functor $J : \mathsf{B} \to \mathrm{Mod}_{U_h}$ from B to the category of U_h-modules. This means that $J_{W(B^{\otimes k})} = J_W(J_{B^{\otimes k}})$. Therefore, in order to show (A.1), we need to check that $J_B \in (\tilde{\mathcal{U}}_q^0)^{\tilde{\otimes} 3}$, and then verify that J_W maps the even part to itself. The first check can be done by a direct computation [7, Sect. 4.3]. The last verification is the content of Corollary 3.2 in [7].

A.2 Proof of Theorem A.1

Generalization of (A.1)

To prove Theorem A.1 we need a generalization of (A.1) or Theorem 4.1 in [7] to tangles with closed components. To state the result let us first introduce two new gradings.

Suppose T is an n-component bottom tangle in a cube, homeomorphic to the 3-ball D^3. Let $\tilde{\mathcal{S}}(D^3 \setminus T)$ be the $\mathbb{Z}[q^{\pm 1/4}]$-module freely generated by the isotopy classes of framed unoriented colored links in $D^3 \setminus T$, including the empty link. For such a link $L \subset D^3 \setminus T$ with m-components colored by $n_1, \dots, n_m$, we define our new gradings as follows. First provide the components of L with arbitrary orientations. Let l_{ij} be the linking number between the ith component of T and the jth component of L, and p_{ij} be the linking number between the ith and the jth components of L. For $X = T \sqcup L$ we put

$$\begin{aligned} &\mathrm{gr}_\varepsilon(X) := (\varepsilon_1, \dots, \varepsilon_n) \in (\mathbb{Z}/2\mathbb{Z})^n, \quad \text{where } \varepsilon_i := \sum_j l_{ij} n'_j \pmod 2, \quad \text{and} \\ &\mathrm{gr}_q(L) := \sum_{1 \le i,j \le m} p_{ij} n'_i n'_j + 2 \sum_{1 \le j \le m} (p_{jj} + 1) n'_j \pmod 4, \\ &\quad \text{where } n'_i := n_i - 1. \end{aligned} \tag{A.2}$$

It is easy to see that the definitions do not depend on the orientation of L.

The meaning of $\mathrm{gr}_q(L)$ is the following: The colored Jones polynomial of L, a priori a Laurent polynomial of $q^{1/4}$, is actually a Laurent polynomial of q after dividing by $q^{\mathrm{gr}_q(L)/4}$; see [17] for this result and its generalization to other Lie algebras.

We further extend both gradings to $\tilde{\mathcal{S}}(D^3 \setminus T)$ by

$$\mathrm{gr}_\varepsilon(q^{1/4}) = 0, \qquad \mathrm{gr}_q(q^{1/4}) = 1 \pmod 4.$$

Recall that the universal invariant J_X can also be defined when X is the union of a bottom tangle and a colored link (see [8, Sect. 7.3]). In [8], it is proved that J_X is adjoint invariant. The generalization of Theorem 4.1 of [7] is the following.

Theorem A.3 *Suppose $X = T \sqcup L$, where T is a n-component bottom tangle with zero linking matrix and L is a framed unoriented colored link with $\mathrm{gr}_\varepsilon(X) = (\varepsilon_1, \dots, \varepsilon_n)$. Then*

$$J_X \in q^{\mathrm{gr}_q(L)/4} \left(\tilde{\mathcal{U}}_q^{\varepsilon_1} \tilde{\otimes} \cdots \tilde{\otimes} \tilde{\mathcal{U}}_q^{\varepsilon_n} \right).$$

Corollary A.4 *Suppose L is colored by a tuple of odd numbers, then*

$$J_X \in (\tilde{\mathcal{U}}_q^0)^{\tilde{\otimes} n}.$$

Since J_X is invariant under the adjoint action, Theorem A.1 follows from Corollary A.4 by repeating Habiro's arguments. □

Hence it remains to prove Theorem A.3. In the proof we will need a notion of a *good morphism.*

Good morphisms

Let $I_m := 1_{\mathsf{b}^{\otimes m}} \in \mathsf{B}(m, m)$ be the identity morphism of $\mathsf{b}^{\otimes m}$ in the cube C. A framed link L in the complement $C \setminus I_m$ is *good* if L is geometrically disjoint from all the up arrows of $\mathsf{b}^{\otimes m}$, i.e. there is a plane dividing the cube into two halves, such that all the up arrows are in one half, and all the down arrows and L are in the other. Equivalently, there is a diagram in which all the up arrows are above all components of L. The union W of I_m and a colored framed good link L is called a *good morphism*. If Y is any bottom tangle so that we can compose $X = WY$, then it is easy to see that $\mathrm{gr}_\varepsilon(X)$ does not depend on Y, and we define $\mathrm{gr}_\varepsilon(W) := \mathrm{gr}_\varepsilon(X)$. Also define $\mathrm{gr}_q(W) := \mathrm{gr}_q(L)$.

