



Integrality of quantum 3-manifold invariants and a rational surgery formula

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ABSTRACT

We prove that the Witten–Reshetikhin–Turaev (WRT) $SO(3)$ invariant of an arbitrary 3-manifold M is always an algebraic integer. Moreover, we give a rational surgery formula for the unified invariant dominating WRT $SO(3)$ invariants of rational homology 3-spheres at roots of unity of order co-prime with the torsion. As an application, we compute the unified invariant for Seifert fibered spaces and for Dehn surgeries on twist knots. We show that this invariant separates Seifert fibered integral homology spaces and can be used to detect the unknot.

Introduction

The Witten–Reshetikhin–Turaev (WRT) invariant was first introduced by Witten using physics heuristic ideas, and then mathematically rigorously by Reshetikhin and Turaev [Tur94]. The invariant, depending on a root ξ of unity, was first defined for the Lie group $SU(2)$, and was later generalized to other Lie groups. The $SO(3)$ version of the invariant was introduced by Kirby and Melvin [KM91]. For this $SO(3)$ version the quantum parameter ξ must be a root of unity of *odd* order. One important result in quantum topology, first proved by Murakami for rational homology spheres [Mur95] and then generalized by Masbaum and Roberts [MR97], is that the WRT $SO(3)$ invariant (also known as quantum $SO(3)$ invariant) $\tau_M(\xi)$ of an arbitrary 3-manifold M is an algebraic integer, when the order of the root of unity ξ is an *odd prime*. The first integrality result for all roots of unity, but for the restricted set of 3-manifolds (integral homology 3-spheres), was obtained by Habiro in [Hab02, Hab06]. Recently, the second author proved [Le05] that, if the order of ξ is co-prime with the cardinality of the torsion of $H_1(M, \mathbb{Z})$, then the $SO(3)$ quantum invariant $\tau_M(\xi) \in \mathbb{Z}[\xi]$. In this paper we remove all the restrictions on the order of ξ .

THEOREM 1. *For every closed 3-manifold and every root ξ of unity of odd order, the quantum $SO(3)$ invariant $\tau_M(\xi)$ belongs to $\mathbb{Z}[\xi]$.*

The integrality has many important applications, among them the construction of an integral topological quantum field theory and representations of mapping class groups over \mathbb{Z} by Gilmer and Masbaum (see e.g. [GM04]). The integrality is also a key property required for the categorification of quantum 3-manifold invariants [Kho05].

Our proof of integrality is inspired by Habiro’s work. In [Hab02, Hab06], Habiro constructed an invariant of integral homology 3-spheres with values in the universal ring

$$\widehat{\mathbb{Z}[q]} = \varprojlim_n \mathbb{Z}[q]/((q; q)_n),$$

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where $(z; q)_n := (1 - z)(1 - qz) \cdots (1 - q^{n-1}z)$ and, for any $f \in \mathbb{Z}[q]$, (f) denotes the ideal generated by f . Habiro’s invariant specializes at a root ξ of unity to $\tau_M(\xi)$.

In [Le05], the second author generalized Habiro’s theory to rational homology 3-spheres. For a rational homology sphere M with $|H_1(M, \mathbb{Z})| = a$, he constructed an invariant I_M which dominates the $SO(3)$ invariants of M at roots of unity of order co-prime to a . Habiro’s universal ring was modified by inverting a and cyclotomic polynomials of order not co-prime to a . Applications of this theory are the new integrality properties of quantum invariants, new results about Ohtsuki series and a better understanding of the relation between L  -Murakami–Ohtsuki (LMO) invariant, Ohtsuki series and quantum invariants.

In this paper we give a rational surgery formula for the unified invariant I_M defined in [Le05] and refine the ring that contains the values of I_M . Let us summarize our main results.

Let $t := q^{1/a}$ and $\mathcal{R}_{a,k} \subset \mathbb{Q}(t)$ be the subring generated over $\mathbb{Z}[t^{\pm 1}]$ by $(t; t)_k / (q; q)_k$. Note that

$$\mathcal{R}_{a,1} \subset \mathcal{R}_{a,2} \subset \cdots \subset \mathcal{R}_a,$$

where $\mathcal{R}_a = \bigcup_{k=1}^{\infty} \mathcal{R}_{a,k}$. The analog of the Habiro ring constructed in [Le05] can be defined as¹

$$\widehat{\mathcal{R}}_a := \varprojlim_n \mathcal{R}_a / ((q; q)_n).$$

Let U_a be the set of all complex roots of unity with orders *odd* and co-prime with a . For every $\xi \in U_a$ and every $f \in \widehat{\mathcal{R}}_a$ one can define an evaluation $\text{ev}_\xi(f) \in \mathbb{C}$, replacing q by ξ ; see § 1.2. It was shown in [Le05] that, if $|H_1(M, \mathbb{Z})| = a$, then $I_M \in \widehat{\mathcal{R}}_a$ and $\text{ev}_\xi(I_M)$, after a simple normalization, is the $SO(3)$ quantum invariants of M at $q = \xi$.

It will be shown that, for $f \in \widehat{\mathcal{R}}_a$, $\text{ev}_\xi(f) \in \mathbb{Z}[\xi/a]$, and, in general, one cannot avoid the denominator. We will single out a subring Γ_a of $\widehat{\mathcal{R}}_a$ such that $\text{ev}_\xi(\Gamma_a) = \mathbb{Z}[\xi]$.

Let

$$x_n := \frac{(q^{n+1}; q)_{n+1}}{1 - q} = \frac{(1 - q^{n+1})(1 - q^{n+2}) \cdots (1 - q^{2n+1})}{1 - q} \in \mathbb{Z}[q].$$

Then $(q; q)_n$ divides x_n , which, in turns, divides $(q; q)_{2n+1}$. Hence the ideals (x_n) and $(q; q)_n$ are cofinal in \mathcal{R}_a , and we have $\widehat{\mathcal{R}}_a := \lim_{\leftarrow n} \mathcal{R}_a / (x_n)$. Every element $f \in \widehat{\mathcal{R}}_a = \lim_{\leftarrow n} \mathcal{R}_a / (x_n)$ can be represented as an infinite series of the form

$$f = \sum_{n=0}^{\infty} f_n(t)x_n, \quad \text{where } f_n(t) \in \mathcal{R}_a. \tag{1}$$

Let Γ_a be the set of all elements f of $\widehat{\mathcal{R}}_a$ that have a presentation (1) such that $f_n(t) \in \mathcal{R}_{a,2n+1}$. It is easy to see that Γ_a is a subring of $\widehat{\mathcal{R}}_a$. The following shows that Γ_a is strictly smaller than $\widehat{\mathcal{R}}_a$: it enjoys stronger integrality.

PROPOSITION 1. *Suppose that $f \in \Gamma_a$ and $\xi \in U_a$, i.e. ξ is a root of unity whose order is odd and co-prime with a . Then $\text{ev}_\xi(f) \in \mathbb{Z}[\xi]$. On the other hand, $\text{ev}_\xi(\widehat{\mathcal{R}}_a) = \mathbb{Z}[\xi/a]$.*

Now we can formulate our next result.

THEOREM 2. *Let M be a rational homology 3-sphere with $|H_1(M, \mathbb{Z})| = a$. Then we have $I_M \in \Gamma_a$.*

In particular, Theorem 2 and Proposition 1 give a new proof of the integrality of $SO(3)$ quantum invariant of rational homology 3-sphere M with $|H_1(M, \mathbb{Z})| = a$ at a root of unity $\xi \in U_a$, a result proved in [Le05].

¹We are grateful to the referee for pointing out that $\widehat{\mathcal{R}}_a$ coincides with the ring $\widehat{\Lambda}_a$ in [Le05].

Further, we compute the unified invariant for Seifert fibered spaces and for Dehn surgeries on twist knots.

THEOREM 3. *The unified invariant separates Seifert fibered integral homology spheres.*

Theorem 3 follows also from [BNL04], where the computations were done for the LMO invariant combined with the sl_2 weight system, i.e. for the Ohtsuki series. By the result of Habiro in [Hab02, Hab06], the Ohtsuki series is just the Taylor expansion of I_M at $q = 1$, which determines I_M .

For a knot K , let $M(K, a)$ denotes the 3-manifold obtained by surgery on the knot K with framing a . In general, there are different K, K' such that $M(K, a) = M(K', a)$ for some a .

THEOREM 4. *Suppose that, for infinitely many $a \in \mathbb{Z}$, the Ohtsuki series of $M(K, a)$ and $M(K', a)$ coincide, i.e. $I_{M(K,a)} = I_{M(K',a)}$. Then K and K' have the same colored Jones polynomial.*

In particular, using the recent deep result of Andersen [And07], that the colored Jones polynomial detects the unknot, we see that (under the assumption of the theorem), if K is the unknot, then so is K' .

1. Quantum invariants

Recall that $q = t^a$. We will use the following notation:

$$\{n\} = q^{n/2} - q^{-n/2}, \quad \{n\}! = \prod_{i=1}^n \{i\}, \quad [n] = \frac{\{n\}}{\{1\}}, \quad \begin{bmatrix} n \\ k \end{bmatrix} = \frac{\{n\}!}{\{k\}!\{n-k\}!}.$$

1.1 The colored Jones polynomial

Suppose that L is a framed, oriented link in S^3 with m ordered components. For every positive integer n there is a unique irreducible sl_2 -module V_n of dimension n . For positive integers n_1, \dots, n_m one can define the quantum invariant $J_L(n_1, \dots, n_m) := J_L(V_{n_1}, \dots, V_{n_m})$ known as the colored Jones polynomial of L (see e.g. [Tur94]). Let us recall here a few well-known formulas. For the unknot U with 0 framing one has

$$J_U(n) = [n] = \{n\}/\{1\}. \tag{2}$$

If L' is obtained from L by increasing the framing of the i th component by 1, then

$$J_{L'}(n_1, \dots, n_m) = q^{(n_i^2-1)/4} J_L(n_1, \dots, n_m). \tag{3}$$

In general, $J_L(n_1, \dots, n_m) \in \mathbb{Z}[q^{\pm 1/4}]$. However, there is a number $b \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$ such that $J_L(n_1, \dots, n_m) \in q^b \mathbb{Z}[q^{\pm 1}]$.

1.2 Evaluation map and Gauss sum

Throughout this paper let ξ be a primitive root of unity of odd order r . We first define, for each ξ , the evaluation map ev_ξ , which replaces q by ξ . Suppose that $f \in \mathbb{Q}[q^{\pm 1/h}]$, where h is co-prime with r , the order of ξ . There exists an integer b , unique modulo r , such that $(\xi^b)^h = \xi$. Then we define

$$ev_\xi f := f|_{q^{1/h} = \xi^b}.$$

Suppose that $t := q^{1/a}$ and \mathbb{N}_a is the set of all positive integers co-prime to a . Denote by $\Phi_s(t)$ the s th cyclotomic polynomial. Recall that

$$1 - t^n = \prod_{s|n} \Phi_s(t).$$

It follows that $(t; t)_k/(q; q)_k$ is the inverse of the product of many $\Phi_s(t)$ with $s \notin \mathbb{N}_a$. Recall that U_a is the set of all complex roots of unity with orders odd and co-prime with a . When $\xi \in U_a$ and $s \notin \mathbb{N}_a$, we have $\Phi_s(\xi) \neq 0$. Thus one can evaluate $\text{ev}_\xi((t; t)_k/(q; q)_k)$. The definition also extends to $\text{ev}_\xi : \widehat{\mathcal{R}}_a \rightarrow \mathbb{C}$, since $\text{ev}_\xi((q; q)_n) = 0$ if $n \geq r$.

Suppose that $f(q; n_1, \dots, n_m)$ is a function of variables q and integers n_1, \dots, n_m . Let

$$\sum_{n_i}^\xi f := \sum_{n_i} \text{ev}_\xi(f),$$

where in the sum all the n_i run over the set of *odd* numbers between 0 and $2r$. A variation $\gamma_d(\xi)$ of the Gauss sum is defined by

$$\gamma_d(\xi) := \sum_n^\xi q^{d(n^2-1)/4}.$$

It is known that, for odd r , $|\gamma_d(\xi)| = \sqrt{r}$, and hence is never 0.

Let

$$F_L(\xi) := \sum_{n_i}^\xi J_L(n_1, \dots, n_m) \prod_{i=1}^m [n_i].$$

The following result is well known (compare [Le05]).

LEMMA 1.1. *For the unknot U^\pm with framing ± 1 , one has $F_{U^\pm}(\xi) \neq 0$. Moreover, one has*

$$F_{U^\pm}(\xi) = \mp 2\gamma_{\pm 1}(\xi) \text{ev}_\xi\left(\frac{q^{\mp 1/2}}{\{1\}}\right). \tag{4}$$

1.3 Definition of $SO(3)$ invariant of 3-manifolds

All 3-manifolds in this paper are supposed to be closed and oriented. Every link in a 3-manifold is framed, oriented, and has components ordered.

Suppose that M is an oriented 3-manifold obtained from S^3 by surgery along a framed, oriented link L . (Note that M does not depend on the orientation of L .) Let σ_+ (respectively, σ_-) be the number of positive (respectively negative) eigenvalues of the linking matrix of L . Suppose ξ is a root of unity of odd order r . Then the quantum $SO(3)$ invariant is defined by

$$\tau_M(\xi) = \tau_M^{SO(3)}(\xi) := \frac{F_L(\xi)}{(F_{U_+}(\xi))^{\sigma_+} (F_{U_-}(\xi))^{\sigma_-}}.$$

For the connected sum, one has $\tau_{M\#N}(\xi) = \tau_M(\xi)\tau_N(\xi)$.

1.4 Laplace transform

In [BBL05], together with Blanchet we developed the Laplace transform method to compute $\tau_M(\xi)$. Here we generalize this method to the case where r is not co-prime with torsion.

Suppose r is an odd number, and d is a positive integer. Let

$$c := (r, d), \quad d_1 := d/c, \quad r_1 := r/c.$$

Let $\mathcal{L}_{d;n} : \mathbb{Z}[q^{\pm n}, q^{\pm 1}] \rightarrow \mathbb{Z}[q^{\pm 1/d}]$ be the $\mathbb{Z}[q^{\pm 1}]$ -linear operator, called the Laplace transform, defined by

$$\mathcal{L}_{d;n}(q^{na}) := \begin{cases} 0 & \text{if } c \nmid a, \\ q^{-a^2/d} & \text{if } a = ca_1. \end{cases} \tag{5}$$

LEMMA 1.2. *Suppose that $f \in \mathbb{Z}[q^{\pm n}, q^{\pm 1}]$. Then one has*

$$\sum_n^\xi q^{d(n^2-1)/4} f = \gamma_d(\xi) \operatorname{ev}_\xi(\mathcal{L}_{d;n}(f)).$$

Proof. It is enough to consider the case when $f = q^{na}$, with a an integer. This case is proven by Lemma 1.3 in the next subsection. \square

The point is that $\mathcal{L}_{d;n}(f)$, unlike the left-hand side $\sum_n^\xi q^{d(n^2-1)/4} f$, does not depend on ξ , and will help us to define a ‘unified invariant’. Note that Lemma 1.2 with $d = \pm 1$ and $f = [n]^2$ implies Lemma 1.1.

1.5 Reduction from r to r_1

Let O_r be the set of all odd integers between 0 and $2r$. This set O_r can be partitioned into r_1 subsets $O_{r;s}$ with $s \in O_{r_1}$, where $O_{r;s}$ is the set of all $n \in O_r$ which are equal to s modulo r_1 . In other words, $O_{r;s} := \{s + 2jr_1, j = 0, 1, \dots, c-1\}$. The point is, the value of $\xi^{d(n^2-1)/4}$ remains the same for all n in the same set $O_{r;s}$. Let $\zeta = \xi^c$; then the order of ζ is r_1 .

LEMMA 1.3. *One has*

$$\gamma_d(\xi) = c\gamma_{d_1}(\zeta), \tag{6}$$

$$\sum_n^\xi q^{d(n^2-1)/4} q^{an} = \begin{cases} 0 & \text{if } c \nmid a, \\ \gamma_d(\xi)\zeta^{-a^2d_1^*} & \text{if } a = ca_1, \end{cases} \tag{7}$$

where d_1 is an integer satisfying $d_1d_1^* \equiv 1 \pmod{r_1}$.

Proof. One has

$$\sum_n^\xi q^{d(n^2-1)/4} q^{an} = \sum_{n \in O_r} \xi^{d(n^2-1)/4} \xi^{an} = \sum_{s \in O_{r_1}} \sum_{n \in O_{r;s}} \xi^{d(n^2-1)/4} \xi^{an}.$$

Using the fact that $\xi^{d(n^2-1)/4}$ remains the same for all n in the same set $O_{r;s}$, we get

$$\sum_n^\xi q^{d(n^2-1)/4} q^{an} = \sum_{s \in O_{r_1}} \xi^{d(s^2-1)/4} \sum_{n \in O_{r;s}} \xi^{an} \tag{8}$$

$$= \sum_{s \in O_{r_1}} \xi^{d(s^2-1)/4} \xi^{sa} \left(\sum_{j=0}^{c-1} \xi^{2ar_1j} \right). \tag{9}$$

Note that (6) follows from (8) with $a = 0$. We also have

$$\sum_{j=0}^{c-1} \xi^{2ar_1j} = \sum_{j=0}^{c-1} (\xi^{2ar_1})^j. \tag{10}$$

If $c \nmid a$, then $(\xi^{2ar_1}) \neq 1$, but is a root of unity of order dividing c ; hence the right-hand side of (10) is 0. It follows that the right-hand side of (9) is also 0, or $\sum_n^\xi q^{d(n^2-1)/4} q^{an} = 0$.

If $c|a$, then the right-hand side of (10) is c . Hence from (9) we have

$$\begin{aligned} \sum_n^\xi q^{d(n^2-1)/4} q^{an} &= c \sum_{s \in O_{r_1}} \xi^{d(s^2-1)/4} \xi^{sa} \\ &= c \sum_{s \in O_{r_1}} \zeta^{d_1(s^2-1)/4} \zeta^{sa_1} = c \sum_n^\zeta q^{d_1(n^2-1)/4} q^{a_1n} \\ &= c\gamma_{d_1}(\zeta)\zeta^{-a_1^2d_1^*}. \end{aligned}$$

The last equality follows by the standard square completion argument. Using (6) we get the result. \square

1.6 Habiro’s cyclotomic expansion of the colored Jones polynomial

In [Hab02, Hab06], Habiro defined a new basis P'_k , $k = 0, 1, 2, \dots$, for the Grothendieck ring of finite-dimensional sl_2 -modules, where

$$P'_k := \frac{1}{\{k\}!} \prod_{i=1}^k (V_2 - q^{(2i-1)/2} - q^{-(2i-1)/2}).$$

For any link L , one has

$$J_L(n_1, \dots, n_m) = \sum_{0 \leq k_i \leq n_i - 1} J_L(P'_{k_1}, \dots, P'_{k_m}) \prod_{i=1}^m \begin{bmatrix} n_i + k_i \\ 2k_i + 1 \end{bmatrix} \{k_i\}!. \tag{11}$$

Since there is a denominator in the definition of P'_k , one might expect that $J_L(P'_{k_1}, \dots, P'_{k_m})$ also has non-trivial denominator. A difficult and important integrality result of Habiro [Hab02, Hab06] is stated below.