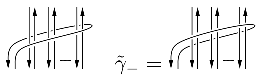
As in the case with $L = \emptyset$, the universal invariant extends to a map $J_W : \mathcal{U}_h^{\otimes m} \to \mathcal{U}_h^{\otimes m}$.

Proof of Theorem A.3

The strategy here is again analogous to the Habiro case: In Proposition A.5 we will decompose X into simple parts: the top is a bottom tangle with zero linking matrix, the next is a good morphism, and the bottom is a morphism obtained by pasting copies of μ_{b}. Since any bottom tangle with zero linking matrix satisfies Theorem A.3 and μ_{b} is the product in $\mathcal{U}_q$, which preserves the gradings, it remains to show that any good morphism preserves the gradings. This is done in Proposition A.6 below. □

Proposition A.5 *Assume $X = T \sqcup L$ where T is a n-component bottom tangle with zero linking matrix and L is a link. Then there is a presentation $X = W_2 W_1 W_0$, where W_0 is a bottom tangle with zero linking matrix, W_1 is a good morphism, and W_2 is obtained by pasting copies of μ_{b}.*

Proof Let us first define $\tilde{\gamma}_\pm \in \mathsf{B}(i, i+1)$ for any $i \in \mathbb{N}$ as follows.

$\tilde{\gamma}_+ =$ $\tilde{\gamma}_- =$ 

If a copy of μ_{b} is directly above $\psi^{\pm 1}_{\mathsf{b},\mathsf{b}}$ or $\gamma_\pm$, one can move μ_{b} down by isotopy and represent the result by pasting copies of $\psi^{\pm 1}_{\mathsf{b},\mathsf{b}}$ and $\tilde{\gamma}_\pm$. It is easy to see that after the isotopy $\gamma_\pm$ gets replaced by $\tilde{\gamma}_\pm$ and $\psi^{\pm 1}_{\mathsf{b},\mathsf{b}}$ by two copies of $\psi^{\pm 1}_{\mathsf{b},\mathsf{b}}$.

Using Proposition A.2 and reordering the basic morphisms so that the μ's are at the bottom, one can see that T admits the following presentation:

$$T = W_2 \tilde{W}_1 (B^{\otimes k})$$

where B is the Borromean tangle, W_2 is obtained by pasting copies of μ_{b} and $\tilde{W}_1$ is obtained by pasting copies of $\psi^{\pm 1}_{\mathsf{b},\mathsf{b}}$, $\tilde{\gamma}_\pm$ and η_{b}.

Let P be the horizontal plane separating $\tilde{W}_1$ from W_2. Let P_+ (P_-) be the upper (lower, respectively) half-space. Note that $W_0 = \tilde{W}_1(B^{\otimes k})$ is a bottom tangle with zero linking matrix lying in P_+ and does not have any minimum points. Hence the pair (P_+, W_0) is homeomorphic to the pair $(P_+, l$ trivial arcs). Similarly, W_2 does not have any maximum points; hence L can be isotoped off P_- into P_+. Since the pair (P_+, W_0) is homeomorphic to the pair $(P_+, l$ trivial arcs) one can isotope L in P_+ to the bottom end points of down arrows. We then obtain the desired presentation. $\square$

Proposition A.6 *For every good morphism W, the operator J_W preserves gradings in the following sense. If $x \in \mathcal{U}_q^{\varepsilon_1} \otimes \cdots \otimes \mathcal{U}_q^{\varepsilon_m}$, then*

$$J_W(x) \in q^{\mathrm{gr}_q(W)/4} \left(\mathcal{U}_q^{\varepsilon'_1} \otimes \cdots \otimes \mathcal{U}_q^{\varepsilon'_m} \right),$$

$$\text{where } (\varepsilon'_1, \ldots, \varepsilon'_m) = (\varepsilon_1, \ldots, \varepsilon_m) + \mathrm{gr}_\varepsilon(W).$$

The rest of the appendix is devoted to the proof of Proposition A.6.