THEOREM 5 [Hab02, Hab06, Theorem 3.3]. *If L is an algebraically split and framed link in S^3 , then*

$$J_L(P'_{k_1}, \dots, P'_{k_m}) \in \frac{\{2k+1\}!}{\{k\}!\{1\}} \mathbb{Z}[q^{\pm 1/2}] = \begin{bmatrix} 2k+1 \\ k \end{bmatrix}_k (q^2)_k \mathbb{Z}[q^{\pm 1/2}],$$

where $k = \max\{k_1, \dots, k_m\}$.

Thus, $J_L(P'_{k_1}, \dots, P'_{k_m})$ is not only integral, but also divisible by $(q)_k$.

Suppose that L is an algebraically split link with 0-framing on each component. Then we have

$$\text{ev}_\xi(J_L(n_1, \dots, n_m)) = \text{ev}_\xi \left(\sum_{k_1, \dots, k_m=0}^{(r-3)/2} J_L(P'_{k_1}, \dots, P'_{k_m}) \prod_{i=1}^m \begin{bmatrix} n_i + k_i \\ 2k_i + 1 \end{bmatrix} \{k_i\}! \right).$$

2. Integrality of quantum invariants for all roots of unity

Throughout this section we assume that $c = (d, r) > 1$, $r/c = r_1$, $d/c = d_1$ and $d_1 d_1^* = 1 \pmod{r_1}$, where r is the order of ξ and d is the order of the torsion part of $H_1(M, \mathbb{Z})$.

2.1 Quantum invariants of links with diagonal linking matrix

The following proposition plays a key role in the proof of integrality.

PROPOSITION 2.1. *For $k \leq (r-3)/2$, we have*

$$\frac{1}{\gamma_{\pm 1}(\xi)} \sum_n^\xi q^{d(n^2-1)/4} \begin{bmatrix} n+k \\ 2k+1 \end{bmatrix} \{k\}!\{n\} \in \mathbb{Z}[\xi]. \tag{12}$$

Proof of Theorem 1 (diagonal case). Suppose that M is obtained from S^3 by surgery along an algebraically split m -component link L with integral framings d_1, d_2, \dots, d_m . Inserting into the definition of $\tau_M(\xi)$ (see § 1.3) the formulas (4) and (11) and using Lemma 1.2, we see that Proposition 2.1 and Theorem 5 imply integrality if $d_i \neq 0$ for all i . If some of d_i are zero, then by the same argument as in § 3.4.2 of [Le05] we have

$$\sum_n^\xi \begin{bmatrix} n+k \\ 2k+1 \end{bmatrix} \{k\}!\{n\} = 2 \text{ev}_\xi(q^{(k+1)(k+2)/4} (q^{k+2}; q)_{r-k-2}).$$

The result now follows from the fact that $\gamma_d(\xi)/\gamma_1(\xi) \in \mathbb{Z}[\xi]$. \square

2.1.1 *Technical results.* This subsection is devoted to the proof of Proposition 2.1.

LEMMA 2.2. (a) Suppose that $x \in \mathbb{Q}(\xi)$ such that $x^2 \in \mathbb{Z}[\xi]$; then $x \in \mathbb{Z}[\xi]$.

(b) Suppose that $x, y \in \mathbb{Z}[\xi]$ such that x^2 is divisible by y^2 ; then x is divisible by y .

Proof. (a) Suppose that $a = x^2$. Then $a \in \mathbb{Z}[\xi]$ and x is a solution of $x^2 - a = 0$. Hence x is integral over $\mathbb{Z}[\xi]$, which is integrally closed. It follows that $x \in \mathbb{Z}[\xi]$.

(b) We have that $(x/y)^2 = x^2/y^2$ is in $\mathbb{Z}[\xi]$. Hence by part (a), we have $x/y \in \mathbb{Z}[\xi]$. \square

Recall that

$$(q^l; q)_m = \prod_{j=l}^{l+m-1} (1 - q^j).$$

Let $\widetilde{(q^l; q)}_m$ be the product on the right-hand side, only with j not divisible by c . Also let $\widehat{(q^l; q)}_m$ be the complement, i.e. $\widehat{(q^l; q)}_m := (q^l; q) / \widetilde{(q^l; q)}_m$. Using $(\xi; \xi)_{r-1} = r$ and $(\xi^c; \xi^c)_{r_1-1} = r_1$, we see that

$$\widetilde{(\xi; \xi)}_{r-1} = c, \tag{13}$$

where $(a; b)_m := (1 - a)(1 - ab) \cdots (1 - ab^{m-1})$. Note that $1 - \xi^j$ is invertible in $\mathbb{Z}[\xi]$ if and only if $(j, r) = 1$. Let

$$z := \widetilde{(\xi; \xi)}_{(r-1)/2} \quad \text{and} \quad z' := (\xi^{(r+1)/2}; \xi)_{(r-1)/2}. \tag{14}$$

Then zz' is the left-hand side of (13), hence $zz' = c$. We use the notation $x \sim y$ if the ratio x/y is a unit in $\mathbb{Z}[\xi]$. Note that $z \sim z'$. This is because $1 - \xi^k \sim 1 - \xi^{r-k}$. Thus we have

$$z^2 \sim c. \tag{15}$$

LEMMA 2.3. We have that $\gamma_d(\xi)/\gamma_1(\xi)$ is divisible by z .

Proof. Using Lemma 2.2(b) and (15), one needs only to show that $(\gamma_d(\xi))^2/(\gamma_1(\xi))^2$ is divisible by c . The values of $\gamma_b(\xi)$ are well known when b is co-prime with r , the order of ξ . In particular, $\gamma_b(\xi) \sim \gamma_1(\xi)$; see [LL96].

Recall that $\zeta = \xi^c$ has order r_1 . Since d_1 and r_1 are co-prime, we have

$$\gamma_{d_1}(\zeta) \sim \gamma_1(\zeta).$$

Using Lemma 1.3, we have

$$\frac{(\gamma_d(\xi))^2}{(\gamma_1(\xi))^2} = c^2 \frac{(\gamma_{d_1}(\zeta))^2}{(\gamma_1(\zeta))^2} \sim c^2 \frac{(\gamma_1(\zeta))^2}{(\gamma_1(\xi))^2}. \tag{16}$$

Using the explicit formula for $\gamma_1(\xi) = \sum_{0 \leq j < r} \xi^{j^2+j}$ (given e.g. by [LL96, Theorem 2.2]), we have that

$$(\gamma_1(\xi))^2 = \pm r \xi^{-2^*} = cr_1 \xi^{-2^*}, \quad (\gamma_1(\zeta))^2 = \pm r_1 \zeta^{-2^*},$$

where 2^* is the inverse of 2. Plugging this in (16), we get the result. \square

For $k, b \in \mathbb{Z}$ we define

$$Y_c(k, b) := (-1)^k \sum_{n=-\lfloor k/c \rfloor}^{\lfloor (k+1)/c \rfloor} (-1)^n \begin{bmatrix} 2k+1 \\ k+nc \end{bmatrix} q^{cbn^2}. \tag{17}$$

LEMMA 2.4. Suppose that $d_1 d_1^* \equiv 1 \pmod{r_1}$, where $r = cr_1$ is the order of ξ ; then we have

$$\sum_n^\xi q^{d(n^2-1)/4} \begin{bmatrix} n+k \\ 2k+1 \end{bmatrix} \{k\}! \{n\} = -2\gamma_d(\xi) \text{ev}_\xi \left(\frac{Y_c(k, -d_1^*) \{k\}!}{\{2k+1\}!} \right).$$

Proof. By Lemma 1.2 we have to compute $\mathcal{L}_{d;n}(\{n\}\{n+k\}/\{n-k-1\})$. Since $\mathcal{L}_{d;n}$ is invariant under $n \rightarrow -n$, one has

$$\mathcal{L}_{d;n}(\{n\}\{n+k\}/\{n-k-1\}) = -2\mathcal{L}_{d;n}(q^{-nk}(q^{n-k}; q)_{2k+1}). \tag{18}$$

By the q -binomial formula we have

$$q^{-nk}(q^{n-k}; q)_{2k+1} = \sum_{j=0}^{2k+1} (-1)^j \begin{bmatrix} 2k+1 \\ j \end{bmatrix} q^{n(j-k)}. \tag{19}$$

Using the definition of $\mathcal{L}_{d;n}$ we get

$$\text{ev}_\xi(\mathcal{L}_{d;n}(\{n\}\{n+k\}/\{n-k-1\})) = -2 \text{ev}_\xi(Y_c(k, -d_1^*)).$$

Multiplying by $\{k\}/\{2k+1\}$, we get the result. □

THEOREM 6. For $b \in \mathbb{Z}$ and $k \leq (r-3)/2$,

$$\frac{\gamma_d(\xi)}{\gamma_1(\xi)} \text{ev}_\xi(Y_c(k, b)) \text{ is divisible by } \text{ev}_\xi\left(\frac{\{2k+1\}}{\{k\}}\right).$$

Here we modify the proof of Theorem 7 in [Le05].