Proof of Proposition A.6

We proceed as follows. Since J_X is invariant under cabling and skein relations, and by Lemma A.8 below, both relations preserve gr_ε and gr_q, we consider the quotient of $\tilde{S}(D^3 \setminus T)$ by these relations known as a skein module of $D^3 \setminus T$. For $T = I_n$, this module has a natural algebra structure, with good morphisms forming a subalgebra. By Lemma A.7 (see also Fig. 4), the basis elements W_γ of this subalgebra are labeled by n-tuples $\gamma = (\gamma_1, \ldots, \gamma_n) \in (\mathbb{Z}/2\mathbb{Z})^n$. It's clear that if the proposition holds for W_{γ_1} and W_{γ_2}, then it holds for $W_{\gamma_1} W_{\gamma_2}$. Hence it remains to check the claim for W_γ's. This is done in Corollary A.10 for basic good morphisms corresponding to γ whose non-zero γ_j's are consecutive. Finally, any W_γ can be obtained by pasting a basic good morphism with few copies of $\psi^{\pm}_{\mathsf{b},\mathsf{b}}$. Since $J_{\psi^\pm}$ preserves gradings (compare (3.15), (3.16) in [7]), the claim follows from Lemmas A.7, A.8 and Proposition A.9 below. $\square$

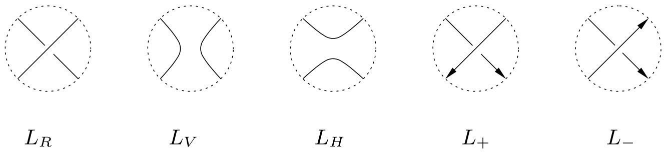


Fig. 3 Tangles involved in the skein relations

Cabling and skein relations

Let us introduce the following relations in $\tilde{\mathcal{S}}(D^3 \setminus T)$.

Cabling relations:

(a) Suppose $n_i = 1$ for some i. The first cabling relation is $L = \tilde{L}$, where $\tilde{L}$ is obtained from L by removing the ith component.
(b) Suppose $n_i \geq 3$ for some i. The second cabling relation is $L = L'' - L'$, where L' is the link L with the color of the ith component switched to $n_i - 2$, and L'' is obtained from L by replacing the ith component with two of its parallels, which are colored with $n_i - 1$ and 2.

Skein relations:

(a) The first skein relation is $U = q^{\frac{1}{2}} + q^{-\frac{1}{2}}$, where U denotes the unknot with framing zero and color 2.
(b) Let L_R, L_V and L_H be unoriented framed links with colors 2 which are identical except in a disc where they are as shown in Fig. 3. Then the second skein relation is $L_R = q^{\frac{1}{4}} L_V + q^{-\frac{1}{4}} L_H$ if the two strands in the crossing come from different components of L_R, and $L_R = \epsilon(q^{\frac{1}{4}} L_V - q^{-\frac{1}{4}} L_H)$ if the two strands come from the same component of L_R, producing a crossing of sign $\epsilon = \pm 1$ (i.e. appearing as in L_ϵ of Fig. 3 if L_R is oriented).

We denote by $S(D^3 \setminus T)$ the quotient of $\tilde{\mathcal{S}}(D^3 \setminus T)$ by these relations. It is known as the *skein module* of $D^3 \setminus T$ (compare [26, 27] and [4]). Recall that the ground ring is $\mathbb{Z}[q^{\pm 1/4}]$.

Using the cabling relations, we can reduce all colors of L in $S(D^3 \setminus T)$ to be 2. Note that the skein module $S(C \setminus I_n)$ has a natural algebra structure, given by putting one cube on the top of the other. Let us denote by A_n the subalgebra of this skein algebra generated by good morphisms.

For a set $\gamma = (\gamma_1, \ldots, \gamma_n) \in (\mathbb{Z}/2\mathbb{Z})^n$ let W_γ be a simple closed curve encircling the end points of those downward arrows with $\gamma_i = 1$. See Fig. 4 for an example.