Proof. The case $b = 0$ is trivial. Let us assume that $b \neq 0$. Separating the case $n = 0$ and combining positive and negative n , we have

$$Y_c(k, b) = (-1)^k \begin{bmatrix} 2k+1 \\ k \end{bmatrix} + (-1)^k \sum_{n=1}^{\lfloor (k+1)/c \rfloor} (-1)^n q^{c b n^2} \left(\begin{bmatrix} 2k+1 \\ k+nc \end{bmatrix} + \begin{bmatrix} 2k+1 \\ k-nc \end{bmatrix} \right).$$

Using

$$\begin{bmatrix} 2k+1 \\ k+cn \end{bmatrix} + \begin{bmatrix} 2k+1 \\ k-cn \end{bmatrix} = \frac{\{k+1\}}{\{2k+2\}} \begin{bmatrix} 2k+2 \\ k+cn+1 \end{bmatrix} (q^{cn/2} + q^{-cn/2})$$

and

$$\begin{bmatrix} 2k+2 \\ k+1 \end{bmatrix} = \begin{bmatrix} 2k+1 \\ k \end{bmatrix} \frac{\{2k+2\}}{\{k+1\}}$$

we get

$$Y_c(k, b) = (-1)^k \begin{bmatrix} 2k+1 \\ k \end{bmatrix} S_N, \tag{20}$$

where $N = k+1$ and

$$S_N = 1 + \sum_{n=1}^{\infty} \frac{q^{Ncn}(q^{-N}; q)_{cn}}{(q^{N+1}; q)_{cn}} (1 + q^{cn}) q^{c b n^2}.$$

For z defined by (14), we show the divisibility of $\text{ev}_\xi(S_N)z$ by $(\xi; \xi)_N$ in § 2.1.2. This implies the result, since $z | (\gamma_d(\xi)/\gamma_1(\xi))$ by Lemma 2.3 and

$$\begin{bmatrix} 2k+1 \\ k \end{bmatrix} \{k+1\} = \frac{\{2k+1\}}{\{k\}}. \tag{□}$$

Proof of Proposition 2.1. Combining Lemma 2.4 with Theorem 6 we get Proposition 2.1. □

2.1.2 *Andrews' identity.* Let α_n, β_n be a Bailey pair as defined in [And85, § 3.4], with $a = 1$. Then for any numbers $b_i, c_i, i = 1, \dots, k$, and positive integer N , we have the identity (3.43)

of [And85]:

$$\begin{aligned} & \sum_{n \geq 0} (-1)^n \alpha_n q^{-\binom{n}{2} + kn + Nn} \frac{(q^{-N})_n}{(q^{N+1})_n} \prod_{i=1}^k \frac{(b_i)_n (c_i)_n}{b_i^n c_i^n} \frac{1}{(q/b_i)_n (q/c_i)_n} \\ &= \frac{(q)_N (q/b_k c_k)_N}{(q/b_k)_N (q/c_k)_N} \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 0} \beta_{n_1} \frac{q^{n_k} (q^{-N})_{n_k} (b_k)_{n_k} (c_k)_{n_k}}{(q^{-N} b_k c_k)_{n_k}} \\ & \times \prod_{i=1}^{k-1} \frac{q^{n_i} ((b_i)_{n_i} / b_i^{n_i}) ((c_i)_{n_i} / c_i^{n_i}) (q/b_i c_i)_{n_{i+1} - n_i}}{(q)_{n_{i+1} - n_i} (q/b_i)_{n_{i+1}} (q/c_i)_{n_{i+1}}}. \end{aligned} \tag{21}$$

A special Bailey pair is given by (see [And85, § 3.5]):

$$\begin{aligned} \alpha_0 &= 1, & \alpha_n &= (-1)^n q^{n(n-1)/2} (1 + q^n) \quad \text{for } n \geq 1, \\ \beta_0 &= 1, & \beta_n &= 0 \quad \text{for } n \geq 1. \end{aligned}$$

Using the decomposition

$$(q^x; q)_{nc} = (q^x; q^c)_n (q^{x+1}; q^c)_n \cdots (q^{x+c-1}; q^c)_n$$

for $x = -N$ and $x = N + 1$, we can identify S_N with the left-hand side of (21) where the parameters are chosen as follows. Let $s = (c + 1)/2$ and $k = b + s$. Suppose $N = mc + t$ with $0 \leq t \leq c - 1$. We consider the limit $b_i, c_i \rightarrow \infty$ for $i = s + 1, \dots, k$. We put $b_s = q^{t-N}$ and $c_s = q^{N+c}$. For $j = 1, 2, \dots, s - 1$, among the integers $\{0, 1, 2, \dots, c - 1\}$ there is exactly one u_j and v_j such that $u_j = j + t \pmod{c}$ and $v_j = -j + t \pmod{c}$. We choose $b_j = q^{u_j - N}$ and $c_j = q^{v_j - N}$ for $j = 1, 2, \dots, s - 1$. The base q in the identity should be replaced by q^c . Therefore, in the rest of this section we put

$$(q^a)_m := (q^a; q^c)_m.$$

The right-hand side of the identity gives us the following expression for S_N :

$$S_N = \sum_{N \geq n_k \geq n_{k-1} \geq \dots \geq n_2 \geq 0} \hat{F}(n_k, \dots, n_2) \tilde{F}(n_k, \dots, n_2), \tag{22}$$

where

$$\begin{aligned} \hat{F}(n_k, \dots, n_2) &\sim \frac{(q^c)_N (q^{-Nc})_{n_k} (q^{Nc+c})_{n_s} (q^{-mc})_{n_s} (q^{mc-Nc})_{n_{s+1}-n_s}}{(q^{-Nc})_{n_{s+1}} (q^{c+mc})_{n_{s+1}} \prod_{i=1}^{k-1} (q^c)_{n_{i+1}-n_i}} \\ & \times \prod_{j=1}^{s-1} (q^{c+2N-v_j-u_j})_{n_{j+1}-n_j}, \\ \tilde{F}(n_k, \dots, n_2) &\sim \prod_j^{s-1} \frac{(q^{u_j-N})_{n_j} (q^{v_j-N})_{n_j}}{(q^{c+N-u_j})_{n_{j+1}} (q^{c+N-v_j})_{n_{j+1}}}. \end{aligned}$$

Here $x \sim y$ means x/y is a unit in $\mathbb{Z}[q^{\pm 1}]$. Note that $c - 2N - v_j - u_j$, which is equal to $2N - 2t \pm c$ or $2N - 2t$, is always a multiple of c .

Observe that $\hat{F}(n_k, \dots, n_2) \neq 0$ if and only if the following inequalities hold:

$$n_k \leq N, \quad n_s \leq \lfloor N/c \rfloor = m \tag{23}$$

(otherwise $(q^{-Nc})_{n_k}$ or $(q^{-mc})_{n_s}$ is zero);

$$n_{s+1} - n_s \leq N - m \tag{24}$$

(otherwise $(q^{mc-Nc})_{n_{s+1}-n_s}$ is zero).

Let us assume that q is a primitive r th root of unity. Then we have in addition

$$N \leq r/c, \quad Nc + cn_s < r \quad (25)$$

(otherwise $(q^c)_N$ or $(q^{Nc+c})_{n_s}$ is zero). Note that if $\hat{F}(n_k, \dots, n_2) \neq 0$ then it is also well defined.

LEMMA 2.5. *Suppose that q is a primitive r th root of unity. Then $z\tilde{F}(n_k, \dots, n_2)$ is divisible by $(\widetilde{q; q})_N$.*

Proof. It suffices to show that z is divisible by $(\widetilde{q; q})_N D$, where D is the denominator of $\tilde{F}(n_k, \dots, n_2)$. Since $n_2 \leq n_3 \leq \dots \leq n_s$, we have

$$D \mid (q^{1+N})_{n_s} (q^{2+N})_{n_s} \cdots (q^{c+N})_{n_s} = (q^{1+N}; q)_{cn_s}$$

and so $(\widetilde{q; q})_N D$ divides $(\widetilde{q; q})_N (q^{1+N}; q)_{cn_s} = (\widetilde{q; q})_{N+cn_s}$, but $N + cn_s \leq (r-1)/2$. Indeed,

$$2N + 2cn_s \leq 3N + cn_s \leq Nc + cn_s < r$$

by (23) and (25). Hence, we obtain

$$(\widetilde{q; q})_{N+cn_s} \mid (\widetilde{q; q})_{(r-1)/2} = z. \quad \square$$

LEMMA 2.6. *For a primitive r th root of unity q , $\hat{F}(n_k, \dots, n_2)$ is divisible by $(\widehat{q; q})_N = (q^c; q^c)_m$.*

Proof. For integer $a \geq b > 0$, using the formula

$$(q^{-ac})_b \sim \frac{(q^c)_a}{(q^c)_{a-b}},$$

we have

$$\frac{(q^{-Nc})_{n_k}}{(q^{-Nc})_{n_{s+1}}} \frac{\prod_{j=1}^{s-1} (q^{c+2N-v_j-u_j})_{n_{j+1}-n_j}}{\prod_{i=1}^{k-1} (q^c)_{n_{i+1}-n_i}} \sim \frac{(q^c)_{N-n_{s+1}}}{(q^c)_{N-n_k}} \frac{\prod_{j=1}^{s-1} (q^{c+2N-v_j-u_j})_{n_{j+1}-n_j}}{\prod_{i=1}^{k-1} (q^c)_{n_{i+1}-n_i}}.$$

The latter, using the fact that $(q^c)_a$ divides $(q^{c+2N-v_j-u_j})_a$, is divisible by $1/(q^c)_{n_{s+1}-n_s}$. Thus $\hat{F}(n_k, \dots, n_2)/(q^c)_m$ is divisible by

$$\frac{(q^c)_{N-m}}{(q^c)_{n_{s+1}-n_s}} \frac{(q^c)_{N+n_s}}{(q^c)_{m+n_{s+1}} (q^c)_{N-m-n_{s+1}+n_s}} (q^{-mc})_{n_s}.$$

Note that in the first factor the denominator divides the numerator due to (24), and in the second factor because of the binomial integrality. \square

2.2 Diagonalization of the linking matrix

We say that a closed 3-manifold is of diagonal type if it can be obtained by integral surgery along an algebraically split link.

PROPOSITION 2.7. *Suppose that M is a closed 3-manifold. There exist lens spaces M_1, \dots, M_k of the form $L(2^l, a)$ such that the connected sum of $(M \# M)$ and these lens spaces is of diagonal type.*

We modify the proof of a similar result in [Le05].

2.2.1 *Linking pairing.* Recall that a *linking pairing* on a finite Abelian group G is a non-singular symmetric bilinear map from $G \times G$ to \mathbb{Q}/\mathbb{Z} . Two linking pairings ν, ν' on respectively G, G' are isomorphic if there is an isomorphism between G and G' carrying ν to ν' . With the obvious block sum, the set of equivalence classes of linking pairings is a semigroup.

One type of linking pairing is given by non-singular square symmetric matrices with integer entries: any such $n \times n$ matrix A gives rise to a linking pairing $\phi(A)$ on $G = \mathbb{Z}^n/A\mathbb{Z}^n$ defined by $\phi(A)(v, v') = v^t A^{-1} v' \in \mathbb{Q} \bmod \mathbb{Z}$, where $v, v' \in \mathbb{Z}^n$. If there is a *diagonal* matrix A such that a linking pairing ν is isomorphic to $\phi(A)$, then we say that ν is of *diagonal type*.

Another type of pairing is the pairing $\phi_{b,a}$, with a, b non-zero co-prime integers, defined on the cyclic group \mathbb{Z}/b by $\phi_{b,a}(x, y) = axy/b \bmod \mathbb{Z}$. It is clear that $\phi_{b,\pm 1}$ is also of the former type, namely, $\phi_{b,\pm 1} = \phi(\pm b)$, where $(\pm b)$ is considered as the 1×1 matrix with entry $\pm b$.

PROPOSITION 2.8. *Suppose that ν is a linking pairing on a finite group G . There are pairs of integers (b_j, a_j) , $j = 1, \dots, s$, with b_j a power of 2 and a_j either -1 or 3 , such that the block sum of $\nu \oplus \nu$ and all the ϕ_{b_j, a_j} is of diagonal type.*

Proof. The following pairings, in three groups, generates the semigroup of linking pairings; see [KK80, Wal63].

Group 1: $\phi(\pm p^k)$, where p is a prime, and $k > 0$.

Group 2: $\phi_{b,a}$ with $b = p^k$ as in group 1, and a is a non-quadratic residue modulo p if p is odd, or $a = \pm 3$ if $p = 2$.

Group 3: E_0^k on the group $\mathbb{Z}/2^k \oplus \mathbb{Z}/2^k$ with $k \geq 1$ and E_1^k on the group $\mathbb{Z}/2^k \oplus \mathbb{Z}/2^k$ with $k \geq 2$.

For explicit formulas of E_0^k and E_1^k , see [KK80]. We will use only a few relations between these generators, taken from [KK80, Wal63].

Any pairing ϕ in group 1 is already diagonal by definition, hence $\phi \oplus \phi$ is also diagonal.

A pairing $\phi = \phi_{b,a}$ in group 2 might not be diagonal, but its double $\phi \oplus \phi$ is always so. Suppose that b is odd; then one of the relations is $\phi_{b,a} \oplus \phi_{b,a} = \phi(b) \oplus \phi(b)$, which is diagonal type. Suppose that b is even; then $b = 2^k$, $a = \pm 3$, and one of the relations says that $\phi_{b,\pm 3} \oplus \phi_{b,\pm 3} = \phi(\mp b) \oplus \phi(\mp b)$.

Thus $\nu \oplus \nu$ is the sum of a diagonal linking pairing and generators of group 3.

Some of the relations concerning group 3 generators are

$$\begin{aligned} E_0^k \oplus \phi_{2^k, -1} &= \phi(2^k) \oplus \phi(-2^k) \oplus \phi(-2^k), \\ E_1^k \oplus \phi_{2^k, 3} &= \phi(2^k) \oplus \phi(2^k) \oplus \phi(2^k). \end{aligned}$$

Thus by adding to $\nu \oplus \nu$ pairings of the form $\phi_{2^k, a}$ with $a = -1$ or $a = 3$, we get a new linking pairing which is diagonal. □

2.2.2 Proof of Proposition 2.7. Every closed 3-manifold M defines a linking pairing, which is the linking pairing on the torsion of $H_1(M, \mathbb{Z})$. The connected sum of 3-manifolds corresponds to the block sum of linking pairings.

First suppose that M is a rational homology 3-sphere, i.e. M is obtained from S^3 by surgery along a framed oriented link L , with non-degenerate linking matrix A . Then the linking pairing on $H_1(M, \mathbb{Z})$ is exactly $\phi(A)$. Also, the lens space $L(b, a)$ has linking pairing $\phi_{b,a}$. Proposition 2.7 follows now from Proposition 2.8 and the well-known fact that, if the linking pairing on $H_1(M, \mathbb{Z})$ is of diagonal type; then M is of diagonal type; see [Oht96, Le05].

The case when M has the higher first Betti number reduces to the case of rational homology 3-spheres just as in [Le05].

2.3 Proof of Theorem 1 (general case)

LEMMA 2.9. *Suppose that $(a, r) = 1$, and $M = L(a, b)$, the lens space. Then $\tau_M(\xi) \in \mathbb{Z}[\xi]$ and, moreover, $\tau_M(\xi)$ is invertible in $\mathbb{Z}[\xi]$.*

Proof. This follows from the explicit formula for the $SO(3)$ invariant of a lens space given below (26). Note that, if $a^*a = 1 \pmod{r}$, then

$$\frac{1 - \xi}{1 - \xi^{a^*}} = \frac{1 - \xi^{aa^*}}{1 - \xi^{a^*}}. \quad \square$$

Proof of Theorem 1 (general case). Choose the lens spaces M_1, \dots, M_k as in Proposition 2.7. Since $N := M \# M \# M_1 \# \dots \# M_k$ is of diagonal type, its $SO(3)$ invariant is in $\mathbb{Z}[\xi]$. Note that the orders of the first homology of M_1, \dots, M_k are powers of 2, and hence co-prime with r . Lemma 2.9 shows that the $SO(3)$ invariant of $M \# M$ is in $\mathbb{Z}[\xi]$, and by Lemma 2.2, the $SO(3)$ invariant of M is in $\mathbb{Z}[\xi]$ too. \square

3. Rational surgery formula

3.1 Hopf chain

Let a, b be co-prime integers with $b > 0$. It is well known that rational surgery with parameter a/b over a link component can be achieved by shackling that component with a framed Hopf chain and then performing integral surgery, in which the framings m_1, m_2, \dots, m_n are related to a/b via the following continued fraction expansion.

$$\frac{a}{b} = - \frac{1}{m_n - \frac{1}{m_{n-1} - \dots - \frac{1}{m_2 - \frac{1}{m_1}}}}$$

Let $D := (F_{U^+}(\xi))^{\sigma_+^H} (F_{U^-}(\xi))^{\sigma_-^H}$ where σ_{\pm}^H is the number of the (positive/negative) eigenvalues of the linking matrix for the Hopf chain. Let $\left(\frac{a}{r}\right)$ be the Jacobi symbol and $s(b, a)$ the Dedekind sum. Recall that

$$s(b, a) := \sum_{i=1}^{|a|-1} \left(\left(\frac{i}{a} \right) \right) \left(\left(\frac{ib}{a} \right) \right), \quad \text{where } ((x)) := x - [x] - 1/2.$$

LEMMA 3.1. *For odd r with $(b, r) = 1$, we have*

$$\begin{aligned} & \frac{\text{ev}_{\xi}([j])}{D} \sum_{j_1, \dots, j_n} \xi \prod_{i=1}^n q^{m_i(j_i^2-1)/4} [j_i] \left(\text{diagram of } n \text{ Hopf links } \right) \\ & = \text{ev}_{\xi} \left(\left(\frac{b}{r} \right) q^{3s(a,b)} \left[\frac{j}{b} \right] q^{a(j^2-1)/(4b)} \right). \end{aligned}$$

Proof. The colored Jones polynomial of the (j_1, j_2) -colored Hopf link is $[j_1 j_2]$. Thus we have to compute

$$\sum_{j_1, \dots, j_n} \xi q^{\sum_i m_i(j_i^2-1)/4} (q^{j_1/2} - q^{-j_1/2}) (q^{j_1 j_2/2} - q^{-j_1 j_2/2}) \dots (q^{j_n j/2} - q^{-j_n j/2}).$$

The result is given by [LL96, Lemma 4.12], where $A = \xi^{1/4}$ has the same order as ξ , because r is odd. Moreover, p and q in [LL96] are related to our parameters as follows: $a = -q$ and $b = p$. Computations analogous to [LL96, Lemmas 4.15–4.21] imply the result. \square

If $(r, a) = 1$, the $SO(3)$ invariant of the lens space $L(a, b)$, which is obtained by surgery along the unknot with rational framing a/b , can be easily computed:

$$\tau_{L(a,b)}(\xi) = \left(\frac{a}{r}\right) \text{ev}_\xi \left(q^{-3s(b,a)} \frac{q^{1/2a} - q^{-1/2a}}{q^{1/2} - q^{-1/2}} \right). \tag{26}$$

Here we used the Dedekind reciprocity law (see e.g. [KM94]), where $\text{sn}(d)$ is the sign of d ,

$$12(s(a, b) + s(b, a)) = \frac{a}{b} + \frac{b}{a} + \frac{1}{ab} - 3 \text{sn}(ab), \tag{27}$$

multiplicativity of the Jacobi symbols $(\frac{ab}{r}) = (\frac{a}{r})(\frac{b}{r})$ and

$$\frac{\gamma_d(\xi)}{\gamma_{\text{sn}(d)}(\xi)} = \left(\frac{|d|}{r}\right) \text{ev}_\xi(q^{(\text{sn}(d)-d)/4}) \tag{28}$$

which holds for any non-zero integer d . Note that $\tau_{L(a,b)}(\xi)$ is invertible in $\mathbb{Z}[\xi]$.