Similarly to the case of Kauffman bracket skein module [4], one can easily prove the following.

Lemma A.7 *The algebra $\mathcal{A}_n$ is generated by 2^n curves W_γ.*

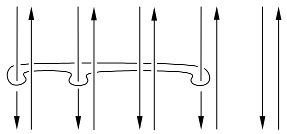


Fig. 4 The element $W_{(1,1,0,1,0)}$

Using linearity, we can extend the definition of J_X to $X = T \sqcup L$, where L is *any element* of $\tilde{S}(D^3 \setminus T)$. It is known that J_X is invariant under the cablings and skein relations (Theorem 4.3 of [12]), hence J_X is defined for $L \in S(D^3 \setminus T)$. Moreover, we have

Lemma A.8 *Both gradings* gr_ε *and* gr_q *are preserved under the cabling and skein relations.*

Proof The statement is obvious for the ε-grading. For the q-grading, notice that

$$\mathrm{gr}_q(L) = 2 \sum_{1 \le i < j \le m} p_{ij} n'_i n'_j + \sum_{1 \le j \le m} p_{jj} n'^2_j + 2 \sum_{1 \le j \le m} (p_{jj} + 1) n'_j,$$

and therefore $\mathrm{gr}_q(L'') \equiv \mathrm{gr}_q(L') \equiv \mathrm{gr}_q(L) \pmod 4$. This takes care of the cabling relations.

Let us now assume that all colors of L are equal to 2 and therefore

$$\mathrm{gr}_q(L) = 2 \sum_{1 \le i < j \le m} p_{ij} + 3 \sum_{i=1}^{m} p_{ii} + 2m.$$

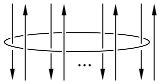
The statement is obvious for the first skein relation. For the second skein relation, choose an arbitrary orientation on L. Let us first assume that the two strands in the crossing depicted in Fig. 3 come from the same component of L_R and that the crossing is positive. Then, L_V and L_H have one positive self-crossing less, and L_V has one link component more than L_R. Therefore

$$\mathrm{gr}_q(q^{\frac{1}{4}} L_V) = \mathrm{gr}_q(L_R) - 3 + 2 + 1 \equiv \mathrm{gr}_q L_R \pmod 4,$$
$$\mathrm{gr}_q(q^{-\frac{1}{4}} L_H) = \mathrm{gr}_q(L_R) - 3 - 1 \equiv \mathrm{gr}_q L_R \pmod 4.$$

It is obvious, that this does not depend on the orientation of L_R. If the crossing of L_R is negative or the two strands do not belong to the same component of L_R, the proof works similar. □

Basic good morphisms

Let Z_n be W_γ for $\gamma = (1, 1, \dots, 1) \in (\mathbb{Z}/2\mathbb{Z})^n$.

$Z_n =$ 

Proposition A.9 *One has a presentation*

$$J_{Z_n} = \sum z_{i_1}^{(n)} \otimes \sum z_{i_2}^{(n)} \otimes \cdots \otimes \sum z_{i_{2n}}^{(n)},$$

such that $z_{i_{2j-1}}^{(n)} z_{i_{2j}}^{(n)} \in v\,\mathcal{U}_q^1$ *for every* $j = 1, \dots, n$.

Corollary A.10 *J_{Z_n} satisfies Proposition A.6.*

Proof Assume $x \in \mathcal{U}_q^{\varepsilon_1} \otimes \cdots \otimes \mathcal{U}_q^{\varepsilon_n}$, then we have

$$J_{Z_n}(x) = \sum z_{i_1}^{(n)} x_1 z_{i_2}^{(n)} \otimes \cdots \otimes \sum z_{i_{2n-1}}^{(n)} x_n z_{i_{2n}}^{(n)}.$$

Hence, by Proposition A.9, we get

$$J_{Z_n}(x) \in q^{1/2} \left(\mathcal{U}_q^{\varepsilon'_1} \otimes \cdots \otimes \mathcal{U}_q^{\varepsilon'_n} \right),$$
$$\text{where } (\varepsilon'_1, \dots, \varepsilon'_m) = (\varepsilon_1, \dots, \varepsilon_n) + (1, 1, \dots, 1).$$

The claim follows now from the fact that $\mathrm{gr}_\varepsilon(Z_n) = (1, 1, \dots, 1)$ and $\mathrm{gr}_q(L) = 2$. □