3.2 Laplace transform

The Laplace transform method, developed in [BBL05], allows us to construct a unified invariant by computing the Laplace transform of

$$\begin{bmatrix} n+k \\ 2k+1 \end{bmatrix} [n],$$

and by proving its divisibility by $\{2k+1\}!/\{k\}!$. Let us show how this strategy works for rational framings.

Suppose that one component of L has rational framing a/b . Then by Lemma 3.1 we have to compute

$$\mathcal{L}_{a/b;n} \left(\begin{bmatrix} n+k \\ 2k+1 \end{bmatrix} \{k\}! \left\{ \frac{n}{b} \right\} \right) = \frac{\{k\}!}{\{2k+1\}!} \mathcal{L}_{a/b;n} \left(\frac{\{n/b\} \{n+k\}!}{\{n-k-1\}!} \right).$$

Let $Y_k(q, n, b) := \{n/b\} \{n+k\}!/\{n-k-1\}!$. One can easily see that $Y_k(q, n, b) = Y_k(q, -n, b)$ and $Y_k(q, n, b) = Y_k(q^{-1}, n, b)$. This implies for $H_k(q, a/b) := \mathcal{L}_{-a/b;n}(Y_k(q, n, b))$ that

$$H_k(q, a/b) = H_k(q^{-1}, -a/b).$$

Therefore, it is sufficient to compute $H_k(q, a/b)$ for $a > 0$.

3.3 Divisibility of the Laplace transform images

PROPOSITION 3.2. For $a, b \in \mathbb{N}$, $(a, r) = 1$, $(b, r) = 1$ and $k \leq (r-3)/2$, we have

$$\sum_n^\xi q^{a(1-n^2)/(4b)} \begin{bmatrix} n+k \\ 2k+1 \end{bmatrix} \{k\}! \left\{ \frac{n}{b} \right\} = 2q^{(b-1)^2/(4ab)} \gamma_{-a/b}(\xi) \text{ev}_\xi(F_k(q, a, b)),$$

where $F_k(q, a, b) \in q^{(3k+2)(k+1)/4} \mathcal{R}_{a,2k+1}$.

A similar formula in the case $b = 1$ was obtained in [Le05]. Proposition 3.2 implies that

$$F_k(q, a, 1) = \frac{\{k\}!}{\{2k+1\}!} Y(k, a), \tag{29}$$

where $Y(k, a)$ was defined in [Le05] as follows:

$$Y(k, a) := \sum_{j=0}^{2k+1} (-1)^j \begin{bmatrix} 2k+1 \\ j \end{bmatrix} q^{(j-k)^2/a}. \tag{30}$$

The proof of Proposition 3.2 is given in Appendix A. In the rest of the section we define $F_k(q, a, b)$. We put

$$C_{k,a,b} = (-1)^k q^{(5k+2)(k+1)/4} t^{(k(k+1)/2)(2b-3)} \frac{(t; t)_{2k+1}}{(q; q)_{2k+1}}.$$

Let w be a primitive root of unity of order a . Let t be the a th primitive root of q , i.e. $t^a = q$. We use the following notation: $(q^x)_y = (q^x; q)_y$, $(t^x)_y = (t^x; t)_y$ and $(w^{\pm i} t^y)_x = (w^i t^y; t)_x (w^{-i} t^y; t)_x$.

Case a is odd. For odd a we define $c = (a - 1)/2$, $l = c + b - 1$, and $x_i = \sum_{j=1}^{c-1+i} m_j$. Then

$$\begin{aligned} \frac{F_k(q, a, b)}{C_{k,a,b}} &:= \sum_{m_1, \dots, m_l \geq 0, x_b \leq k} (-1)^{m_1} t^{-m_1(m_1+1)/2 + \sum_{i=1}^b x_i(x_i-1)} \\ &\times \frac{(t^{4k+2})_{m_{c-1}+2m_{c-2}+\dots+(c-1)m_1}}{(t^{2k})_{m_l+2m_{l-1}+\dots+lm_1}} \\ &\times \frac{(q^{-k})_{k-m_1} (t^{2k+2})_{m_1} (t^{2k+2})_{m_2} \dots (t^{2k+2})_{m_c}}{(t)_{k-x_b} (t)_{m_2} (t)_{m_3} \dots (t)_{m_l}} \\ &\times \frac{(w^{\pm 2} t^{-2k-1})_{m_1} (w^{\pm 3} t^{-2k-1})_{m_1+m_2} \dots (w^{\pm c} t^{-2k-1})_{x_0}}{(w^{\pm 2} t^{m_1+1})_{m_2} (w^{\pm 3} t^{m_1+1})_{m_2+m_3} \dots (w^{\pm c} t^{m_1+1})_{x_1-m_1}}. \end{aligned} \tag{31}$$

Case a is even. For even a we define $c = a/2 - 1$, $l = c + b$, and $x_i = \sum_{j=1}^{c+i} m_j$. Then

$$\begin{aligned} \frac{F_k(q, a, b)}{C_{k,a,b}} &:= \sum_{m_1, \dots, m_l \geq 0, x_b \leq k} (-1)^{m_1} t^{-m_1(m_1+1)/2 + \sum_{i=1}^b x_i(x_i-1)} \frac{(t^{4k+2})_{m_{c-1}+2m_{c-2}+\dots+(c-1)m_1}}{(t^{2k})_{m_l+2m_{l-1}+\dots+lm_1}} \\ &\times \frac{(t^{3k+1})_{x-1} (-w^c t^{4k+2})_{m_c} (q^{-k})_{k-m_1} (t^{2k+2})_{m_1} \dots (t^{2k+2})_{m_{c-1}} (-w^{-c} t^{k+1})_{m_c} (-w^c t^{2k+2})_{m_{c+1}}}{(t)_{k-x_b} (t)_{m_2} (t)_{m_3} \dots (t)_{m_l}} \\ &\times \frac{(w^{\pm 2} t^{-2k-1})_{m_1} (w^{\pm 3} t^{-2k-1})_{m_1+m_2} \dots (w^c t^{-2k-1})_{x-1} (w^{-c} t^{-2k-1})_{x_0} (-t^{-2k-1})_{x_0}}{(w^{\pm 2} t^{m_1+1})_{m_2} (w^{\pm 3} t^{m_1+1})_{m_2+m_3} \dots (w^c t^{m_1+1})_{x_0-m_1} (-t^{-k+x-1})_{m_c} (w^{-c} t^{m_1+1})_{x_1-m_1} (-t^{m_1+1})_{x_1-m_1}}. \end{aligned} \tag{32}$$

Example. We consider the following:

$$F_k(q, 1, b) := q^{-(3k-2)(k+1)/4} q^{kb(k+1)} \sum_{m_1, m_2, \dots, m_b \geq 0, \sum m_i = k} q^{\sum_{i=1}^{b-1} (x_i^2 - (2k+1)x_i)} \frac{(q)_k}{\prod_{i=1}^b (q)_{m_i}} \tag{33}$$

where $x_p = \sum_{i=1}^p m_i$.

Note that (33) coincides up to units with the formula for the coefficient $c'_{k,b}$ in the decomposition of ω^b computed in [Mas03, (46)]. (The same coefficient (up to units) appears also in the cyclotomic expansion of the Jones polynomial of twist knots (35).) This is because surgery on the $(-1/b)$ -framed component can be achieved by replacing this component by b (-1) -framed copies. Indeed, changing the variables in (33) as $s_1 = k - m_1$, $s_2 = k - m_1 - m_2, \dots, s_{b-1} = k - m_1 - \dots - m_{b-1}$, we get

$$F_k(q, 1, b) = q^{(k+2)(k+1)/4} \sum_{k \geq s_1 \geq s_2 \geq \dots \geq s_{b-1} \geq 0} q^{s_1^2 + s_2^2 + \dots + s_{b-1}^2 + s_1 + \dots + s_{b-1}} \frac{(q)_k}{\prod_{i=1}^{b-1} (q)_{s_i - s_{i+1}}}.$$

4. Unified invariant

In this section we assume that $(r, a) = 1$, where r is the order of the root of unity ξ and $a = |H_1(M, \mathbb{Z})|$.

Let $M = L(a, b)$ be a lens space with $a > 0$. Then the unified invariant I_M was defined in [Le05] as follows:

$$I_M := q^{3s(1,a)-3s(b,a)} \frac{1 - q^{-1/a}}{1 - q^{-1}}.$$

Note that $3(s(1, a) - s(b, a)) \in \mathbb{Z}$ and I_M is invertible in Λ_a .

For an arbitrary rational homology sphere M with $a = |H_1(M, \mathbb{Z})|$, it was shown in [Le05] that there are lens spaces M_1, \dots, M_l such that $M' = (\#_{i=1}^l M_i) \# M$ can be obtained by surgery on an algebraically split link and I_{M_j} are invertible in Λ_a . Then we can define

$$I_M = I_{M'} \left(\prod_{i=1}^l I_{M_i} \right)^{-1}.$$

It remains to define I_M when M is given by surgery along an algebraically split link L . Assume that L has m components with non-zero rational framings $a_1/b_1, \dots, a_m/b_m$. Then we have $|H_1(M, \mathbb{Z})| = a$ for $a = \prod_i a_i$. Let L_0 be L with all framings switched to zero.