A.3 Proof of Proposition A.9

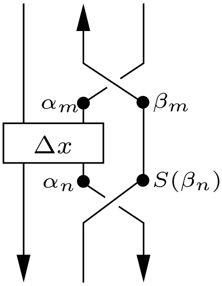
The statement holds true for $J_{Z_1} = C \otimes \mathrm{id}_\uparrow$. Now Lemma 7.4 in [8] states that applying Δ to the ith component of the universal quantum invariant of a tangle is the same as duplicating the ith component. Using this fact we represent

$$J_{Z_{n+1}} = (1_{\mathsf{b}^{\otimes n-1}} \otimes \Phi \otimes \mathrm{id}_\uparrow)(J_{Z_n}), \tag{A.3}$$

where Φ is defined as follows. For $x \in \mathcal{U}_q$ with $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$, we put

$$\Phi(x) := \sum_{(x), m, n} x_{(1)} \otimes \beta_m S(\beta_n) \otimes \alpha_n x_{(2)} \alpha_m$$

where the R-matrix is given by $R = \sum_l \alpha_l \otimes \beta_l$. See figure below for a picture.



We are left with the computation of the ε-grading of each component of $\Phi(x)$.

In $\mathcal{U}_q$, in addition to the ε-grading, there is also the K-grading, defined by $|K| = |K^{-1}| = 0$, $|e| = 1$, $|F| = -1$. In general, the co-product Δ does not preserve the ε-grading. However, we have the following.

Lemma A.11 *Suppose $x \in \mathcal{U}_q$ is homogeneous in both ε-grading and K-grading. Then we have a presentation*

$$\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)},$$

where each $x_{(1)}, x_{(2)}$ are homogeneous with respect to the ε-grading and K-grading. In addition, for $x \in \mathcal{U}_q^\varepsilon$, we have $x_{(2)} \in \mathcal{U}_q^\varepsilon$ and $x_{(1)} K^{-|x_{(2)}|} \in \mathcal{U}_q^\varepsilon$.

Proof If the statements hold true for $x, y \in \mathcal{U}_q$, then they hold true for xy. Therefore, it is enough to check the statements for the generators e, $\tilde{F}^{(1)}$, and K, for which they follow from explicit formulas of the co-product. $\square$

Lemma A.12 *Suppose $x \in \mathcal{U}_q$ is homogeneous in both ε-grading and K-grading. There is a presentation*

$$\Phi(x) = \sum x_{i_0} \otimes x_{i_1} \otimes x_{i_2}$$

such that each x_{i_j} is homogeneous in both ε-grading and K-grading, and for $x \in \mathcal{U}_q^\varepsilon$, x_{i_2} and $x_{i_0} x_{i_1}$ belong to $\mathcal{U}_q^\varepsilon$.

Proof We put $D = \sum D' \otimes D'' := v^{\frac{1}{2} H \otimes H}$. Using (see e.g. [7])

$$R = D \left(\sum_n q^{\frac{1}{2} n(n-1)} \tilde{F}^{(n)} K^{-n} \otimes e^n \right),$$

we get

$$\begin{aligned}\Phi(x) &= \sum_{(x),n,m} q^{\frac{1}{2}(m(m-1)+n(n-1))} x_{(1)} \otimes D_2'' e^m S(D_1'' e^n)\\ &\quad \otimes D_1' \tilde{F}^{(n)} K^{-n} x_{(2)} D_2' \tilde{F}^{(m)} K^{-m}\\ &= \sum_{(x),n,m} (-1)^n q^{-\frac{1}{2}m(m+1)-n(|x_{(2)}|+1)} x_{(1)} \otimes e^m e^n K^{-|x_{(2)}|}\\ &\quad \otimes \tilde{F}^{(n)} x_{(2)} \tilde{F}^{(m)},\end{aligned}$$

where we used $(\mathrm{id}\otimes S)D = D^{-1}$ and $D^{\pm 1}(1\otimes x) = (K^{\pm|x|}\otimes x)D^{\pm 1}$ for homogeneous $x \in \mathcal{U}_q$ with respect to the K-grading. Now, the claim follows from Lemma A.11. □

By induction on n in (A.2), given that $C \in v\mathcal{U}_q^1$, Lemma A.12 implies Proposition A.9. □

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