THEOREM 7. For M as above, the unified invariant is given by the following formula:

$$I_M = q^{(a-1)/4} \sum_{k_i=0}^{\infty} J_{L_0}(P'_{k_1}, \dots, P'_{k_m}) \prod_{i=1}^m \text{sn}(a_i) q^{1/(2a_i)-3s(b_i, a_i)} F_{k_i}(q^{-\text{sn}(a_i)}, |a_i|, b_i). \tag{34}$$

Moreover, we have

$$\left(\frac{a}{r} \right) \tau_M(\xi) = \text{ev}_\xi(q^{(1-a)/4} I_M).$$

Proof. Note first that, if $b_i = 1$ for all i , our formula coincides with [Le05, (21)]. It follows from (29) and

$$q^{(3\text{sn}(a_i)-a_i)/4} q^{3s(1, a_i)} = q^{1/(2a_i)}.$$

Here we used that

$$3s(1, a) = \frac{1}{2a} + \frac{a - 3\text{sn}(a)}{4}$$

by the reciprocity law (27).

Let us collect the coefficients in the definition of τ_M . From Lemmas 1.1 and 3.1, Proposition 3.2 and (28) we have

$$q^{3s(a_i, b_i) - (b_i - 1)^2 / (4a_i b_i) + 3\text{sn}(a_i) / 4 - a_i / (4b_i)} = q^{-3s(b_i, a_i)} q^{1/(2a_i)}.$$

The Corollary 0.3(d) in [Le05] allows us to drop the conditions $(b_i, r) = 1$, because I_M is determined by its values at any infinite sequence of roots of prime power order from U_a . \square

4.1 Proof of Theorem 2

The statement holds trivially if $M = L(a, b)$. Indeed, we have $m = 1$, $f_k = 0$ for $k > 0$, and $f_0 = q^{3s(1, a) - 3s(b, a)} (1 - t) / (1 - q) \in \Gamma_a$.

The general case follows from (34), Proposition 3.2 and Theorem 5. Note that multiplication of I_M by the inverse of $I_{L(a, b)}$ multiplies all f_{k_i} by an element of $\mathbb{Z}[t^{\pm 1}]$. Moreover, I_M does not contain fractional powers of $q^{1/a}$ (compare the proof of Lemma 4.2 in [Le05]).

4.2 On the ring $\widehat{\mathcal{R}}_a$

Here we present the proof (of the referee) that $\widehat{\mathcal{R}}_a$ coincides with the ring $\widehat{\Lambda}_a$ of [Le05]. The ring $\widehat{\Lambda}_a$ in [Le05] is obtained from \mathcal{R}_a by first inverting a , and then completing using $(q; q)_n$:

$$\widehat{\Lambda}_a := \varprojlim_n \mathcal{R}_a[1/a]/((q; q)_n).$$

To prove that $\widehat{\Lambda}_a = \widehat{\mathcal{R}}_a$ one needs only to show that a is invertible in $\widehat{\mathcal{R}}_a$.

Suppose that p is a prime factor of a . For any integer $l \in \mathbb{N}_a$ we have $\Phi_{pl}(t) \in (p, \Phi_l(t))$ in $\mathbb{Z}[t^{\pm 1}]$. Since $pl \notin \mathbb{N}_a$, Φ_{pl} is invertible in \mathcal{R}_a . Therefore $(p, \Phi_l(t)) = 1$ in \mathcal{R}_a , or p is invertible in $\mathcal{R}_a/(\Phi_l)$. Since

$$t^m - 1 = \prod_{l|m} \Phi_l(t),$$

it follows that if, $m \in \mathbb{N}_a$, then p is invertible in $\mathcal{R}_a/((t^m - 1)^j)$ for every $j \geq 1$. Hence p is invertible in the completion of \mathcal{R}_a with respect to the directed system of ideals $\{(t^m - 1)^j \mathcal{R}_a\}_{j \geq 1, m \in \mathbb{N}_a}$. Note that $\{(q; q)_n \mathcal{R}_a\}_{n \geq 1}$ and $\{(t^m - 1)^j \mathcal{R}_a\}_{j \geq 1, m \in \mathbb{N}_a}$ are cofinal, hence they define the same completion. This completes the proof.

4.3 Proof of Proposition 1

First note that, if $f = 1/a$, then $f \in \widehat{\mathcal{R}}_a$, and $\text{ev}_\xi(f) = 1/a$. It follows that $\text{ev}_\xi \widehat{\mathcal{R}}_a = \mathbb{Z}[\xi/a]$.

Assume that the order of $\xi \in U_a$ is r . Suppose that $f \in \Gamma_a$ has a presentation given by formula (1). Since $\text{ev}_\xi(x_n) = 0$ if $2n + 1 \geq r$, we have

$$\text{ev}_\xi(f) = \sum_{n=0}^{(r-3)/2} \text{ev}_\xi(f_n(t)) \text{ev}_\xi(x_n).$$

Since $x_n \in \mathbb{Z}[q^{\pm 1}]$, then $\text{ev}_\xi(x_n) \in \mathbb{Z}[\xi]$. We will show that $\text{ev}_\xi(f_n(t)) \in \mathbb{Z}[\xi]$ for $n \leq (r - 3)/2$.

Note that

$$f_n(t) \in \mathcal{R}_{a,k} = \mathbb{Z}[t^{\pm 1}] \left[\frac{(t; t)_k}{(q; q)_k} \right],$$

with $k = r - 1$. Since $k < r$, $\text{ev}_\xi((q; q)_k) \neq 0$. Hence, with an integer a^* such that $aa^* \equiv 1 \pmod{r}$, we have

$$\text{ev}_\xi \left(\frac{(t; t)_k}{(q; q)_k} \right) = \prod_{j=1}^k \frac{1 - \xi^{a^* j}}{1 - \xi^j} \in \mathbb{Z}[\xi].$$

It follows that $\text{ev}_\xi(f_n(t)) \in \mathbb{Z}[\xi]$ for every $f_n(t) \in \mathcal{R}_{a,k}$. This completes the proof of Proposition 1.

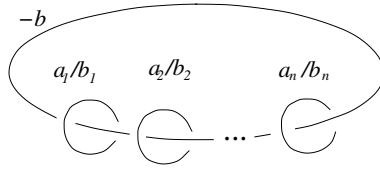
5. Applications

In this section we compute the unified invariant I_M for Seifert fibered spaces and for a/b surgeries on twist knots.

5.1 Seifert fibered spaces with a spherical base

Let $M = L(b; a_1/b_1, \dots, a_n/b_n)$ be the Seifert fibered space with base space S^2 and with n exceptional fibers with orbit invariants (a_i, b_i) (with $a_i > 0$, $0 \leq b_i \leq a_i$, $(a_i, b_i) = 1$), and with bundle invariant $b \in \mathbb{Z}$.

It is well known [Mon85] that M is a rational homology sphere if $e := b + \sum b_i/a_i \neq 0$ and $|H_1(M, \mathbb{Z})| = |e| \prod_i a_i$. Furthermore, M can be obtained by surgery on the following (rationally framed) link.



THEOREM 8. *Let $M = L(b; a_1/b_1, \dots, a_n/b_n)$ be as above. Assume that $e \neq 0$, and $|H_1(M, \mathbb{Z})| = d$. Then we have*

$$I_M = q^{(d-1)/4} \frac{q^{(e-3\text{sn}(e))/4} q^{-3 \sum_i s(b_i, a_i)}}{\{1\}} \mathcal{L}_{-e; j} \left(\frac{\prod_{i=1}^n \{j/a_i\}}{\{j\}^{n-2}} \right).$$

Proof. The linking matrix of the surgery link has n positive eigenvalues and the sign of the last eigenvalue is equal to $-\text{sn}(e) = -\text{sn}(b)$. Let us color the rationally framed components of the surgery link by j_i , $i = 1, \dots, n$, and the $(-b)$ -framed component by j .

The main ingredient of the proof is the following computation. Using Lemmas 1.1 and 3.1 we have

$$\left(\frac{b_i}{r} \right) \frac{q^{3s(a_i, b_i)}}{F_{U+(\xi)}} \sum_{\xi} q^{a_i(j_i^2-1)/(4b_i)} \left[\frac{j_i}{b_i} \right] [j j_i] = \left(\frac{a_i}{r} \right) q^{-3s(b_i, a_i)} q^{-b_i(j^2-1)/(4a_i)} \left[\frac{j}{a_i} \right].$$

Applying finally the Laplace transform $\mathcal{L}_{-e; j}$ and collecting the factors, we get the result. \square

5.2 Proof of Theorem 3

Note that $M = L(b; a_1/b_1, \dots, a_n/b_n)$ is an integral homology sphere if $e^{-1} = \pm \prod_i a_i$. Then M is uniquely determined by the pairwise co-prime integers a_i . (Knowing the a_i and e , one can compute the b_i and b using the Chinese remainder theorem.)

Suppose for simplicity that $e > 0$. Rewriting

$$\frac{1}{\{j\}^{n-2}} = (-1)^{n-2} q^{(n-2)/2} \sum_{k=0} c_k q^k$$

with $c_k \in \mathbb{Z}$, we see that the image of the Laplace transform is the sum of the following terms:

$$(-1)^{n-2} c_k q^{(\prod_i a_i/4)(\pm 1/a_1 \pm 1/a_2 \dots \pm 1/a_n + 2k + n - 2)^2}.$$

The leading term in I_M for $k \rightarrow \infty$ behaves asymptotically like

$$q^{k^2 \prod_i a_i + k(n-2) \prod_i a_i + k \sum_i a_1 \dots \hat{a}_i \dots a_n},$$

where \hat{a}_i means delete a_i . This allows us to determine the a_i . In the case $e < 0$, we have the same asymptotic expression after replacing q by q^{-1} .

5.3 Dehn surgeries on twist knots

Let K_p be the twist knot with p twists. Masbaum [Mas03] calculated the P'_n colored Jones polynomial of this knot. For $p > 0$ we have

$$J_{K_p}(P'_n) = q^{n(n+3)/2} \sum_{i_1, i_2, \dots, i_p \geq 0, \sum_j i_j = p} q^{\sum_i (s_i^2 + s_i)} \frac{(q)_n}{\prod_{j=1}^p (q)_{i_j}}, \tag{35}$$

where $s_k = \sum_{j=1}^k i_j$. The formula for negative p can be obtained from the given one by sending $p \rightarrow -p$, $q \rightarrow q^{-1}$, forgetting the factor $q^{n(n+3)/2}$ and multiplying the result by $(-1)^n$.

COROLLARY 5.1. *Let $M_{a/b}$ be obtained by (a/b) surgery in S^3 on the twist knot K_p . Then we have*

$$I_{M_{a/b}} := q^{(a-1)/4} \operatorname{sn}(a) q^{-3s(b,a)+1/(2a)} \sum_{n=0}^{\infty} J_{K_p}(P'_n) F_n(q^{-\operatorname{sn}(a)}, |a|, b). \quad (36)$$

5.4 Proof of Theorem 4

Assume that K and K' are 0-framed. We expand the function $Q_K(N) := J_K(N)[N]$ around $q = 1$ into power series. Suppose that $q = e^h$; then we have

$$Q_K(N)|_{q=e^h} = \sum_{2j \leq n+2} c_{j,n}(K) N^j h^n.$$

It is known that $c_{j,n}$ is zero if j is odd. Applying the Laplace transform, we have to replace N^{2j} by $(-2)^j (2j-1)! / a^j h^j$ (see [Le03]). Therefore, the following expression coincides (up to some standard factor) with the Ohtsuki series:

$$\sum_{2j \leq n+2} c_{2j,n}(K) (-2)^j (2j-1)! a^{-j} h^{n-j}.$$

From the fact that the Ohtsuki series for $M(K, a)$ and $M(K', a)$ coincide, we derive

$$\sum_{2j \leq n+2} (c_{2j,n}(K) - c_{2j,n}(K')) (-2)^j (2j-1)! a^{-j} h^{n-j} = 0.$$

Because the last system of equations should hold for infinitely many $a \in \mathbb{Z}$, we have $c_{2j,n}(K) = c_{2j,n}(K')$ and $J_K(N) = J_{K'}(N)$ for any $N \in \mathbb{N}$.

Appendix A

The main technical ingredient we use in the proof of Proposition 3.2 is the Andrews' generalization of Watson's identity [And75, Theorem 4, p. 199]:

$$\begin{aligned} & {}_{2p+4}\phi_{2p+3} \left[\begin{matrix} \alpha, t\sqrt{\alpha}, -t\sqrt{\alpha}, b_1, c_1, \dots, b_p, c_p, t^{-N} \\ \sqrt{\alpha}, -\sqrt{\alpha}, \alpha t/b_1, \alpha t/c_1, \dots, \alpha t/b_p, \alpha t/c_p, \alpha t^{N+1} \end{matrix} ; t, \frac{\alpha^p t^{p+N}}{b_1 c_1 \dots b_p c_p} \right] \\ &= \frac{(\alpha t)_N (\alpha t/b_p c_p)_N}{(\alpha t/b_p)_N (\alpha t/c_p)_N} \sum_{m_1, \dots, m_{p-1} \geq 0} \frac{(b_p)_{\sum_i m_i} (c_p)_{\sum_i m_i} (t^{-N})_{\sum_i m_i}}{(b_p c_p t^{-N} / \alpha)_{\sum_i m_i}} \\ & \times \prod_{i=1}^{p-1} \frac{t^{m_i} (\alpha t)^{(p-i-1)m_i} (\alpha t/b_i c_i)_{m_i} (b_i)_{\sum_{j < i} m_j} (c_i)_{\sum_{j < i} m_j}}{(t)_{m_i} (\alpha t/b_i)_{\sum_{j \leq i} m_j} (\alpha t/c_i)_{\sum_{j \leq i} m_j} (b_i c_i)_{\sum_{j < i} m_j}}, \end{aligned} \quad (A.1)$$

where

$${}_r\phi_s \left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; t, z \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \dots (a_r)_n}{(t)_n (b_1)_n \dots (b_s)_n} [(-1)^n t^{\binom{n}{2}}]^{1+s-r} z^n \quad (A.2)$$

are the basic q -hypergeometric series and $(a)_n := (a; t)_n$.

Proof of Proposition 3.2. We have to compute $\mathcal{L}_{-a/b;n}(\{n/b\} \{n+k\}! / \{n-k-1\}!)$. Note that

$$\{n/b\} \{n+k\}! / \{n-k-1\}! = q^{-n/2-nk-n/(2b)} (1 - q^{n/b}) (q^{n-k})_{2k+1}.$$

Using the q -binomial theorem and (5) (with $c = 1$) we get

$$q^{(2bk+b+1)^2/(4ba)} \sum_{j=0}^{\infty} \frac{(q^{-2k-1})_j}{(q)_j} q^{(b/a)j^2 + (1-2b/a)kj + (1-(b+1)/a)j} (1 - q^{(2j-2k-1)/a}).$$

We put $t := q^{1/a}$ and choose a primitive a th root of unity w . Then using

$$(q^x; q)_j = (t^x; t)_j (wt^x; t)_j (w^2t^x; t)_j \cdots (w^{a-1}t^x; t)_j,$$

we can rewrite the previous sum as follows:

$$\sum_{j=0}^{2k+1} \frac{(t^{-2k-1})_j (wt^{-2k-1})_j \cdots (w^{a-1}t^{-2k-1})_j}{(t)_j (wt)_j \cdots (w^{a-1}t)_j} t^{bj^2 + (a-2b)kj + (a-b-1)j} (1 - t^{2j-2k-1}). \quad (\text{A.3})$$

The main point is that (A.3) is equal to $(1 - t^{-2k-1})$ times the left-hand side of the generalized Watson identity (A.1) with the specialization of parameters described below. We consider the limit $\alpha \rightarrow t^{-2k-1}$. We set $p = \max\{b, a + b - 2\}$, $b_i, c_i \rightarrow \infty$ for $i = a - 1, \dots, p - 1$; and $b_p \rightarrow t^{-k}$, $c_p \rightarrow \infty$ and $N \rightarrow \infty$.

Case when a is odd. We put $c = (a - 1)/2$; $b_i = w^i t^{-2k-1}$, $c_i = w^{-i} t^{-2k-1}$ for $i = 1, \dots, c$; and $b_i, c_i \rightarrow t^{-k}$ for $i = c + 1, \dots, a - 2$.

Case when a is even. We put $c = a/2 - 1$. Let $p = a + b - 2$, $b_i = w^i t^{-2k-1}$, $c_i = w^{-i} t^{-2k-1}$ for $i = 1, \dots, c - 1$; $b_c = w^c t^{-2k-1}$, $c_c = -t^{-k}$; $b_{c+1} = -t^{-2k-1}$, $c_{c+1} = w^{-c} t^{-2k-1}$; and $b_i, c_i \rightarrow t^{-k}$ for $i = c + 2, \dots, a - 2$.

To simplify (A.1) we use the following limits:

$$\begin{aligned} \lim_{c \rightarrow \infty} \frac{(c)_n}{c^n} &= (-1)^n t^{n(n-1)/2}, & \lim_{c \rightarrow \infty} \left(\frac{t}{c}\right)_n &= 1, \\ \lim_{c_1, c_2 \rightarrow \infty} \frac{(c_1)_n (c_2)_n}{(t^{-N} c_1 c_2)_n} &= (-1)^n t^{n(n-1)/2} t^{Nn}, & \lim_{\alpha \rightarrow t^{-2k-1}} \frac{(\alpha t)_\infty}{(\sqrt{\alpha t})_\infty} &= 2(t^{-2k})_k. \end{aligned}$$

Finally, the formulas below allow us to separate the factor $(t)_{2k+1}/(q)_{2k+1}$:

$$\begin{aligned} \frac{\{2k+1\}!}{\{k\}!} &= q^{-(3k+2)(k+1)/4} (-1)^{k+1} (q^{k+1})_{k+1}, \\ (q)_j &= (-1)^{(k-j)} q^{(j-k)(k+j+1)/2} \frac{(q)_k}{(q^{-k})_{k-j}}, \\ (t^{-k})_j &= (-1)^j t^{-kj+j(j-1)/2} \frac{(t)_k}{(t)_{k-j}}. \end{aligned}$$

The next lemma implies the result. □

LEMMA A.1. *We have*

$$\frac{F_k(q, a, b)}{C_{k,a,b}} \in \mathbb{Z}[t^{\pm 1}].$$

Proof. First note that $F_k(q, a, b)$ does not depend on w , because w does not occur in the left-hand side of the identity.

Suppose a is odd. Let $z := x_1 - m_1 = m_2 + m_3 + \cdots + m_c$. Let us complete $(w^c t^{m_1+1})_z$ to $(q^{m_1+1})_z$ by multiplying the numerator and the denominator of (31) with

$$(t^{m_1+1})_z (w^\pm t^{m_1+1})_z (w^{\pm 2} t^{m_1+m_2})_{z-m_2} \cdots (w^{\pm(c-1)} t^{m_1+z-m_c})_{m_c}.$$

Now up to units the denominator of (31) is equal to $(q^{m_1+1})_{x_1-m_1} (t)_{k-x_b} (t)_{m_2} (t)_{m_3} \cdots (t)_{m_l}$, which divides the numerator. The even case is similar. □

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